Acta Math. Univ. Comenianae
Vol. LXI, 2(1992), pp. 143–165

ON THE NEUMANN OPERATOR OF THE ARITHMETICAL MEAN

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We shall identify the Euclidean plane $\mathbb{R}^2$ with the set $\mathbb{C}$ of all complex numbers. If $z \in \mathbb{C}$, then $\operatorname{Re} z$, $\operatorname{Im} z$ and $\overline{z}$ denote the real part, the imaginary part and the complex conjugate of z, respectively. The scalar product of vectors $u, v \in \mathbb{R}^2$ will be denoted by $\langle u, v\rangle$ $(= \operatorname{Re} u\overline{v})$. We shall be engaged with logarithmic potentials in the plane derived from the classical kernel defined for $x, z \in \mathbb{R}^2$ by

$$h_z(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|z-x|}, & \text{if } x \neq z, \\ +\infty, & \text{if } x = z. \end{cases}$$

The symbol λ_k $(k \in \{1, 2\})$ will denote the k-dimensional Hausdorff measure (with the usual normalization, so that $\lambda_k([0,1]^k) = 1$). For $M \subset \mathbb{R}^2$ we use the symbols ∂M, $\operatorname{int} M$ and $\operatorname{cl} M$ to denote the boundary, the interior and the closure of M, respectively. For $M \neq \emptyset$ we denote by $\mathcal{C}(M)$ the Banach space of all bounded continuous functions on M with the supremum norm, by 1_M the constant function equal to 1 on M, by $\operatorname{Const}(M) = \{\alpha 1_M; \alpha \in \mathbb{R}\}$ the class of all constant functions on M. $\mathcal{C}_0^{(1)}$ will stand for the class of all continuously differentiable functions with a compact support in $\mathbb{R}^2$, for bounded M we write $\mathcal{C}^{(1)}(M) = \{\varphi|_M : \varphi \in \mathcal{C}_0^{(1)}\}$ for the class of all restrictions to M of functions in $\mathcal{C}_0^{(1)}$. Throughout, $K \subset \mathbb{R}^2$ will be a fixed non-void compact set which is massive at each $z \in K$ in the sense that each disk

$$B_r(z) = \{x \in \mathbb{R}^2; |x - z| < r\}$$

with radius $r > 0$ and center z in K intersects K in a set of positive Lebesgue measure:

$$\lambda_2[B_r(z) \cap K] > 0\,.$$

This is the only à priori restriction we impose on K; it is by no means essential in connection with boundary value problems (cf. Remark 1.14 and 2.3 in [**8**]) but it will allow us to avoid some technical complications.

Received September 17, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 31B20, 47A53.

Put $G = \mathbb{R}^2 \setminus K$ and denote by $\mathcal{C}^*(\partial K)$ the space of all finite signed Borel measures supported by ∂K. For each $\mu \in \mathcal{C}^*(\partial K)$ the potential

$$\mathcal{U}\mu(x) = \int_{\partial K} h_z(x)\, d\mu(z) \tag{1}$$

defines a harmonic function of the variable x on $\mathbb{R}^2 \setminus \partial K$ such that, for each bounded Borel $P \subset \mathbb{R}^2 \setminus \partial K$, the gradient of (1) is integrable over P:

$$\int_P |\mathrm{grad}\, \mathcal{U}\mu(x)|\, d\lambda_2(x) < +\infty\,.$$

This property makes it possible to introduce the so-called weak normal derivative of $\mathcal{U}\mu$, to be denoted by $N^G\mathcal{U}\mu$, which is defined as a linear functional over $\mathcal{C}_0^{(1)}$ by the formula

$$\langle N^G\mathcal{U}\mu, \varphi\rangle = \int_G \langle \mathrm{grad}\ \varphi(x), \mathrm{grad}\ \mathcal{U}\mu(x)\rangle\, d\lambda_2(x), \quad \varphi \in \mathcal{C}_0^{(1)}\,; \tag{2}$$

if the boundary $\partial G = \partial K$ is smooth and n denotes the unit normal exterior to G, and if $\mathcal{U}\mu$ extends smoothly from G to $\mathrm{cl}\, G$, then the right-hand side in (2) transforms by divergence theorem into

$$\int_{\partial K} \varphi \frac{\partial \mathcal{U}\mu}{\partial n}\, d\lambda_1$$

so that $N^G\mathcal{U}\mu$ is a natural weak characterization of the normal derivative $\frac{\partial \mathcal{U}\mu}{\partial n}$ (compare [**21**]). Transforming the integral occurring in (2) by Fubini's theorem we get, for any $\varphi \in \mathcal{C}_0^{(1)}$,

$$\langle N^G\mathcal{U}\mu, \varphi\rangle = \int_{\partial K} W\varphi(z)\, d\mu(z)\,,$$

where

$$W\varphi(z) = \int_G \langle \mathrm{grad}\ \varphi(x), \mathrm{grad}\ h_z(x)\rangle\ d\lambda_2(x)\,. \tag{3}$$

We shall consider (3) as a function of the variable $z \in K$. It is easily seen (cf. §2 in [**8**]) that, for $z \in K$, (3) depends on $\varphi|_{\partial K}$ only and represents a continuous function on K which is harmonic on $\mathrm{int}\, K$; this function $W\varphi$ will be called the double layer potential of density φ. Note also that, for any fixed $\mu \in \mathcal{C}^*(\partial K)$, the weak normal derivate $N^G\mathcal{U}\mu$ has support contained in ∂K in the sense that $\langle N^G\mathcal{U}\mu, \varphi\rangle = 0$ whenever ∂K does not meet the support of $\varphi \in \mathcal{C}_0^{(1)}$ (cf. 1.2 in [**8**]).

For $\emptyset \neq M \subset K$ denote by $W_M\varphi = (W\varphi)\big|_M$ the restriction to M of the double layer potential $W\varphi$. Then

$$(4_K) \qquad W_K : \varphi\big|_{\partial K} \to W_K\varphi \qquad (\mathcal{C}^{(1)}(\partial K) \to \mathcal{C}(K))$$

and

$$(4_{\partial K}) \qquad W_{\partial K} : \varphi\big|_{\partial K} \to W_{\partial K}\varphi \qquad (\mathcal{C}^{(1)}(\partial K) \to \mathcal{C}(\partial K))$$

are linear operators form $\mathcal{C}^{(1)}(\partial K)$ to $\mathcal{C}(K)$ and from $\mathcal{C}^{(1)}(\partial K)$ to $\mathcal{C}(\partial K)$, respectively. Since

$$(5) \qquad W_K 1_{\partial K} = 1_K$$

(cf. [**8**, p. 60]), we have W_K (Const $(\partial K) \subset$ Const (K) which makes it possible to consider the operators induced on the factor space $\mathcal{C}^{(1)}(\partial K)/_{\mathrm{Const}\,(\partial K)}$ to be denoted by the same symbols

$$(6_K) \qquad W_K : \mathcal{C}^{(1)}(\partial K)/_{\mathrm{Const}\,(\partial K)} \to \mathcal{C}(K)/_{\mathrm{Const}\,(K)}$$

and

$$(6_{\partial K}) \qquad W_{\partial K} : \mathcal{C}^{(1)}(\partial K)/_{\mathrm{Const}\,(\partial K)} \to \mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}\,.$$

Necessary and sufficient geometric condition is known (cf. [**1**], [**9**]; see also the exposition in [**8**], [**14**]) guaranteeing extendability of the operators (4_K), $(4_{\partial K})$ to bounded linear operators defined on the whole $\mathcal{C}(\partial K)$ and of the operators (6_K), $(6_{\partial K})$ to bounded linear operators acting on $\mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}$. As pointed out by M. Chlebík ([**6**]), results in geometric measure theory ([**3**]) permit to formulate this condition (occurring in an equivalent form in [**8**] and [**14**]; cf. proof of Lemma 3 below) in terms of the essential boundary

$$\partial_e K = \{z \in \mathbb{R}^2; \limsup_{r\to 0+} \lambda_2[B_r(z) \cap K]/_{r^2} > 0,\ \limsup_{r\to 0+} \lambda_2[B_r(z) \cap G]/_{r^2} > 0\}$$

as follows. Denoting for θ in

$$\Gamma \equiv \{\theta \in \mathbb{R}^2; |\theta| = 1\}$$

and fixed $z \in \mathbb{R}^2$ by $n^K(z,\theta)$ the total number of points in

$$\{z + t\theta;\ t > 0\} \cap \partial_e K$$

($0 \le n^K(z,\theta) \le +\infty$) we arrive at a λ_1-measurable function $\theta \mapsto n^K(z,\theta)$ which makes it possible to introduce the integral

$$v^K(z) := \frac{1}{\pi} \int_\Gamma n^K(z,\theta)\, d\lambda_1(\theta)\,.$$

Then finiteness of the quality

$$V^K := \sup\{v^K(z);\ z \in \partial K\} \tag{7}$$

is necessary and sufficient for extendability of W_K to a bounded linear operator on $\mathcal{C}(\partial K)$ to $\mathcal{C}(K)$ (or, equivalently, on $\mathcal{C}(\partial K)/\text{Const}\,(\partial K)$ to $\mathcal{C}(K)/\text{Const}\,(\partial K)$) and, which is the same, extendability of $W_{\partial K}$ to a bounded operator on $\mathcal{C}(\partial K)$ (or, equivalently, on $\mathcal{C}(\partial K)/\text{Const}\,(\partial K)$). The same condition

$$V^K < +\infty \tag{8}$$

is necessary and sufficient to guarantee the existence, for each $\mu \in \mathcal{C}^*(\partial K)$, of a (uniquely determined) finite signed Borel measure $\nu_\mu \in \mathcal{C}^*(B)$ representing $N^G\mathcal{U}\mu$ in the sense that

$$\langle N^G\mathcal{U}\mu, \varphi\rangle = \int_{\partial K} \varphi\, d\nu_\mu, \quad \forall\, \varphi \in \mathcal{C}_0^{(1)}\,;$$

under the assumption (8) the arising operator $N^G\mathcal{U}\colon \mu \mapsto \nu_\mu$ is bounded on $\mathcal{C}^*(\partial K)$ and is adjoint to $W_{\partial K}$ acting on $\mathcal{C}(\partial K)$:

$$N^G\mathcal{U} = W^*_{\partial K}\,. \tag{9}$$

Assuming (8) we define the operator of the arithmetical mean, to be denoted by $T^K \equiv T$, by the equation

$$\frac{1}{2}(I + T^K) = W_{\partial K}\,, \tag{10}$$

where I is the identity operator. Then (5) implies

$$T 1_{\partial K} = 1_{\partial K}\,. \tag{11}$$

The norm of T on $\mathcal{C}(\partial K)$ is precisely evaluated by

$$\|T^K\| = V^K \tag{12}$$

(cf. [**8**], 2.25; note that our normalization of $v^K(z)$ is different from that used in [**8**], so that our V^K coincides with $2V^G$ in [**8**]). The attempt to represent the solution of the Dirichlet problem for K with a prescribed boundary condition $g \in \mathcal{C}(\partial K)$

in the form of the double layer potential $W_K f$ with an unknown $f \in \mathcal{C}(\partial K)$ leads to the equation

$$(I+T)f = 2g\,. \tag{13}$$

In view of (9), the attempt to find, for a given $\nu \in \mathcal{C}^*(\partial K)$, another $\mu \in \mathcal{C}^*(\partial K)$ whose potential $\mathcal{U}\mu$ solves the weak Neumann problem $N^G\mathcal{U}\mu = \nu$ for G results in the adjoint equation

$$(I+T)^*\mu = \nu \tag{14}$$

for the unknown μ. It follows from (12) that $\|T^K\| \geq 1$ where the sign of equality holds iff K is convex (cf. [**8**], Theorem 3.1). If we consider T^K on the quotient space $\mathcal{C}(\partial K)/_{\textstyle\mathrm{Const}\,(\partial K)}$, then the quotient norm of T^K, to be denoted by $\|T^K\|_0$, may become less that 1. Let us recall that the norm of the class containing $f \in \mathcal{C}(\partial K)$ in $\mathcal{C}(\partial K)/_{\textstyle\mathrm{Const}\,(\partial K)}$ is given by $\frac{1}{2}\mathrm{osc}\, f(\partial K)$, where

$$\mathrm{osc}\, f(\partial K) = \max f(\partial K) - \min f(\partial K)\,.$$

Hence $\|T^K\|_0$ is the least constant $q \geq 0$ for which

$$\mathrm{osc}\,(T^K f)(\partial K) \leq q\, \mathrm{osc}\, f(\partial K)\,, \quad \forall\, f \in \mathcal{C}(\partial K)\,.$$

This constant was called the configuration constant of K by Carl Neumann who was able to prove for convex K that $\|T^K\|_0 < 1$ iff K is different from triangles and quadrangles ([**18**]) (H. Lebesgue [**12**] observed later that $\|T_K^2\| < 1$ for all convex bodies $K \subset \mathbb{R}^2$) which permitted to establish convergence (in the operator norm) of the Neumann series for the inverse of $I + T^K$ on $\mathcal{C}(\partial K)/_{\textstyle\mathrm{Const}\,(\partial K)}$. Note that, in view of (9)–(11), $(T^K)^*$ maps the subspace

$$\mathcal{C}_0^*(\partial K) := \{\mu \in \mathcal{C}^*(\partial K) :\ \mu(\partial K) = 0\}$$

of all balanced signed measures in $\mathcal{C}^*(\partial K)$ into itself and $\mathcal{C}_0^*(\partial K)$ may be identified with the adjoint space to $\mathcal{C}(\partial K)/_{\textstyle\mathrm{Const}\,(\partial K)}$. Hence $\|T^K\|_0$ equals the norm of the operator $(T^K)^*$ restricted to $\mathcal{C}_0^*(\partial K)$. For general K no simple evaluation of $\|T^K\|_0$ comparable with the formula (12) for $\|T^K\|$ seems to be known. Nevertheless, geometric estimates of the configuration constant $\|T^K\|_0$ can be obtained which permit to establish the inequality $\|T^K\|_0 < 1$ for many concrete highly non-convex compact $K \subset \mathbb{R}^2$. We shall prove the following theorems and some of their consequences.

Theorem 1. *Let B_1, B_2 be disjoint λ_1-measurable subsets of ∂K and suppose that with each $z \in B \equiv B_1 \cup B_2$ there is associated a disk $B(z) = B_{r(z)}(\zeta(z))$ of radius $r(z) = |z - \zeta(z)|$ such that $K \cap B(z) = \emptyset$ for $z \in B_1$, $K \subset \operatorname{cl} B(z)$ for $z \in B_2$ and $z \mapsto r(z)$ is λ_1-measurable. If*

$$\lambda_1(\partial K \setminus B) = 0 \quad \textit{and} \quad \int_B \frac{d\lambda_1(z)}{r(z)} < +\infty\,,$$

then

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(z)}{r(z)} - \int_{B_2} \frac{d\lambda_1(z)}{r(z)}\right). \tag{15}$$

Theorem 2. *Suppose that with each $z \in B_0 \subset B$ there is associated a disk $B(z) = B_{r(z)}(\zeta(z)) \subset K$ of radius $r(z) = |z - \zeta(z)|$. If $z \mapsto r(z)$ is λ_1-measurable,*

$$\lambda_1(\partial K \setminus B_0) = 0 \quad \textit{and} \quad \int_{B_0} \frac{d\lambda_1(z)}{r(z)} < +\infty\,,$$

then

$$\|T^K\|_0 \le \frac{1}{2\pi}\int_{B_0} \frac{d\lambda_1(z)}{r(z)} - 1\,. \tag{16}$$

We shall also show that the sign of equality holds in (15) and (16) if ∂K is a circular polygon of a certain type. The proofs depend on a series of lemmas.

Lemma 1. *Let $B \subset \partial G$, $\lambda_1(\partial G \setminus B) = 0$, $\delta > 0$ and suppose that with each $z \in B$ there is associated an $r(z) > 0$ and $\theta(z) \in \Gamma$ such that*

$$\{z + t\theta;\ 0 < t < r(z),\, \theta \in \Gamma,\, |\theta - \theta(z)| < \delta\} \subset G\,.$$

If $z \mapsto r(z)$ is λ_1-measurable and $\int_B r^{-a}(z)\,d\lambda_1(z) < \infty$ for some $a \in [0, \infty[$, then $\lambda_1(\partial G) < \infty$.

Proof. Fix $R > 0$ large enough to have $K \subset B_R(0)$ and put $\Omega = G \cap B_R(0)$, so that $\partial\Omega = \partial G \cup \{\zeta;\ |\zeta| = R\}$. Assumptions of our lemma guarantee that with each $z \in C \equiv B \cup \{\zeta;\ |\zeta| = R\}$ it is possible to associate a circular sector $\{z + t\theta;\, 0 < t < r(z),\, \theta \in \Gamma,\, |\theta - \theta(z)| < \delta_0\} \subset \Omega$, where $0 < \delta_0 \le \delta$, $z \mapsto r(z)$ is λ_1-measurable on C and $\int_C r^{-a}(z)\,d\lambda_1(z) < \infty$. Put $C_1 = \{z \in C;\, r(z) \ge 1\}$, $C_2 = C \setminus C_1$. Clearly,

$$\lambda_1(C_2) \le \int_{C_2} r^{-a}(z)\,d\lambda_1(z) \le \int_C r^{-a}(z)\,d\lambda_1(z) < \infty\,,$$

so that it is sufficient to verify that $\lambda_1(C_1) < \infty$. Let $\mathscr{S}$ by the system of all circular sectors of the form

$$S(z,\theta_z,\delta_0) \equiv \{z+t\theta;\, 0<t<1, \theta\in\Gamma, |\theta-\theta_z|<\delta_0\}$$

with $z \in C_1$, $\theta_z \in \Gamma$ such that $S(z,\theta_z,\delta_0) \subset \Omega$. Let $S = \cup\mathscr{S}$, which is an open bounded set. If $S_1, \ldots, S_k$ are mutually different components of S, then each of them must contain a sector isometric with $S(0,1,\delta_0)$, whence

$$k\lambda_2(S(0,1,\delta_0)) \le \sum_{j=1}^{k} \lambda_2(S_j) \le \lambda_2(S), \quad k \le \lambda_2(S)/\lambda_2(S(0,1,\delta_0))\,.$$

We see that S has only finitely many components $S_1, \ldots, S_k$. We shall show that each S_j has the cone property in the following sense: There is an $r>0$ such that with each $z \in \partial S_j$ it is possible to associate a $\theta_z \in \Gamma$ with

$$B_r(z) \cap S(z,\theta_z,r) \subset S_j\,. \tag{17}$$

Let $z \in \partial S_j$, $\mathscr{S}_j = \{D \in \mathscr{S};\, D \subset S_j\}$. There is a sequence $x_n \in S_j$ with $\lim_{n\to\infty} x_n = z$. Since $S_j = \cup\mathscr{S}_j$, for each n there is a $D_n \in \mathscr{S}_j$ with $x_n \in D_n$. Denote by z_n the vertex of D_n and by $\theta^n \equiv \theta_{z_n}$ the corresponding vector in Γ determining $D_n = S(z_n,\theta^n,\delta_0)$. Since $\{z_n\} \subset \partial\Omega$ which is compact, passing to subsequences, if necessary, we may achieve that $\lim_{n\to\infty} z_n = y \in \partial\Omega$ and $\lim_{n\to\infty}\theta^n = \tilde\theta \in \Gamma$ for suitable y and $\tilde\theta$. Writing $\tilde D = S(y,\tilde\theta,\delta_0)$ we observe that

$$\tilde D \subset \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} D_n \subset \bigcap_{k=1}^{\infty} \mathrm{cl} \bigcup_{n=k}^{\infty} D_n \subset \mathrm{cl}\,\tilde D\,,$$

so that $\tilde D \subset S_j \subset \Omega$, $\tilde D \in \mathscr{S}_j$. As $x_n \in D_n$ tend to z, we have $z \in \mathrm{cl}\,\tilde D$. Since $z \in \partial S_j$ while $\tilde D \subset S_j$, we see that $z \in \partial\tilde D$. It remains to realize that $\tilde D$ is isometric with $S(0,1,\delta_0)$, so that there is an $r>0$ (depending on δ_0 only) such that with each $\tilde z \in \partial\tilde D$ it is possible to associate a $\theta_{\tilde z} \in \Gamma$ with $S(\tilde z,\theta_{\tilde z},r)\cap B_r(\tilde z) \subset \tilde D$; this is in particular true for $\tilde z = z$, so that the cone property (17) of S_j has been verified. Now we recall the following result established in [**4**]:

If $\mathcal{U}$ is a bounded domain having the cone property, then there are open sets $\mathcal{U}_1,\ldots,\mathcal{U}_p$ with $\cup_{i=1}^p \mathcal{U}_i = \mathcal{U}$ such that each $\mathcal{U}_i$ has locally lipschitzian boundary (and, in particular, $\lambda_1(\partial\mathcal{U}_i) < \infty$); consequently, $\lambda_1(\partial\mathcal{U}) \le \sum_{i=1}^p \lambda_1(\partial\mathcal{U}_i) < \infty$.

Applying this to $\mathcal{U} = S_j$ $(j=1,\ldots,k)$ we get $\lambda_1(\partial S) \le \sum_{j=1}^k \lambda_1(\partial S_j) < \infty$. Since $C_1 \subset \partial S$, $\lambda_1(C_1) < \infty$ has been verified and the proof is complete. $\square$

Lemma 2. *Denote by $\hat{\partial}K$ the set of all $y \in \mathbb{R}^2$, for which there exists $n^K(y) \in \Gamma$ (which is called the Federer exterior normal of K at y and is uniquely determined) such that*

$$\lim_{r\to 0+} r^{-2}\lambda_2[B_r(y)\cap\{x\in K;\ \langle x-y, n^K(y)\rangle > 0\}] \\ = \lim_{r\to 0+} r^{-2}\lambda_2[B_r(y)\cap\{x\in G;\ \langle x-y, n^K(y)\rangle < 0\}] = 0\,.$$

If $y \in \hat{\partial}K$, $z \in \partial K\setminus\{y\}$, $\zeta(y)\in\mathbb{R}^2$ and $|y-\zeta(y)| = r(y) > 0$, then the following implications hold:

$$B_{r(y)}(\zeta(y)) \subset K \implies -\langle \operatorname{grad} h_z(y), n^K(y)\rangle \\ = \frac{1}{4\pi r(y)} + \frac{r^2(y)-|z-\zeta(y)|^2}{4\pi r(y)|y-z|^2} \le \frac{1}{4\pi r(y)}\,, \tag{18}$$

$$K \subset \operatorname{cl} B_{r(y)}(\zeta(y)) \implies -\langle \operatorname{grad} h_z(y), n^K(y)\rangle \\ = \frac{1}{4\pi r(y)} + \frac{r^2(y)-|z-\zeta(y)|^2}{4\pi r(y)|y-z|^2} \ge \frac{1}{4\pi r(y)}\,, \tag{19}$$

$$K \cap B_{r(y)}(\zeta(y)) = \emptyset \implies -\langle \operatorname{grad} h_z(y), n^K(y)\rangle \\ = -\frac{1}{4\pi r(y)} - \frac{r^2(y)-|z-\zeta(y)|^2}{4\pi r(y)|y-z|^2} \ge -\frac{1}{4\pi r(y)}\,. \tag{20}$$

Proof. If $y \in \hat{\partial}K$ and the assumptions from (18) or (19) are valid, then

$$n^K(y) = \frac{y-\zeta(y)}{r(y)}\,,$$

while

$$\frac{y-\zeta(y)}{r(y)} = -n^K(y)$$

under the assumption occurring in (20). Since calculation yields

$$-\Big\langle \operatorname{grad} h_z(y), \frac{y-\zeta(y)}{r(y)}\Big\rangle = \frac{1}{2\pi}\left\langle \frac{y-z}{|y-z|^2}, \frac{y-\zeta(y)}{r(y)}\right\rangle \\ = \frac{1}{2\pi r(y)}\cdot\frac{|y-\zeta(y)|^2 - \langle z-\zeta(y), y-\zeta(y)\rangle}{|y-z|^2} \\ = \frac{1}{2\pi r(y)}\cdot\frac{|y-\zeta(y)|^2 - 2\langle z-\zeta(y), y-\zeta(y)\rangle + |z-\zeta(y)|^2}{2|y-z|^2} \\ + \frac{r^2(y)-|z-\zeta(y)|^2}{4\pi r(y)|y-z|^2} \\ = \frac{1}{4\pi r(y)} + \frac{r^2(y)-|z-\zeta(y)|^2}{4\pi r(y)|y-z|^2}\,.$$

It remains to note that $r^2(y) - |z - \zeta(y)|^2 \leq 0$ under the assumptions occurring in (18), (20), while $r^2(y) - |z - \zeta(y)|^2 \geq 0$ under the assumption occurring in (19).□

Lemma 3. *If the assumptions of Theorem* 1 *are fulfilled, then*

$$V^K = \|T^K\| < \infty\,.$$

Proof. Lemma 1 shows that $\lambda_1(\partial K) < \infty$, so that K has finite perimeter $P(K)$ in the sense of 2.10 in [**8**] (see 4.5 in [**3**]). For $y \in \hat{\partial} K$ the vector $n^K(y) \in \Gamma$ has been defined in Lemma 2; we shall further put $n^K(y) = 0$ ($\in \mathbb{R}^2$) for $y \in \mathbb{R}^2 \setminus \hat{\partial} K$. Then the vector-valued function $y \mapsto n^K(y)$ is defined on $\mathbb{R}^2$ and is Borel measurable (cf. Remark 2.14 in [**8**]), so that we may introduce

$$2 \int_{\partial K} |\langle n^K(y), \operatorname{grad}\, h_z(y)\rangle|\, d\lambda_1(y) \equiv v^K(z)$$

(which agrees with the quantity occurring in (28) in [**8**] up to the multiplicative factor 2). Then a necessary and sufficient condition for extendability of $W_{\partial K}$ (defined so far on $\mathcal{C}^{(1)}(\partial K)$ only) to a bounded linear operator on $\mathcal{C}(\partial K)$ consists in finiteness of the quantity

$$V^K \equiv \sup\{v^K(z);\ z \in \partial K\}$$

which is then equal to the norm of the operator T^K defined by (10) (cf. §2 in [**8**], in particular 2.19–2.25; notice that our V^K coincides with $2V^G$ occurring in [**8**]). We should remark that the quantity $v^K(z)$ can be equivalently defined by various expressions, one of them being

$$v^K(z) = \frac{1}{\pi} \int_{\Gamma} n_{\infty}^K(\theta, z)\, d\lambda_1(\theta)\,,$$

where $n_{\infty}^K(\theta, z)$ is the number of so-called hits of the half-line

$$H_z(\theta) = \{z + t\theta;\ t > 0\}$$

on K in the sense of 1.7 in [**8**] (note that, according to 1.11 in [**8**], $\theta \mapsto n_{\infty}^K(\theta, z)$ is a Baire function of the variable $\theta \in \Gamma$). As pointed out by M. Chlebík [**6**], methods of geometric measure theory [**3**] permit to show that $n_{\infty}^K(\theta, z)$ coincides with the total number of points in $H_z(\theta) \cap \partial_e K$ for λ_1-a.e. $\theta \in \Gamma$, so that $v^K(z)$ has the same meaning as described in the introduction. Fix now an arbitrary $z \in \partial K$ and consider $\delta > 0$ such that

$$\lambda_1(\partial B_\delta(z) \cap \partial K) = 0 \tag{21}$$

(as $\lambda_1(\partial K) < \infty$, all but countable many values $\delta > 0$ enjoy this property). Under the conditions of Theorem 1, for λ_1-a.e. $y \in \hat{\partial} K$ either the assumption in (19) or that occurring in (20) is fulfilled; accordingly,

$$-\langle \text{grad } h_z(y), n^K(y)\rangle \geq -\frac{1}{4\pi r(y)}, \quad \lambda_1\text{-a.e. } y \in \hat{\partial} K\,. \tag{22}$$

Put $Q = K - B_\delta(z)$. Employing (21) we see that λ_1-a.e. $y \in \hat{\partial} Q \cap \partial B_\delta(z)$ belongs to $\hat{\partial} Q \cap \operatorname{int} K \subset \partial B_\delta(z) \cap \operatorname{int} K$, so that $n^Q(y) = \frac{z-y}{\delta}$ and

$$\langle \text{grad } h_z(y), n^Q(y)\rangle = \frac{1}{2\pi\delta}, \quad \lambda_1\text{-a.e. } y \in \hat{\partial} Q \cap \partial B_\delta(z)\,. \tag{23}$$

Noting that $n^Q(\cdot) = n^K(\cdot)$ on $\hat{\partial} Q \setminus \partial B_\delta(z) \subset \hat{\partial} K$ we get by (22), (23)

$$\begin{aligned}
\frac{1}{2} v^Q(z) &= \int_{\hat{\partial} Q} |\langle \text{grad } h_z(y), n^Q(y)\rangle|\, d\lambda_1(y) \\
&\leq \int_{\hat{\partial} Q \cap \partial B_\delta(z)} \left[\frac{1}{\pi\delta} - \langle \text{grad } h_z(y), n^Q(y)\rangle\right] d\lambda_1(y) \\
&\quad + \int_{\hat{\partial} Q \setminus \partial B_\delta(z)} \left[\frac{1}{4\pi r(y)} - \langle \text{grad } h_z(y), n^Q(y)\rangle\right] d\lambda_1(y) \\
&\quad + \int_{\hat{\partial} Q \setminus \partial B_\delta(z)} \frac{1}{4\pi r(y)}\, d\lambda_1(y) \\
&\leq -\int_{\hat{\partial} Q} \langle \text{grad } h_z(y), n^Q(y)\rangle\, d\lambda_1(y) + \frac{1}{\pi\delta} \cdot 2\pi\delta + 2\int_{\partial K} \frac{1}{4\pi r(y)}\, d\lambda_1(y) \\
&= 2 + \frac{1}{2\pi} \int_{\partial K} \frac{1}{r(y)}\, d\lambda_1(y)\,,
\end{aligned}$$

where we have used the fact that $y \mapsto h_z(y)$ is harmonic in some neighbourhood of $\operatorname{cl} Q$, whence it follows by the divergence theorem for sets with finite perimeter (cf. p. 49 in [**8**]) that

$$\int_{\hat{\partial} Q} \langle \text{grad } h_z(y), n^Q(y)\rangle\, d\lambda_1(y)\rangle = 0\,.$$

Since $\partial K \setminus B_\delta(z) \subset \partial Q$ and $n^K(\cdot) = n^Q(\cdot)$ holds λ_1-a.e. on $\partial K \setminus B_\delta(z)$ by (21), we arrive at

$$\int_{\partial K \setminus B_{\delta(z)}} |\,\langle \text{grad } h_z(y), n^K(y)\rangle\,|\, d\lambda_1(y) \leq \frac{1}{2} v^Q(z) \leq 2 + \frac{1}{2\pi} \int_{\partial K} \frac{1}{r(y)}\, d\lambda_1(y)\,,$$

whence we get making $\delta \to 0^+$ (with δ obeying (21))

$$v^K(z) = 2\int_{\partial K} |\langle \text{grad } h_z(y), n^K(y)\rangle|\, d\lambda_1(y) \leq 4 + \frac{1}{\pi} \int_{\partial K} \frac{1}{r(y)}\, d\lambda_1(y)\,.$$

Since $z \in \partial K$ has been arbitrarily chosen, we have

$$V^K \leq 4 + \pi^{-1} \int_{\partial K} r^{-1}(y)\, d\lambda_1(y) < \infty$$

and the proof is complete. □

Lemma 4. *If the assumptions of Theorem 2 are fulfilled, then*

$$V^K = \|T^K\| < \infty\,.$$

Proof. Choose $R > 0$ large enough to have $K \subset B_R(0)$ and put $L = \mathrm{cl}\,[B_R(0) \setminus K]$. If K satisfies the assumptions of Theorem 2, then L satisfies the assumptions of Theorem 1 (where K is replaced by L) and Lemma 3 implies $V^L < \infty$. It remains to observe that $V^K \leq V^L$. □

Lemma 5. *Let $V^K < \infty$. Then the density*

$$d_K(z) = \lim_{r\to 0+} \frac{\lambda_2[K \cap B_r(z)]}{\lambda_2[B_r(z)]}$$

is well defined for any $z \in \mathbb{R}^2$. Denoting by δ_z the Dirac unit point-mass concentrated at z define for any $z \in \partial K$ the signed Borel measure τ_z on ∂K by

$$d\tau_z(y) = [1 - 2d_K(z)]d\delta_z(y) - 2\langle n^K(y), \mathrm{grad}\ h_z(y)\rangle d\lambda_1(y)\,. \tag{24}$$

Then

$$T^K f(z) = \int_{\partial K} f\, d\tau_z, \quad z \in \partial K, \quad f \in \mathcal{C}(\partial K)\,. \tag{25}$$

Proof. See §3 in [**8**] (p. 73). □

Lemma 6. *Let $V^K < \infty$ and let D be a dense subset of ∂K. Let us agree to denote by $\|\nu\|$ the total variation of an arbitrary signed Borel measure ν on ∂K. Then*

$$\|T^K\|_0 = \frac{1}{2}\sup\{\|\tau_u - \tau_v\|;\ u, v \in D\} \tag{26}$$

and for each signed Borel measure μ on ∂K the following estimate holds

$$\|T^K\|_0 \leq \sup\{\|\tau_z - \mu\|;\ z \in D\}\,. \tag{27}$$

Proof. If $f \in \mathcal{C}(\partial K)$, then we denote by $\|f\|_0 = \frac{1}{2}\mathrm{osc}\, f(\partial K)$ the norm in $\mathcal{C}(\partial K)/\mathrm{Const}\,(\partial K)$ of the class containing f. Hence

$$\begin{aligned}
\|T^K\|_0 &= \sup\left\{\frac{1}{2}\mathrm{osc}\, T^K f(\partial K);\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \leq 1\right\}\\
&= \frac{1}{2}\sup\left\{\left|\int_{\partial K} f\, d\tau_u - \int_{\partial K} f\, d\tau_v\right|;\ u, v \in D,\ f \in D,\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \leq 1\right\}\\
&= \sup\left\{\left|\int_{\partial K} f\, d(\tau_u - \tau_v)\right|;\ u, v \in D,\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \leq \frac{1}{2}\right\}\,.
\end{aligned}$$

In view of (11) we have $\int_{\partial K} d(\tau_u - \tau_v) = 0$, so that the last expression transforms into

$$\|T^K\|_0 = \sup\left\{\left|\int_{\partial K} f\, d(\tau_u - \tau_v)\right|;\, u, v \in D,\, f \in \mathcal{C}(\partial K),\, \|f\| \le \frac{1}{2}\right\}$$
$$= \frac{1}{2}\sup\{\|\tau_u - \tau_v\|;\, u, v \in D\}$$

which is (26). Given $f \in \mathcal{C}(\partial K)$ we have for any $\gamma \in \mathbb{R}$:

$$\|T^K f\|_0 \le \|T^K f - \gamma 1_{\partial K}\| = \sup\left\{\left|\int_{\partial K} f\, d\tau_z - \gamma\right|;\, z \in D\right\}\,.$$

Choosing $\gamma = \int_{\partial K} f\, d\mu$ we arrive at

$$\|T^K f\|_0 \le \sup\left\{\left|\int_{\partial K} f\, d(\tau_z - \mu)\right|;\, z \in D\right\} \le \|f\| \sup\{\|\tau_z - \mu\|;\, z \in D\}\,.$$

In this inequality we replace f by $f - \alpha 1_{\partial K}$ for any $\alpha \in \mathbb{R}$. Since

$$\|T^K f\|_0 = \|T^K f - \alpha 1_{\partial K}\|_0$$

we get

$$\|T^K f\|_0 \le \|f - \alpha 1_{\partial K}\| \cdot \sup\{\|\tau_z - \mu\|;\, z \in D\}, \quad \alpha \in \mathbb{R}\,,$$

so that

$$\|T^K f\|_0 \le \|f\|_0 \cdot \sup\{\|\tau_z - \mu\|;\, z \in D\}, \quad f \in \mathcal{C}(\partial K)\,,$$

and (27) follows. □

We are in position to present proofs of Theorems 1, 2 stated above.

Proof of Theorem 1. We know from Lemma 3 that $V^K < \infty$. Define a signed Borel measure μ on ∂K putting for each Borel set $M \subset \partial K$

$$\mu(M) = \frac{1}{2\pi}\left(\int_{M \cap B_2} \frac{d\lambda_1(y)}{r(y)} - \int_{M \cap B_1} \frac{d\lambda_1}{r(y)}\right)\,.$$

Fix $z \in \hat{\partial} K$, so that $d_K(z) = \frac{1}{2}$. Using (24), (19), (20) we get

$$\|\tau_z - \mu\| = \int_{B_1}\left[-2\left\langle \operatorname{grad}\, h_z(y), n^K(y)\right\rangle + \frac{1}{2\pi r(y)}\right] d\lambda_1(y)$$
$$+ \int_{B_2}\left[-2\left\langle \operatorname{grad}\, h_z(y), n^K(y)\right\rangle - \frac{1}{2\pi r(y)}\right] d\lambda_1(y)$$
$$= \int_{\partial K} d\tau_z(y) + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(y)}{r(y)} - \int_{B_2} \frac{d\lambda_1(y)}{r(y)}\right)$$
$$= T^K 1_{\partial K}(z) + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(y)}{r(y)} - \int_{B_2} \frac{d\lambda_1(y)}{r(y)}\right)\,,$$

which in combination with (11), (27) completes the proof, because $\hat{\partial} K$ is dense in ∂K thanks to our assumption that K is massive at each point of ∂K (cf. [**8**], p. 54 and isoperimetric lemma on p. 50). $\square$

Proof of Theorem 2. Lemma 4 shows that $V^K < \infty$. Fix again an arbitrary $z \in \hat{\partial} K$ and define now the signed measure μ on Borel sets $M \subset \partial K$ by

$$\mu(M) = \frac{1}{2\pi} \int_{M \cap B_0} \frac{d\lambda_1(y)}{r(y)} .$$

It follows from (24), (18) that

$$\begin{aligned} \|\mu - \tau_z\| &= \int_{B_0} \left[\frac{1}{2\pi r(y)} + 2 \left\langle \operatorname{grad} h_z(y), n^K(y) \right\rangle \right] d\lambda_1(y) \\ &= \frac{1}{2\pi} \int_{B_0} \frac{d\lambda_1(y)}{r(y)} - T^K 1_{\partial K}(z) \end{aligned}$$

which together with (11), (27) proves (16), because $\hat{\partial} K$ is dense in ∂K as observed above. $\square$

Notation. We now specialize to the case that K is bounded by a simple oriented circular polygon

$$\partial K = \bigcup_{m=1}^{n} C_m \cup \{z_m\} ,$$

where C_m is an open oriented circular arc situated on the boundary of a disk $B_{r_m}(\zeta_m)$ and z_m is the initial point of C_m; for $m < n$ the end-point of C_m coincides with z_{m+1}, the end-point of C_n is z_1. Further suppose that for $1 \le k < m \le n$ either $C_k \cap \partial B_{r_m}(\zeta_m) = \emptyset$ or else $C_k \subset \partial B_{r_m}(\zeta_m) \setminus C_m$. We put

$$\begin{aligned} \alpha_m &= \lambda_1(C_m)/r_m , & \mathcal{A}_0 &= \{m;\, B_{r_m}(\zeta_m) \subset K\} , \\ \mathcal{A}_1 &= \{m;\, B_{r_m}(\zeta_m) \cap K = \emptyset\} , & \mathcal{A}_2 &= \{m;\, K \subset \operatorname{cl} B_{r_m}(\zeta_m)\} \end{aligned}$$

and adopt the following assumption:

$$\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 = \{1, \ldots, n\} .$$

Then we may state the following result.

Theorem 3. *Let i run over $\mathcal{A}_0$, j run over $\mathcal{A}_1$ and k run over $\mathcal{A}_2$. If $\mathcal{A}_0 = \emptyset$, then*

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi} \left(\sum_j \alpha_j - \sum_k \alpha_k \right) , \tag{28}$$

where the sign of equality holds in case $n \le 4$*. If* $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$*, then*

$$\|T^K\| \le \frac{1}{2\pi}\sum_{i=1}^{n}\alpha_i - 1\,, \tag{29}$$

where again the sign of equality holds provided $n \le 4$*; now the condition*

$$\operatorname{int} K \setminus \bigcup_{i=1}^{n} B_{r_i}(\zeta_i) \equiv \bigcap_{i=1}^{n}[\operatorname{int} K \setminus B_{r_i}(\zeta_i)] \neq \emptyset \tag{30}$$

implies that

$$\frac{1}{2\pi}\sum_{i=1}^{n}\alpha_i - 1 \ge 1 \tag{31}$$

(*so that in case* $n \le 4$ *the operator* T^K *cannot be contractive on* $\mathcal{C}(\partial K)/\mathrm{Const}\,(\partial K)$ *in view of the equality in* (29)), *while the conditions*

$$\bigcap_{i=1}^{n}[\operatorname{int} K \setminus B_{r_i}(\zeta_i)] = \emptyset, \qquad \bigcap_{i=1}^{n} B_{r_i}(\zeta_i) \neq \emptyset \tag{32}$$

together imply the inequality

$$\frac{1}{2\pi}\sum_{i=1}^{n}\alpha_1 - 1 < 1 \tag{33}$$

(*guaranteeing contractivity of* T^K *on* $\mathcal{C}(\partial K)/\mathrm{Const}\,(\partial K)$).

Corollary 1. *If* $\mathcal{A}_0 = \emptyset = \mathcal{A}_1$*, then* (28) *implies the inequality*

$$\|T^K\|_0 \le 1 - \frac{1}{2\pi}\sum_{k=1}^{n}\alpha_k$$

guaranteeing contractivity of T^K *on* $\mathcal{C}(\partial K)/\mathrm{Const}\,(\partial K)$*. If* $\mathcal{A}_0 = \emptyset = \mathcal{A}_2$ *and* $n \le 4$ *then the equality*

$$\|T^K\|_0 = 1 + \frac{1}{2\pi}\sum_{k=1}^{n}\alpha_k$$

holds, so that T^K *cannot be contractive on* $\mathcal{C}(\partial K)/\mathrm{Const}\,(\partial K)$*.*

The proof will depend on the following lemma.

Lemma 7. *Put for any* $m \in \{1, \dots, n\}$

$$\sigma_m = \begin{cases} 1, & \text{in case } K \cap B_{r_m}(\zeta_m) \neq \emptyset\,, \\ -1, & \text{in case } K \cap B_{r_m}(\zeta_m) = \emptyset\,. \end{cases}$$

If $z \in C_m$, *then*

$$-2 \int_{\partial K \setminus C_m} \langle \text{grad } h_z(y), n^K(y) \rangle \, d\lambda_1(y) = 1 - \frac{1}{2\pi} \sigma_m \alpha_m, \quad m \in \{1, \dots, n\}; \tag{34}$$

further we have

$$-2 \int_{\partial K \setminus C_1 \setminus C_n} \langle \text{grad } h_{z_1}(y), n^K(y) \rangle \, d\lambda_1(y) = 2d_K(z_1) - \frac{1}{2\pi} \sigma_1 \alpha_1 - \frac{1}{2\pi} \sigma_n \alpha_n\,, \tag{35}$$

$$-2 \int_{\partial K \setminus C_{m-1} \setminus C_m} \langle \text{grad } h_{z_m}(y), n^K(y) \rangle \, d\lambda_1(y) = 2d_K(z_m) - \frac{1}{2\pi} \sigma_{m-1} \alpha_{m-1} - \frac{1}{2\pi} \sigma_m \alpha_m \quad \textit{for } 1 < m \leq n\,. \tag{36}$$

Proof. If $z \in C_m$, then (11), (25), (24) yield

$$-2 \int_{\partial K} \left\langle n^K(y), \text{grad } h_z(y) \right\rangle \, d\lambda_1(y) = \int_{\partial K} d\tau_z(y) + [2d_K(z) - 1] = 2d_K(z)\,. \tag{37}$$

From Lemma 2 we get for $y, z \in C_m$, $y \neq z$

$$- \left\langle \text{grad } h_z(y), n^K(y) \right\rangle = \frac{\sigma_m}{4\pi r_m}\,,$$

whence

$$-2 \int_{C_m} \left\langle n^K(y), \text{grad } h_z(y) \right\rangle \, d\lambda_1(y) = \frac{1}{2\pi} \sigma_m \alpha_m\,, \tag{38}$$

which together with (37) implies (34).

If $y \in C_1$, then Lemma 2 combined with $|z_1 - \zeta_1| = r_1$ yields again

$$- \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle = \frac{\sigma_1}{4\pi r_1}\,,$$

whence

$$-2\int_{C_1} \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle \, d\lambda_1(y) = \frac{1}{2\pi}\sigma_1\alpha_1 \,. \tag{39}$$

Similarly we get from Lemma 2 for $y \in C_n$ in view of $|z_1 - \zeta_n| = r_n$

$$-\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle = \frac{\sigma_n}{4\pi r_n}\,,$$

so that

$$-2\int_{C_n} -\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle \, d\lambda_1(y) = \frac{1}{2\pi}\sigma_n\alpha_n \,. \tag{40}$$

Combining (37), (39), (40) we get (35). Similar reasoning proves (36). □

Proof of Theorem 3. Assuming $\mathcal{A}_0 = \emptyset$ put $B_1 = \cup C_j$ $(j \in \mathcal{A}_1)$, $B_2 = \cup C_k$ $(k \in \mathcal{A}_2)$, $B = B_1 \cup B_2$, $B(z) = B_{r_m}(\zeta_m)$ for $z \in C_m$ $(1 \le m \le n)$. Then $\partial K \setminus B = \{z_1, \ldots, z_n\}$ and Theorem 1 implies

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi}\left(\sum_j \lambda(C_j)/_{r_j} - \sum_k \lambda(C_k)/_{r_k}\right)$$

which is (28). Now we shall verify that the sign of equality holds in (28) provided $1 \le n \le 4$. This is clear when $n = 1$, because then $\mathcal{A}_3 = \emptyset$, $\alpha_1 = 2\pi$ and $0 \le \|T^K\|_0 \le 1 - \frac{1}{2\pi}\alpha_1 = 0$. Let now $n = 2$ and fix $u \in C_1$, $v \in C_2$. According to Lemma 2 we have for $y \in C_1$

$$-\left\langle \text{grad } h_u(y), n^K(y)\right\rangle = \frac{\sigma_1}{4\pi r_1}\,, \quad -\left\langle \text{grad } h_v(y), n^K(y)\right\rangle - \frac{\sigma_1}{4\pi r_1} \ge 0\,,$$

while for $y \in C_2$

$$-\left\langle \text{grad } h_v(y), n^K(y)\right\rangle = \frac{\sigma_2}{4\pi r_2}\,, \quad -\left\langle \text{grad } h_u(y), n^K(y)\right\rangle - \frac{\sigma_2}{4\pi r_2} \ge 0\,.$$

Hence we get by (24)

$$\begin{aligned}
\|\tau_u - \tau_v\| &= -\int_{C_1}\left[\frac{\sigma_1}{2\pi r_1} + 2\left\langle \text{grad } h_v(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\
&\quad - \int_{C_2}\left[\frac{\sigma_2}{2\pi r_2} + 2\left\langle \text{grad } h_u(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\
&= -\frac{\sigma_1}{2\pi}\frac{\lambda_1(C_1)}{r_1} - 2\int_{\partial K\setminus C_2} \left\langle \text{grad } h_v(y), n^K(y)\right\rangle \, d\lambda_1(y) \\
&\quad - 2\int_{\partial K\setminus C_1} \left\langle \text{grad } h_u(y), n^K(y)\right\rangle \, d\lambda_1(y) - \frac{\sigma_2}{2\pi}\frac{\lambda_1(C_2)}{r_2}\,.
\end{aligned}$$

Using (34) we arrive at

$$\begin{aligned}\|\tau_u - \tau_v\| &= -\frac{\sigma_1}{2\pi}\alpha_1 + \big(1 - \frac{\sigma_2}{2\pi}\alpha_2\big) + \big(1 - \frac{\sigma_1}{2\pi}\alpha_1\big) - \frac{\sigma_2}{2\pi}\alpha_2 \\ &= 2\big(1 - \frac{1}{2\pi}\sigma_1\alpha_1 - \frac{1}{2\pi}\sigma_2\alpha_2\big)\,.\end{aligned}$$

Hence we get by (26)

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_u - \tau_v\| = 1 - \frac{1}{2\pi}(\sigma_1\alpha_1 + \sigma_2\alpha_2)$$

which is the inequality opposite to (28) for $n = 2$.

Next we shall consider the case $n = 3$. Observing that

$$\big\langle \text{grad } h_{z_1}(y), n^K(y)\big\rangle = \big\langle \text{grad } h_{z_3}(y), n^K(y)\big\rangle \quad \text{for } y \in C_3$$

by Lemma 2, we get from (24) and this lemma

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| = {} & |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\ & - \int_{C_1} \left[\frac{\sigma_1}{2\pi r_1} + 2\left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\ & - \int_{C_2} \left[\frac{\sigma_2}{2\pi r_2} + 2\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\ \geq {} & [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] - \frac{1}{2\pi}\sigma_1\alpha_1 \\ & - 2\int_{\partial K \setminus C_2 \setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle\, d\lambda_1(y) - \frac{1}{2\pi}\sigma_2\alpha_2 \\ & - 2\int_{\partial K \setminus C_1 \setminus C_3} \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\, d\lambda_1(y)\,.\end{aligned}$$

Employing (36) and (35) we obtain

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| \geq {} & [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] - \frac{1}{2\pi}\sigma_1\alpha_1 + \Big[2d_K(z_3) - \frac{1}{2\pi}\sigma_2\alpha_2 \\ & - \frac{1}{2\pi}\sigma_3\alpha_3\Big] - \frac{1}{2\pi}\sigma_2\alpha_2 + \Big[2d_K(z_1) - \frac{1}{2\pi}\sigma_1\alpha_1 - \frac{1}{2\pi}\sigma_3\alpha_3\Big] \\ = {} & 2\left(1 - \sum_{m=1}^{3} \frac{1}{2\pi}\sigma_m\alpha_m\right)\,,\end{aligned}$$

whence it follows by (26) that

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq 1 - \frac{1}{2\pi}\sum_{m=1}^{3} \sigma_m\alpha_m$$

which gives the inequality opposite to (28) for $n = 3$.

Finally we shall treat the case $n = 4$. We obtain from (24) and Lemma 2

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| = |1 - 2d_K(z_1)| &+ |1 - 2d_K(z_3)| \\ &- \sum_{m=2}^{3} \int_{C_m} \left[2 \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle + \frac{\sigma_m}{2\pi r_m} \right] d\lambda_1(y) \\ &- \sum_{m \in \{1,4\}} \int_{C_m} \left[2 \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle + \frac{\sigma_m}{2\pi r_m} \right] d\lambda_1(y) \\ \geq [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] &- \sum_{m=1}^{4} \frac{1}{2\pi} \sigma_m \alpha_m \\ &- 2 \int_{\partial K \setminus C_1 \setminus C_4} \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle d\lambda_1(y) \\ &- 2 \int_{\partial K \setminus C_2 \setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle d\lambda_1(y) \,.\end{aligned}$$

Applying (35), (36) we finally get

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| \geq [1 - 2d_K(z_1)] &+ [1 - 2d_K(z_3)] \\ &- \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m + \left[2d_K(z_1) - \frac{1}{2\pi} \sigma_1 \alpha_1 - \frac{1}{2\pi} \sigma_4 \alpha_4 \right] \\ &+ \left[2d_K(z_3) - \frac{1}{2\pi} \sigma_2 \alpha_2 - \frac{1}{2\pi} \sigma_3 \alpha_3 \right] = 2 \left(1 - \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m \right)\end{aligned}$$

which again yields the inequality

$$\|T^K\|_0 \geq 1 - \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m$$

opposite to (28) for $n = 4$.

The first part of Theorem 3 dealing with the inequality (28) concerning the case $\mathcal{A}_0 = \emptyset$ is completely proved. We now proceed to the case $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$ and put $B_0 = \cup_{i=1}^{n} C_i$. Then $\partial K \setminus B_0 = \{z_1, \ldots, z_n\}$ and letting again $B(z) = B_{r_m}(\zeta_m)$ for $z \in C_m$ $(1 \leq m \leq n)$ we get from Theorem 2

$$\|T^K\|_0 \leq \frac{1}{2\pi} \sum_{i=1}^{n} \lambda(C_i) \big/ r_i - 1 \,,$$

which is the inequality (29). It remains to discuss the case $1 \leq n \leq 4$. If $n = 1$ then $\alpha_1 = 2\pi$ and $\|T^K\|_0 = 0$ as in the first part of the proof. If $n = 2$ we again choose $u \in C_1$, $v \in C_2$ and get by (24) and Lemma 2

$$\begin{aligned}\|\tau_u - \tau_v\| &= \int_{C_1} \left[2\left\langle \text{grad } h_v(y), n^K(y)\right\rangle + \frac{1}{2\pi r_1}\right] d\lambda_1(y) \\ &\quad + \int_{C_2} \left[2\left\langle \text{grad } h_u(y), n^K(y)\right\rangle + \frac{1}{2\pi r_2}\right] d\lambda_1(y) \\ &= \frac{1}{2\pi}(\alpha_1 + \alpha_2) + 2\int_{\partial K \setminus C_2} \left\langle \text{grad } h_v(y), n^K(y)\right\rangle\, d\lambda_1(y) \\ &\quad + 2\int_{\partial K \setminus C_1} \left\langle \text{grad } h_u(y), n^K(y)\right\rangle\, d\lambda_1(y)\,.\end{aligned}$$

Hence it follows by (34) that

$$\|\tau_u - \tau_v\| = \frac{1}{2\pi}(\alpha_1 + \alpha_2) - 1 + \frac{1}{2\pi}\alpha_1 - 1 + \frac{1}{2\pi}\alpha_2 = \frac{1}{\pi}(\alpha_1 + \alpha_2) - 2$$

which together with (26) implies

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_u - \tau_v\| = \frac{1}{2\pi}(\alpha_1 + \alpha_2) - 1\,,$$

so that equality holds in (29) for $n = 2$. If $n = 3$, then (24) and Lemma 2 imply

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &= |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\ &\quad + \int_{C_1} \left[2\left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_1}\right] d\lambda_1(y) \\ &\quad + \int_{C_2} \left|2\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_2}\right] d\lambda_1(y) \\ &\geq 2d_K(z_1)] + 2d_K(z_3) - 2 + \frac{1}{2\pi}\alpha_1 + \frac{1}{2\pi}\alpha_2 \\ &\quad + 2\int_{\partial K \setminus C_2 \setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle\, d\lambda_1(y) \\ &\quad + 2\int_{\partial K \setminus C_1 \setminus C_3} \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\, d\lambda_1(y)\,.\end{aligned}$$

Using (36), (35) we get

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &\geq 2d_K(z_1) + 2d_K(z_3) - 2 + \frac{1}{2\pi}(\alpha_1 + \alpha_2) \\ &\quad - 2d_K(z_3) + \frac{1}{2\pi}(\alpha_2 + \alpha_3) - 2d_K(z_1) + \frac{1}{2\pi}(\alpha_1 + \alpha_3) \\ &= \frac{1}{\pi}(\alpha_1 + \alpha_2 + \alpha_3)\,,\end{aligned}$$

whence

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq \frac{1}{2\pi}\sum_{i=1}^{3} \alpha_i - 1$$

by (26), which shows that equality holds in (29) for $n = 3$. Finally, if $n = 4$ we obtain similarly from (24) and Lemma 2

$$\begin{aligned}
\|\tau_{z_1} - \tau_{z_3}\| = {} & |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\
& + \sum_{i=2}^{3} \int_{C_i} \left[2 \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_i}\right] d\lambda_1(y) \\
& + \sum_{i\in\{1,4\}} \int_{C_i} \left[2 \left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_i}\right] d\lambda_1(y) \\
\geq {} & 2d_K(z_1) - 1 + 2d_K(z_3) - 1 + \sum_{i=1}^{4} \frac{1}{2\pi}\alpha_i \\
& + 2\int_{\partial K\setminus C_1\setminus C_4} \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle d\lambda_1(y) \\
& + 2\int_{\partial K\setminus C_2\setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle d\lambda_1(y) \\
= {} & \frac{1}{\pi}\sum_{i=1}^{4} \alpha_i - 2 \quad \text{(see (35) and (36))},
\end{aligned}$$

so that by (26) we have again

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq \frac{1}{2\pi}\sum_{i=1}^{4} \alpha_i - 1$$

which yields equality in (29) for $n = 4$.

Now we assume (30) together with $\mathcal{A}_0 = \{1, \dots, n\}$ and choose $z_0 \in \operatorname{int} K \setminus \cup_{i=1}^{n} B_{r_i}(\zeta_i)$. Denote by $\triangle \arg[y - z_0;\, y \in C_i]$ the increment of the argument of $y - z_0$ as y describes the oriented arc C_i. Assuming, as we may, that the Jordan curve ∂K arising as the union of the oriented arcs $\operatorname{cl} C_1, \dots, \operatorname{cl} C_n$ is positively oriented we get

$$\begin{aligned}
2\pi = \sum_{i=1}^{n} \triangle \arg[y - z_0;\, y \in C_i] &= \sum_{i=1}^{n} \int_{C_i} \frac{\left\langle n^K(y), y - z_0\right\rangle}{|y - z_0|^2}\, d\lambda_1(y) \\
&= -2\pi \sum_{i=1}^{n} \int_{C_i} \left\langle n^K(y), \text{grad } h_{z_0}(y)\right\rangle d\lambda_1(y)\,.
\end{aligned}$$

We have seen in the proof of (18) in Lemma 2 that for $i \in \{1, \dots, n\}$ and any $z_0 \notin \partial K$

$$\Big(y \in C_i, B_{r_i}(\zeta_i) \subset K\Big) \implies -\left\langle \text{grad } h_{z_0}(y), n^K(y)\right\rangle = \frac{1}{4\pi r_i} + \frac{r_i^2 - |z_0 - \zeta_i|^2}{4\pi r_i |y - z_0|^2}, \tag{41}$$

whence we get noting that $|z_0 - \zeta_i| \geq r_i$ for $i \in \{1, \dots, n\}$

$$2\pi \leq \frac{1}{2} \sum_{i=1}^{n} \int_{C_i} \frac{d\lambda_1(y)}{r_i} = \frac{1}{2} \sum_{i=1}^{n} \alpha_i$$

which proves (31).

Finally suppose that (32) holds together with $\mathcal{A}_0 = \{1, \dots, n\}$ and choose $z_0 \in \cap_{i=1}^n B_{r_i}(\zeta_i) \subset \text{int}\, K$. Keeping the assumption that ∂K is positively oriented we obtain from (41) in view of $|z_0 - \zeta_i| < r_i$ $(1 \leq i \leq n)$ by the above reasoning

$$\begin{aligned} 2\pi &= -2\pi \sum_{i=1}^{n} \int_{C_i} \left\langle n^K(y), \text{grad } h_{z_0}(y)\right\rangle \, d\lambda_1(y) \\ &> \frac{1}{2} \sum_{i=1}^{n} \int_{C_i} \frac{d\lambda_1(y)}{r_i} = \frac{1}{2} \sum_{i=1}^{n} \alpha_i \end{aligned}$$

which is (33). The proof of Theorem 3 is complete. □

Corollary 2. *If $n = 2$ in Theorem 3 then T^K is always contractive on $\mathcal{C}(\partial K)/\text{Const}\,(\partial K)$ if both C_1 and C_2 are convex w.r. to K (i.e. $\sigma_1 = 1 = \sigma_2$); if only C_1 is convex while C_2 is concave (i.e. $\sigma_1 = 1 = -\sigma_2$), then $\|T^K\|_0 < 1$ iff $\alpha_1 > \alpha_2$.*

Remark. If $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$ and $\text{int}\, K \subset \cup_{i=1}^n B_{r_i}(\zeta_i)$ then, as we have seen in Theorem 3,

$$\bigcap_{i=1}^{n} B_{r_i}(\zeta_i) \neq \emptyset \tag{42}$$

is sufficient for $\|T^K\|_0 < 1$; to see that (42) is not necessary consider $\alpha \in]0, \pi/2[$ and form the region

$$K = \text{cl}\, B_1(-2\cos\alpha) \cup \text{cl}\, B_1(0) \cup \text{cl}\, B_1(2\cos\alpha)$$

whose boundary consists of four circular arcs

$$\begin{aligned} C_1 &= \{-2\cos\alpha + \exp i\theta; \alpha < \theta < 2\pi - \alpha\} && (\text{so that } \alpha_1 = 2\pi - 2\alpha), \\ C_2 &= \{\exp i\theta; -\pi + \alpha < \theta < -\alpha\} && (\text{so that } \alpha_2 = \pi - 2\alpha), \\ C_3 &= \{+2\cos\alpha + \exp i\theta; -\pi + \alpha < \theta < \pi - \alpha\} && (\text{so that } \alpha_3 = 2\pi - 2\alpha), \\ C_4 &= \{\exp i\theta; \alpha < \theta < \pi - \alpha\} && (\text{so that } \alpha_4 = \pi - 2\alpha), \end{aligned}$$

and their end-points $z_1, \dots, z_4$. Elementary considerations show that (42) holds iff $\alpha > \pi/3$ while the equality occurring in (29) (Theorem 3) for $n = 4$ tells us that $\|T^K\|_0 < 1$ iff $\alpha > \pi/4$.

Comments. The estimate $\|T^K\|_0 < 1$ guarantees convergence of the Neumann series for the inverse of $I \pm T^K$ in the operator norm; it is not indispensable for the convergence of the Neumann series $\sum_{n=0}^{\infty}(-1)^n(T^K)^n g$ (corresponding to an individual $g \in \mathcal{C}(\partial K)$) to the solution f of the equation $(I + T^K)f = g$ in $\mathcal{C}(\partial K)$ (cf. [**20**], [**15**]). Nevertheless, evalation or estimates of $\|T^K\|_0$ are useful in connection with iterative techniques connected with the equations of the type (13), (14) (cf. [**7**], [**19**]). C. Neumann started investigation of the quantity $\|T^K\|_0$ (which he called the configuration constant of K) in order to get a proof for the existence of the solution of the Dirichlet problem for any continuous boundary condition g prescribed on the boundary of a convex region K ([**17**]); Dirichlet's principle used for this purpose previously by Riemann lost credit after Weierstrass' criticism concerning attaining minima in variational problems. C. Neumann's first proof dealing with the inequality $\|T^K\|_0 < 1$ for convex regions $K \subset \mathbb{R}^2$ different from triangles and quadrangles was only sketchy (as he himself admitted cf. [**18**], p. 759) and was followed by a detailed and correct proof in [**18**], §6 (which was known in his time – cf. [**5**]). This contribution was forgotten later and after Lebesgue's criticism [**12**] of Neumann's first proof (which apparently contained the same gap connected with attaining minima as Riemann's reasoning based on the Dirichlet principle) there remained a common belief that Neumann's proof of $\|T^K\|_0 < 1$ for general convex $K \subset \mathbb{R}^2$ different from triangles and quadrangles was insufficient (cf. [**16**], [**2**], chap. 8, p. 572); Neumann's original proof has been included in [**11**], characterization of convex bodies in higher dimensional spaces for which the operator of the arithmetical mean is contractive is presented in [**10**], where also historical comments are included. We refer the reader to [**13**] for the description of the role played by the Neumann operator in the development of the theory of integral equations.

References

1. Burago Yu. D. and Maz'ya V. G., *Nekotoryje voprosy teorii potenciala i teorii funkcij dlja oblastej s nereguljarnymi granicami*, Zapiski naučnych seminarov LOMI **3** (1967).

2. Dieudonné J., *Geschichte der Mathematik 1700–1900*, Vieweg and Sohn, Braunschweig/ Wiesbaden, 1985.

3. Federer H., *Geometric measure theory*, Springer-Verlag, 1969.

4. Gagliardo E., *Proprietà di alcuni classi di funziono in piu variabli*, Ricerche Mat. **7** (1958), 102–137.

5. Hölder O., *Nachruf auf Carl Neumann*, Ber. Verh. Ges. Wiss. Leipzig, Math.-Phys. Klasse **77** (1925), 154–180.

6. Chlebík M., *Tricomiho potenciály*, Thesis, Mathematical Institute of Czechoslovak Academy of Sciences, Praha, 1988 (in Slovak).

7. Kleinman R. E. and Wendland W. L., *On Neumann's method for the exterior Neumann problem for the Helmholtz equation*, Journal of Mathematical Analysis and Applications **57** (1977), 170–202.
8. Král J., *Integral operators in potential theory,* Lecture Notes in Mathematics vol. 823, Springer-Verlag, 1980.
9. ———, *The Fredholm method in potential theory*, Trans. Amer. Math. Soc. **125** (1966), 511–547.
10. Král J. and Netuka I., *Contractivity of C. Neumann's operator in potential theory*, Journal of the Mathematical Analysis and its Applications **61** (1977), 607–619.
11. Král J., Netuka I. and Veselý J., *Teorie potenciálu IV*, Stát. pedagog. nakl., Praha, 1977 (in Czech).
12. Lebesgue H., *Sur la méthode de Carl Neumann*, J. Math. Pures Appl. 9^e série **XVI** (1937), 205–217, 421–423.
13. Leis R., *Zur Entwicklung der angewandten Analysis und mathematischen Physik in den letzten hundert Jahren.* Ein Jahrhundert Mathematik 1890–1990, Festschrift zum Jubiläum der DMV (herausgeben von G. Fischer, F. Hirzebruch, W. Scharlau und W. Torning), Dokumente zur Geschichte der Mathematik Bd. 6, DMV, Braunschweig 1990.
14. Maz'ya V. G., *Boundary integral equations,* in "Analysis IV", Encyclopaedia of Mathematical Science vol. 27, Springer-Verlag, 1991.
15. Medková D., *On the convergence of Neumann series for noncompact operators*, Czechoslovak Math. J. **41 (116)** (1991), 312–316.
16. Monna A. F., *Dirichlet's principle, A mathematical comedy of errors and its influence on the development of analysis*, Oostholk, Scheltema and Holkema, Utrecht, 1975.
17. Neumann C., *Untersuchungen über das logarithmische und Newtonsche Potential*, B. G. Teubner, Leipzig, 1877.
18. ———, *Über die Methode des aritmethmetischen Mittels*, Hirzel, Leipzig, 1887 (erste Abhandlung), 1888 (zweite Abhandlung).
19. Roach G. F., *An introduction to iterative techniques for potential problems,* in Proc. "Potential Theory – Surveys and Problems" (J. Král, J. Lukeš, I. Netuka, J. Veselý, eds.), Lecture Notes in Math., vol 1344, Springer-Verlag, 1988.
20. Suzuki N., *On the convergence of Neumann series in Banach space*, Math. Ann. **220** (1976), 143–146.
21. Young L. C., *A theory of boundary values*, Proc. London Math. Soc. (3) **14A** (1965), 300–314.

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