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A NOTE ON CONTINUOUS RESTRICTIONS OF LINEAR MAPS BETWEEN BANACH SPACES

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Abstract. This note is devoted to the answers to the following questions asked by V. I. Bogachev, B. Kirchheim and W. Schachermayer:
1. Let $T: l_1 \to X$ be a linear map into the infinite dimensional Banach space X. Can one find a closed infinite dimensional subspace $Z \subset l_1$ such that $T\big|_Z$ is continuous?
2. Let $X = c_0$ or $X = l_p$ $(1 < p < \infty)$ and let $T: X \to X$ be a linear map. Can one find a dense subspace Z of X such that $T\big|_Z$ is continuous?

This paper continues investigations of [**2**]. In [**2**] the following proposition was proved:

Let X be a separable Banach space not containing l_1 isomorphically. If Y is an arbitrary infinite dimensional Banach space then there exists a linear map $T: X \to Y$ such that there does not exist any closed infinite dimensional subspace Z of X such that the restriction of T to Z is continuous.

At the end of [**2**] the following question was proposed:

Let $T: l_1 \to X$ be a linear map into the infinite dimensional Banach space X. Can one find a closed infinite dimensional subspace $Z \subset l_1$ such that $T\big|_Z$ is continuous?

We answer this question in Proposition 1.

It was shown in [**2**] that if $X = l_p$ $(1 \le p < \infty)$ or $X = c_0$ then for every linear map $T: X \to X$ there is an infinite dimensional subspace Z of X such that $T\big|_Z$ is continuous. If $X = l_1$ then there is a dense subspace Z of X such that the restriction $T\big|_Z$ is continuous. The authors of [**2**] asked: can one find a dense subspace Z of X such that $T\big|_Z$ is continuous in other cases? We answer this question for $p \ne 2$ in Proposition 4.

We use standard Banach space terminology and notation as may be found in [**4**].

Propositon 1. *Let X and Y be infinite dimensional Banach spaces and let X be separable. Then there exists a linear map $T: X \to Y$ such that there does not exist any closed infinite dimensional subspace Z of X such that the restriction of T to Z is continuous.*

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Proof. Let $\{x_\alpha\}_{\alpha\in[0,1]}$ be a Hamel basis of X. Let us denote by e_α ($\alpha \in [0,1]$) the unit vectors of the space $l_1([0,1])$. Let us introduce a linear map $A_1\colon X \to l_1([0,1])$ in the following way: we represent $x \in X$ as a finite linear combination $x = \sum_\alpha a_\alpha x_\alpha$ and set $A_1(x) = \sum_\alpha a_\alpha e_\alpha$.

It is well-known that $l_1([0,1])$ embeds isometrically into l_∞. Let $A_2 : l_1([0,1]) \to l_\infty$ be one of the isometric embeddings.

Let $\{y_i\}_{i=1}^\infty$ be a minimal sequence in Y. We define the map $A_3\colon l_\infty \to Y$ by the equality

$$A_3(\{a_i\}_{i=1}^\infty) = \sum_{i=1}^\infty \frac{a_i y_i}{2^i\|y_i\|}\,.$$

It is clear that A_3 is a linear continuous injective map. Let us define $T\colon X \to Y$ by the equality $T = A_3A_2A_1$.

Assume that $T\big|_Z$ is continuous, where Z is an infinite dimensional closed subspace of X. Let B be an infinite dimensional convex compact in Z. Then the set $T(B)$ is also closed. By the continuity of A_2 and A_3 the set $(A_3A_2)^{-1}T(B)$ is also closed. Since A_2 and A_3 are injective we have $(A_3A_2)^{-1}T(B) = A_1(B)$.

Our construction is such that $A_1(B)$ consists of finite linear combinations of the unit vectors of $l_1([0,1])$. On the other hand, $A_1(B)$ is a closed convex set. It is easy to see that such subset is contained in some finite dimensional subspace of $l_1([0,1])$. This contradicts with injectivity of A_1. □

Proposition 2. *The restriction of A_1 (introduced in Proposition 1) to every subspace Z of X with uncountable Hamel basis is discontinuous.*

Proof. Let $\{z_\lambda\}_{\lambda\in\Lambda}$ be a Hamel basis of Z. Let us prove the following claim: the set of those e_α ($\alpha \in [0,1]$) which enter into the decompositions of vectors $\{A_1z_\lambda\}_{\lambda\in\Lambda}$ is uncountable.

Suppose that it is not the case. Since the set of finite subsets of a countable set is countable there exists an infinite (even uncountable) subset $\Omega \subset \Lambda$ and a finite set $D = \{e_{\alpha(1)}, \dots, e_{\alpha(n)}\}$ such that the decompositions of $\{A_1z_\lambda\}_{\lambda\in\Omega}$ use unit vectors of D only. This contradicts with injectivity of A_1.

This claim implies that A_1Z is nonseparable. Since Z is separable then the restriction $A_1\big|_Z$ is discontinuous. □

Remark 3. Using Proposition 2 we are able to finish the proof of Proposition 1 using the arguments from the proof of Proposition 10 in [**2**].

Proposition 4. *Let $X = l_p$ ($1 < p < \infty$, $p \neq 2$) or $X = c_0$. Then there exists a linear map $T\colon X \to X$ for which there does not exist a dense subspace Z of X such that the restriction of T to Z is continuous.*

Proof. We shall give arguments only in the case when $X = l_p$ ($1 < p < \infty$, $p \neq 2$). The case $X = c_0$ can be considered similarly using the result of [**3**] instead of the results of [**1**] and [**5**].

It is known ([**1**], [**5**]) that for every p $(1 < p < \infty,\ p \neq 2)$ we can find an infinitely increasing sequence $\{c_n\}_{n=1}^{\infty}$ of positive reals and a positive real number $\alpha < \infty$ such that for some sequence $\{m(n)\}_{n=1}^{\infty}$ of positive integers there exist surjective linear operators $\psi_n : l_p^{m(n)} \to l_p^n$ $(n \in \mathbb{N})$ such that

$$(1)\qquad (\forall\, x \in l_p^n)(\|x\| \ \leq\ \inf\{\|y\| : y \in l_p^{m(n)},\ \psi_n y = x\} \ \leq\ \alpha\|x\|)$$

and we have $\|S\| > c_n$ for every linear operator $S\colon l_p^n \to l_p^{m(n)}$ for which $\psi_n S$ is the identity map on l_p^n.

Let us define the operator

$$\psi\colon \Big(\sum_{n=1}^{\infty} \oplus\, l_p^{m(n)}\Big)_p \to \Big(\sum_{n=1}^{\infty} \oplus\, l_p^{n}\Big)_p$$

by the equality $\psi(\{x_n\}_{n=1}^{\infty}) = \{\psi_n x_n\}_{n=1}^{\infty}$. Let us note that both of the spaces in the definition of ψ are isometric to l_p. Inequality (1) implies that ψ is surjective.

Let $Y \subset X = l_p$ be an arbitrary algebraic complement of $\ker\psi$. Then $\psi\big|_Y$ is a bijective map onto X. Hence the inverse T of $\psi\big|_Y$ maps X into X.

Let us show that $T\big|_Z$ is discontinuous for every dense subspace Z of X. Suppose the contrary. Let Z be a dense subspace of X such that $T\big|_Z$ is continuous. Let $n \in \mathbb{N}$ be such that $c_n > 2\|T\big|_Z\|$. Let us denote by $P_n\colon X \to X$ the projections corresponding to the decomposition $X = (\sum_{n=1}^{\infty} \oplus l_p^n)$ (i.e. $P_n(\{x_k\}_{k=1}^{\infty}) = \{0,\dots,0,x_n,0,\dots\}$), and by $Q_n\colon X \to X$ the projections corresponding to the decomposition $X = (\sum_{n=1}^{\infty} \oplus l_p^{m(n)})_p$.

Let $\{e_k^n\}_{k=1}^n$ be the unit vector basis of $P_n(X)$. Let $\{z_k\}_{k=1}^n \subset Z$ be such that $\|z_k - e_k^n\| < 2^{-n}/n$ $(k = 1,\dots,n)$. It is easy to check that for every collection $\{a_k\}_{k=1}^n$ of scalars we have

$$\Big\|\sum_k a_k z_k\Big\| \leq 2\Big\|\sum_k a_k P_n z_k\Big\|.$$

It is clear that $\{P_n z_k\}_{k=1}^n$ is a basis of $P_n(X)$. Let us introduce the operator $S\colon P_n(X) \to Q_n(X)$ by the equalities $SP_n z_k = Q_n T z_k$ $(k = 1,\dots,n)$. Since $\psi Q_n T z_k = P_n z_k$ then $\|S\| > c_n$, i.e. for some collection $\{a_k\}_{k=1}^n$ of scalars we have

$$\Big\|\sum_{k=1}^n a_k Q_n T z_k\Big\| > c_n\Big\|\sum_{k=1}^n a_k P_n z_k\Big\|.$$

Therefore

$$\begin{aligned}\|T\big|_Z\|\,\Big\|\sum_{k=1}^n a_k z_k\Big\| &\geq \Big\|\sum_k a_k T z_k\Big\| \geq \Big\|\sum_k a_k Q_n T z_k\Big\| \\ &> c_n\Big\|\sum_k a_k P_n z_k\Big\| \geq \Big(\frac{c_n}{2}\Big)\Big\|\sum_k a_k z_k\Big\|.\end{aligned}$$

We obtain a contradiction. The proposition is proved. □

Remark 5. Using the arguments from Proposition 5 of [**2**] it is easy to prove the following statement. If Banach spaces X and Y are such that there exists a surjective strictly singular continuous linear map $\psi\colon Y \to X$ then there exists a linear map $T\colon X \to Y$ such that there does not exist an infinite dimensional subspace Z of X such that the restriction of T to Z is continuous.

Added in proof. Recently the author learnt that the first question was also solved by G. Godefroy.

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