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ON THE VOLUME OF THE DOUBLE STOCHASTIC MATRICES

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1. Introduction and Notation

Let $\prod_n$ be the group of permutation matrices in $\mathbb{R}^n$. Then

$$D_n := co\left(\Pi : \Pi \in \prod_n\right)$$

is a convex set in $\mathbb{R}^{n^2}$. It is well known that D_n is the set of all double stochastic matrices, i.e.

$$D_n = \left\{T = (t_{ij})_{i,j=1}^n : \sum_{i=1} t_{ij} = \sum_{j=1} t_{ij} = 1 \;\forall\, i,j \in \{1,\dots,n\},\, t_{ij} \in [0,1]\right\}$$

The volume of D_n is somehow related to a Kahane type inequality (cf. [**S**]) for the group of permutations, more precisely: let $(x_{j,k})$ be a double sequence in some Banach space X, if

$$\left(\frac{\mathrm{Vol}_k(D_n)}{\mathrm{Vol}_k(B_k^2)}\right)^{\frac{1}{k}} \geq c\sqrt{n}$$

where k is the dimension of $D_n \subseteq \mathbb{R}^{n^2}$, then the L^1-norm and the norm associated with $\psi_1(t) := e^t - 1$ of $\|\sum_{j,k} x_{j,k}\pi_{j,k}\|$ (the expectation being taken with respect to the normalized counting measure on the group of all signed permutation matrices $(\pi_{j,k})$ i.e. $\pi_{j,k} \in \{-1,0,1\}$) are equivalent. Conversely, if the L^1-norm and the norm associated with $\psi_2(t) := e^{t^2} - 1$ are equivalent, then the volume of D_n must satiesfy the above inequality up to some logarithmic factor. We prove that such an inequality can not hold. We also include a proof of an upper estimate for the volume of a convex polytope all of whose vertices are at a given distance from the origin. Though this result is known we could not find a reference.

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It is easy to see that the subspace E of $\mathbb{R}^{n^2}$ defined by

$$E = \bigcap_{j=1}^{2n} \left\{ x \in \mathbb{R}^{n^2} : \langle x, N_j \rangle = 0 \right\}$$

where

$$\begin{aligned}
N_1 &= (\underbrace{1, \dots, 1}_{n}, 0, \dots, 0), \\
N_2 &= (\underbrace{0, \dots, 0}_{n}, \underbrace{1, \dots, 1}_{n}, 0, \dots, 0), \dots \\
N_n &= (0, \dots, 0, \underbrace{1, \dots, 1}_{n}), \\
N_{n+1} &= (1, \underbrace{0, \dots, 0}_{n-1}, 1, \underbrace{0, \dots, 0}_{n-1}, 1, \dots), \\
N_{n+2} &= (0, 1, \underbrace{0, \dots, 0}_{n-1}, 1, \underbrace{0, \dots, 0}_{n-1}, 1, \dots), \dots \\
N_{2n} &= (\underbrace{0, \dots, 0}_{n-1}, 1, \underbrace{0, \dots, 0}_{n-1}, 1, \dots)
\end{aligned}$$

has dimension $(n-1)^2$. Thus the dimension of D_n is $(n-1)^2$.

2. The Basic Estimates

In order to estimate the $(n-1)^2$-dimensional volume of D_n we need some results. The first one is due to Vaaler [**V**] (a generalization of this result can be found in [**M-P**]).

Lemma 2.1. *Let E be a k-dimensional subspace of $\mathbb{R}^n$. Then*

$$\mathrm{Vol}_k(B_n^\infty \cap E) \geq 2^k$$

where B_n^∞ is the cube $[-1,1]^n$.

The next result is a classical inequality of Urysohn (for an elementary proof we refer to [**P**]).

Lemma 2.2. *Let B be a convex symmetric body in $\mathbb{R}^n$. Then*

$$\left(\frac{\mathrm{Vol}_n(B)}{\mathrm{Vol}_n(B_n^2)} \right)^{\frac{1}{n}} \leq \left(\int_{S^{n-1}} ||x||_{B^*}^2 \, d\lambda(x) \right)^{\frac{1}{2}}$$

where B_n^2 is the unit ball of ℓ_n^2, B^* is the polar of B and λ is the normalized Lebesgue measure on S^{n-1}.

It is well known that the latter integral can be expressed as a gaussian integral, i.e.

$$\left(\int_{S^{n-1}} ||x||_{B^*}^2 \, d\lambda(x)\right)^{\frac{1}{2}} = \frac{1}{\sqrt{n}} \left(\int_{\mathbb{R}^n} ||x||_{B^*}^2 \, d\gamma_n(x)\right)^{\frac{1}{2}}$$

where γ_n is the canonical gaussian probability measure on $\mathbb{R}^n$.

Lemma 2.3. *Let $g_1, \dots, g_k$ be not necessarily independent gaussian variables with mean zero. Then*

$$\left(\mathbf{E} \sup_{j \le k} |g_j|^2\right)^{\frac{1}{2}} \le c(1 + \log k)^{\frac{1}{2}} \sup_{j \le k} ||g_j||_2$$

For a proof we refer to [**P**].

Lemmas 2.2 and 2.3 immediately imply the following

Proposition 2.4. *Let $x_1, \dots, x_k$ be unit vectors in $\mathbb{R}^n$. Then*

$$\left(\frac{\mathrm{Vol}_n(co(x_1, \dots, x_k))}{\mathrm{Vol}_n(B_n^2)}\right)^{\frac{1}{n}} \le c\sqrt{\frac{\log k}{n}}$$

Proof. By Lemma 2.2 we have

$$\left(\frac{\mathrm{Vol}_n(B)}{\mathrm{Vol}_n(B_n^2)}\right)^{\frac{1}{n}} \le \frac{1}{\sqrt{n}} \left(\mathbf{E} || \sum_{j=1} g_j e_j ||_{B^*}^2\right)^{\frac{1}{2}}$$

where B is the absolutely convex hull of $\{x_1, \dots, x_n\}$ and $(g_j)_{j=1}^n$ are independent standard gaussian variables. Since

$$||x||_{B^*} = \sup_{i \le k} |\langle x_i, x\rangle|$$

we get from Lemma 2.3

$$\begin{aligned} \mathbf{E} || \sum_{j=1} g_j e_j ||_{B^*}^2 &= \mathbf{E} \sup_{i \le k} | \sum_{j=1} g_j \langle x_i, e_j \rangle |^2 \\ &\le c(1 + \log k) \end{aligned}$$

□

Theorem 2.5. *There exists an absolute constant c such that the following inequalities hold.*

$$\frac{2}{n} \leq \left(\mathrm{Vol}_{(n-1)^2}(D_n)\right)^{\frac{1}{(n-1)^2}} \leq \frac{c}{n}\sqrt{\log n}$$

Proof. Let P_0 be the $n \times n$ matrix with the constant entry $\frac{1}{n}$ in each place. Then we conclude by Lemma 2.1

$$\begin{aligned} \mathrm{Vol}_{(n-1)^2}(D_n) &= \mathrm{Vol}_{(n-1)^2}\left([0,1]^{n^2} \cap (E + P_0)\right) \\ &= \mathrm{Vol}_{(n-1)^2}\left(\left[-\frac{1}{n}, 1 - \frac{1}{n}\right]^{n^2} \cap E\right) \\ &\geq \mathrm{Vol}_{(n-1)^2}\left(\left[-\frac{1}{n}, \frac{1}{n}\right]^{n^2} \cap E\right) \\ &\geq \left(\frac{2}{n}\right)^{(n-1)^2} \end{aligned}$$

Thus the left hand side inequality is established. As for the right hand side observe that for any permutation matrix Π:

$$\Pi - P_0||_{HS} = \left((n^2 - n)\frac{1}{n^2} + n\left(1 - \frac{1}{n}\right)^2\right)^{\frac{1}{2}} = \sqrt{n-1}$$

Since the number of permutation matrices is $n!$ we deduce from the above proposition

$$\left(\frac{\mathrm{Vol}_{(n-1)^2}(D_n)}{\mathrm{Vol}_{(n-1)^2}(B^2_{(n-1)^2})}\right)^{\frac{1}{(n-1)^2}} \leq \sqrt{n-1}c\sqrt{\frac{\log n!}{(n-1)^2}} \leq c_1\sqrt{\log n}$$

Hence

$$(Vol\, D_n)^{\frac{1}{(n-1)^2}} \leq \frac{c_2}{n}\sqrt{\log n}$$

□

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