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UNIFORMLY BEST LINEAR–QUADRATIC ESTIMATOR IN A SPECIAL STRUCTURE OF THE REGRESSION MODEL

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Abstract. The paper shows the uniformly best linear-quadratic unbiased estimator of the covariance matrix element related to the repeated measurement in a regression model where dispersions depend quadratically on mean value parameters. Its consistency with respect to increasing number of repeated measurements is also investigated.

Introduction

Let us have the well known regression model $(\widetilde{\mathbf{Y}}, \mathbf{X}_1\boldsymbol{\beta}, \widetilde{\boldsymbol{\Sigma}})$, where the observation $\widetilde{\mathbf{y}} \in \mathcal{R}^q$ is a realization of the normally distributed random vector $\widetilde{\mathbf{Y}}$ with mean value $\mathcal{E}(\widetilde{\mathbf{Y}}) = \mathbf{X}_1\boldsymbol{\beta}$ ($\mathbf{X}_1$ is a known $q \times k$ design matrix, $\boldsymbol{\beta} \in \mathcal{R}^k$ is a vector of unknown parameters) and covariance matrix $\widetilde{\boldsymbol{\Sigma}}$. A large class of measurement devices has its dispersion characteristic of the form $\sigma^2(a + b|\varphi|)^2$, where σ^2, a and b are known positive constants and φ is the true measured value (see e.g. [**1,** p. 28], [**3,** p. 456, 914]). If we assume independent measurements, we obtain $\widetilde{\boldsymbol{\Sigma}}$ of the form

$$\widetilde{\boldsymbol{\Sigma}} = \sigma^2\widetilde{\boldsymbol{\Sigma}}(\boldsymbol{\beta}) = \sigma^2 \begin{pmatrix} (a + b|\mathbf{e}_1'\mathbf{X}_1\beta|)^2 & 0 & \dots & 0 \\ 0 & (a + b|\mathbf{e}_2'\mathbf{X}_1\beta|)^2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & & (a + b|\mathbf{e}_q'\mathbf{X}_1\beta|)^2 \end{pmatrix}.$$

Further let us suppose that the rank of the matrix $\mathbf{X}_1$ is

$$R(\mathbf{X}_1) = q \le k\,,$$

(there are less linearly independent observations as unknown parameters) but one measurement (say s-th measurement) is repeated $n - q$ times.

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We obtain the model

$$(\mathbf{Y}, \mathbf{X}\beta, \mathbf{\Sigma}), \tag{1}$$

where

$$\mathbf{X}_{n,k} = \begin{pmatrix} \mathbf{I}_{q,q} \\ \mathbf{e}'_s \\ \vdots \\ \mathbf{e}'_s \end{pmatrix} \mathbf{X}_1,$$

$$\begin{aligned}\mathbf{\Sigma}_{n,n} &= \sigma^2 \mathbf{\Sigma}(\beta) \\ &= \sigma^2 \begin{pmatrix} (a+b|\mathbf{e}'_1\mathbf{X}\beta|)^2 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & & & & & & \\ 0 & & (a+b|\mathbf{e}'_s\mathbf{X}\beta|)^2 & & & & & \\ \vdots & & & \ddots & & & & \\ 0 & & & & (a+b|\mathbf{e}'_q\mathbf{X}\beta|)^2 & & & \\ 0 & & & & & (a+b|\mathbf{e}'_s\mathbf{X}\beta|)^2 & & \\ \vdots & & & & & & \ddots & \\ 0 & & & & & & & (a+b|\mathbf{e}'_s\mathbf{X}\beta|)^2 \end{pmatrix},\end{aligned}$$

$s \in \{1, 2, \dots, q\}$, $n > q$ and $\mathbf{e}'_i$ is the transpose of the i-th unit vector.

Continuated the considerations concerning the special case of the regression model with dispersions depending on mean value parameters stated in [**4**], [**5**] and [**6**] there arises the problem of finding the UBLQUE (uniformly best linear-quadratic unbiased estimator) of the functional $\sigma^2(a + b|\mathbf{e}'_s\mathbf{X}\beta|)^2$ of parameter β in model (1).

Based on analyses and methods in [**5**] and [**6**] it can be shown that the $\beta_\circ$-LBLQUE ($\beta_\circ$-locally best linear-quadratic unbiased estimator) of $\sigma^2(a + b|\mathbf{e}'_s\mathbf{X}\beta|)^2$ exists iff the linear system (2) is soluble. In this case is the desired $\beta_\circ$-LBLQUE given by (3) and (4). Of course the UBLQUE of $\sigma^2(a + b|\mathbf{e}'_s\mathbf{X}\beta|)^2$ in model (1) is related to such a solution of (2) that does not depend on $\beta_\circ$.

In this paper is proved that the "natural" estimator (9) based only on the repeated measurements is the desired UBLQUE of $\sigma^2(a + b|\mathbf{e}'_s\mathbf{X}\beta|)^2$ (in Corollary 4). An easy consequence is the consistency of (9) with respect to increasing number n of the repeated s-th measurement. This forms a base for further investigations with repetitions of more than one measurement.

UBLQUE

First let us introduce some denotations.

$\mathcal{X}'_{(s)}$ is the $(k^2+q)\times n^2$ matrix

$$\begin{pmatrix} \mathbf{X}'\otimes\mathbf{X}' \\ \mathbf{e}'_1\otimes\mathbf{e}'_1 \\ \vdots \\ \mathbf{e}'_{s-1}\otimes\mathbf{e}'_{s-1} \\ \mathbf{e}'_{s+1}\otimes\mathbf{e}'_{s+1} \\ \vdots \\ \mathbf{e}'_q\otimes\mathbf{e}'_q \\ \mathbf{e}'_s\otimes\mathbf{e}'_s+\sum_{j=q+1}^n \mathbf{e}'_j\otimes\mathbf{e}'_j \end{pmatrix},$$

where $\otimes$ means the Kronecker product (see e.g. [**2,** p. 11]) and $\mathbf{I}^\star$ is the nonsingular matrix with next property:

$$\forall\{\mathbf{A}_{n,n}\}\qquad \mathbf{I}^\star vec\,\mathbf{A}=vec\,\mathbf{A}'$$

$(vec\,\mathbf{A}_{n,n}=(a_{11},a_{21},\dots,a_{n1},a_{12},a_{22},\dots,a_{n2},\dots,a_{1n},a_{2n},\dots,a_{nn})')$.

Further

$$\mathcal{C}_n=((\mathbf{I}_{q,q},\mathbf{e}_s,\dots,\mathbf{e}_s)\otimes(\mathbf{I}_{q,q},\mathbf{e}_s,\dots,\mathbf{e}_s))(\mathbf{I}+\mathbf{I}^\star)(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))$$
$$\cdot(\mathbf{I}+\mathbf{I}^\star)(\begin{pmatrix}\mathbf{I}_{q,q}\\ \mathbf{e}'_s\\ \vdots\\ \mathbf{e}'_s\end{pmatrix}\otimes\begin{pmatrix}\mathbf{I}_{q,q}\\ \mathbf{e}'_s\\ \vdots\\ \mathbf{e}'_s\end{pmatrix})$$

and

$$\mathcal{A}'_{k^2+q-1,q^2}=\begin{pmatrix} \mathbf{X}'_1\otimes\mathbf{X}'_1 \\ \mathbf{e}'_1\otimes\mathbf{e}'_1 \\ \vdots \\ \mathbf{e}'_{s-1}\otimes\mathbf{e}'_{s-1} \\ \mathbf{e}'_{s+1}\otimes\mathbf{e}'_{s+1} \\ \vdots \\ \mathbf{e}'_q\otimes\mathbf{e}'_q \end{pmatrix}.$$

(We only note that $\mathcal{A}'$ is a matrix of full rank in columns.)

It holds

$$\mathcal{X}'_{(s)}=\begin{pmatrix}\mathcal{A}'((\mathbf{I}_{q,q},\mathbf{e}_s,\dots,\mathbf{e}_s)\otimes(\mathbf{I}_{q,q},\mathbf{e}_s,\dots,\mathbf{e}_s))\\ \mathbf{e}'_s\otimes\mathbf{e}'_s+\sum_{j=q+1}^n\mathbf{e}'_j\otimes\mathbf{e}'_j\end{pmatrix}.$$

As stated in Introduction it can be shown (based on analyses and methods in [**5**] and [**6**]) that there exists in model (1) the $\boldsymbol{\beta}_o$-LBLQUE of the functional $\sigma^2(a+b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}|)^2$ of parameter $\boldsymbol{\beta}$ iff

$$\mathcal{X}'_{(s)}(\mathbf{I}+\mathbf{I}^\star)(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)\otimes\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))(\mathbf{I}+\mathbf{I}^\star)\mathcal{X}_{(s)}\delta=\begin{pmatrix}\mathbf{O}_{k^2+q-1,1}\\ 8\end{pmatrix}\tag{2}$$

is soluble. In this case $\mathbf{a}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the desired $\boldsymbol{\beta}_\circ$-LBLQUE, where

$$\mathbf{a} = -(\mathbf{A} + \mathbf{A}')\mathbf{X}\boldsymbol{\beta}_\circ \tag{3}$$

and

$$vec\mathbf{A} = \frac{1}{2}(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ))\mathcal{X}_{(s)}\delta. \tag{4}$$

It is easy to seen that equation (2) can be rewritten as

$$\begin{pmatrix} \mathcal{A}'\mathcal{C}_n\mathcal{A} & \mathbf{g} \\ \mathbf{g}' & g \end{pmatrix} \delta = \begin{pmatrix} \mathbf{O} \\ 8 \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} \mathbf{g}'_{1,k^2+q-1} &= (\mathbf{e}'_s \otimes \mathbf{e}'_s + \sum_{j=q+1}^{n} \mathbf{e}'_j \otimes \mathbf{e}'_j)(\mathbf{I} + \mathbf{I}^\star)(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)) \\ &\qquad \cdot (\mathbf{I} + \mathbf{I}^\star)(\begin{pmatrix} \mathbf{I}_{q,q} \\ \mathbf{e}'_s \\ \vdots \\ \mathbf{e}'_s \end{pmatrix} \otimes \begin{pmatrix} \mathbf{I}_{q,q} \\ \mathbf{e}'_s \\ \vdots \\ \mathbf{e}'_s \end{pmatrix})\mathcal{A} \\ &= 4(n-q+1)(a + b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}_\circ|)^{-4}(\mathbf{e}'_s\mathbf{X}_1 \otimes \mathbf{e}'_s\mathbf{X}_1, 0, \ldots, 0) \end{aligned}$$

and

$$\begin{aligned} g &= (\mathbf{e}'_s \otimes \mathbf{e}'_s + \sum_{j=q+1}^{n} \mathbf{e}'_j \otimes \mathbf{e}'_j)(\mathbf{I} + \mathbf{I}^\star)(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ) \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_\circ)) \\ &\qquad \cdot (\mathbf{I} + \mathbf{I}^\star)(\mathbf{e}_s \otimes \mathbf{e}_s + \sum_{j=q+1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j) \\ &= 4(n-q+1)(a + b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}_\circ|)^{-4}. \end{aligned}$$

Let us denote

$$\delta_{k^2+q} = \frac{2(a + b|\mathbf{e}'_s\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)} \begin{pmatrix} \mathbf{X}'_1(\mathbf{X}_1\mathbf{X}'_1)^{-1}\mathbf{e}_s \otimes \mathbf{X}'_1(\mathbf{X}_1\mathbf{X}'_1)^{-1}\mathbf{e}_s \\ 0 \\ \vdots \\ 0 \\ 1-n+q \end{pmatrix}. \tag{6}$$

Lemma 1. *It holds*

$$(\mathcal{A}'\mathcal{C}_n\mathcal{A} \quad \mathbf{g})\delta = \mathbf{O}_{k^2+q-1,1} \ .$$

Proof. As

$$\mathcal{A}\begin{pmatrix}\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\otimes\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\\0\\\vdots\\0\end{pmatrix}_{k^2+q-1,1}\cdot\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)}=\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)}(\mathbf{e}_s\otimes\mathbf{e}_s)_{q^2,1}$$

and

$$\begin{aligned}&\mathcal{C}_n(\mathbf{e}_s\otimes\mathbf{e}_s)\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)}\\&=2(\mathbf{I}+\mathbf{I}^\star)_{q^2,q^2}\cdot\left(\begin{pmatrix}(a+b|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&0&\dots&0\\\vdots&\ddots&&\\0&(n-q+1)(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&\dots&0\\\vdots&\ddots&&\\0&\dots&(a+b|\mathbf{e}_q'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&\end{pmatrix}\right.\\&\qquad\left.\otimes\begin{pmatrix}(a+b|\mathbf{e}_1'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&0&\dots&0\\\vdots&\ddots&&\\0&(n-q+1)(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&\dots&0\\\vdots&\ddots&&\\0&\dots&(a+b|\mathbf{e}_q'\mathbf{X}\boldsymbol{\beta}_\circ|)^{-2}&\end{pmatrix}\right)\\&\qquad\cdot(\mathbf{e}_s\otimes\mathbf{e}_s)\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)}\\&=-8(\mathbf{e}_s\otimes\mathbf{e}_s)\frac{n-q+1}{n-q}\,,\end{aligned}$$

we obtain

$$\mathcal{A}'\mathcal{C}_n\mathcal{A}\begin{pmatrix}\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\otimes\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\\0\\\vdots\\0\end{pmatrix}\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)}$$

$$=-\begin{pmatrix}\mathbf{X}'\otimes\mathbf{X}'\\\mathbf{e}_1'\otimes\mathbf{e}_1'\\\vdots\\\mathbf{e}_q'\otimes\mathbf{e}_q'\end{pmatrix}(\mathbf{e}_s\otimes\mathbf{e}_s)\frac{8(n-q+1)}{n-q}\tag{7}$$

$$=-8\frac{n-q+1}{n-q}\begin{pmatrix}\mathbf{X}_1'\mathbf{e}_s\otimes\mathbf{X}_1'\mathbf{e}_s\\0\\\vdots\\0\end{pmatrix}_{k^2+q-1,1}.$$

Further

$$\mathbf{g}\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)} = 8\frac{n-q+1}{n-q}\begin{pmatrix}\mathbf{X}_1'\mathbf{e}_s\otimes\mathbf{X}_1'\mathbf{e}_s\\ 0\\ \vdots\\ 0\end{pmatrix}_{k^2+q-1,1}. \tag{8}$$

After an easy calculation (by the help of (7) and (8)) we can finish the proof of the lemma. □

Lemma 2. *It holds*

$$(\mathbf{g}' \quad g)\delta = 8\ .$$

Proof. As

$$\mathbf{g}'\begin{pmatrix}\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\otimes\mathbf{X}_1'(\mathbf{X}_1\mathbf{X}_1')^{-1}\mathbf{e}_s\\ 0\\ \vdots\\ 0\end{pmatrix}\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)(1-n+q)} = -\frac{8}{n-q}$$

and

$$g\frac{2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}_\circ|)^4}{(n-q)} = \frac{8(n-q+1)}{n-q},$$

we easy obtain

$$(\mathbf{g}' \quad g)\delta = -\frac{8}{n-q} + \frac{8(n-q+1)}{n-q} = 8.$$

The lemma is proved. □

According to Lemma 1 and Lemma 2, δ (given in (6)) is a solution to (5), or, equivalently to (2). Thus we obtain the next corollary:

Corollary 3. *The random variable*

$$\frac{1}{n-q}\sum_{j=s,j=q+1}^{n}(Y_j-\overline{Y})^2, \tag{9}$$

where

$$\overline{Y} = \frac{1}{n-q+1}\sum_{j=s,j=q+1}^{n} Y_j$$

is the $\boldsymbol{\beta}_\circ$-LBLQUE *of* $\sigma^2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|)^2$.

Proof. As δ (given in (6)) is a solution to (2), $\mathbf{a}'\mathbf{Y}+\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2(a+b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|)^2$, where (3) and (4) hold.

It is easy to see that

$$(\mathbf{e}_i' \otimes \mathbf{e}_j') vec\, \mathbf{A}$$

is

$$0 \text{ for } i \notin \{s, q+1, \ldots, n\} \text{ or } j \notin \{s, q+1, \ldots, n\},$$

$$\frac{1}{n-q}\left(-\frac{1}{n-q+1}\right) \text{ for } i \in \{s, q+1, \ldots, n\},\ j \in \{s, q+1, \ldots, n\},\ i \neq j$$

and

$$\frac{1}{n-q}\left(1-\frac{1}{n-q+1}\right) \text{ for } i \in \{s, q+1, \ldots, n\},\ j = i\,.$$

So

$$\begin{aligned}
\mathbf{Y}'\mathbf{A}\mathbf{Y} &= (\mathbf{Y}' \otimes \mathbf{Y}') vec\, \mathbf{A} \\
&= \frac{1}{n-q}\left(-\frac{1}{n-q+1}\right) \sum_{i=s, i=q+1}^{n} \sum_{j=s, j=q+1}^{n} Y_i Y_j \\
&\quad + \frac{1}{n-q}(1 - \frac{1}{n-q+1}) \sum_{i=s, i=q+1}^{n} Y_i^2 \\
&= \frac{1}{n-q} \sum_{j=s, j=q+1}^{n} (Y_j - \overline{Y})^2\,.
\end{aligned}$$

We see that for $\mathbf{A}$ given by (4) is $\mathbf{A} = \mathbf{A}'$. After a short computation also $\mathbf{X}'\mathbf{A} = \mathbf{O}$ is obvious. So (9) is the $\boldsymbol{\beta}_\circ$-LBLQUE of $\sigma^2(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|)^2$. □

As (9) does not depend on the $\boldsymbol{\beta}_\circ$, we finally have

Corollary 4. *The random variable*

$$\frac{1}{n-q} \sum_{j=s, j=q+1}^{n} (Y_j - \overline{Y})^2,$$

with

$$\overline{Y} = \frac{1}{n-q+1} \sum_{j=s, j=q+1}^{n} Y_j$$

is in model (1) *the* UBLQUE (*uniformly best linear-quadratic unbiased estimator*) *of* $\sigma^2(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|)^2$.

We only note that evidently (9) is a consistent estimator with dispersion

$$\mathcal{D}_{\boldsymbol{\beta}} = \frac{2\sigma^4(a + b|\mathbf{e}_s'\mathbf{X}\boldsymbol{\beta}|)^4}{n-q}\,.$$

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