Acta Math. Univ. Comenianae
Vol. LXI, 2(1992), pp. 209–217

MINIMAL SIZE OF A GRAPH WITH DIAMETER 2 AND GIVEN MAXIMAL DEGREE, II

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Abstract. Let $F_2(n, \lfloor pn \rfloor)$ be the minimal size of a graph on n vertices with diameter 2 and maximal degree $\lfloor pn \rfloor$. The asymptotic behaviour of $F_2(n, \lfloor pn \rfloor)$ is considered for $2/5 < p < 5/12$.

1. Introduction

Denote by $H_2(n, \lfloor pn \rfloor)$ the family of undirected graphs of order n, diameter 2 and maximal degree $\lfloor pn \rfloor$ $(0 < p < 1)$ and put

$$F_2(n, \lfloor pn \rfloor) = \min_{G \in H_2(n, \lfloor pn \rfloor)} e(G)$$

where $e(G)$ is the size of G. Denote further by

$$f(p) = \lim_{n \to \infty} F_2(n, \lfloor pn \rfloor) .$$

The function $f(p)$ was introduced in [**1**] and in [**5**] the existence of the limit (conjectured in [**2**]) was proved for all values of p except of a sequence tending to 0. It is also showed in [**5**] that for a given p, $f(p)$ can be determined using linear programming. However, this procedure is too slow to enable us to solve the problem even for relatively large values of p.

In [**2**] the values of $f(p)$ for $p > 1/2$ were determined. Further, in [**4**] it was shown that if a projective plane of order t exists, then $f(p) = t+1$ for $(t+1)/(t^2 + t + 1) < p < 1/t$, hence putting $t = 2$ we get $f(p) = 3$ if $3/7 < p < 1/2$. In [**6**] $f(p)$ is determined for $5/12 < p < 3/7$. Thus for all $p > 5/12$ the values of $f(p)$ are known.

In this paper we determine $f(p)$ for smaller p. In fact, we prove here the following result conjectured in [**5**].

Theorem.

$$f(p) = 8 - 11p \qquad \text{for} \quad 2/5 < p < 5/12 . \tag{1}$$

Received May 16, 1991.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05C35.

We shall use here very often methods and results of [**5**]. In those cases when our assertions can be proved by a slight modification of those in [**5**], the proofs will be omitted here. On the other hand, the following lemma is used here in exactly the same way as in [**5**].

Lemma 1 ([**5**]). *Let a set U with $|U| \geq 8$ be given. Let Z be a system of triples of distinct elements of U. If every element of U is contained in some triple of Z and any two triples of Z intersect then there exist $x, y \in U$ such that every triple of Z contains at least one of x, y (we say that x, y cover Z).*

2. The Main Inequality

We prove now that if p fulfils (1) and n is sufficiently large then for every $G \in H_2(n, \lfloor pn \rfloor)$ we have

$$e(G) \geq 8n - 11\lfloor pn \rfloor - 8\left\{4\binom{192}{3} + 128\right\}\sqrt{n}\,. \tag{2}$$

Let $I = 4\binom{192}{3} + 128$. We shall proceed indirectly: suppose there exists a graph $G_0 \in H_2(n, \lfloor pn \rfloor)$ with

$$e(G_0) < 8n - 11\lfloor pn \rfloor - 8I\sqrt{n}\,. \tag{3}$$

Denote by V the set of all vertices of G_0, and by A the set of vertices of degree at least $\sqrt{n}$. According to (3) we have

$$|A| < 16\sqrt{n}\,. \tag{4}$$

Denote further by B the set of vertices of degree at most 7 adjacent to 3 vertices of A (due to (1), no vertex of degree less than 3 exists in G_0), and by C the set of such vertices adjacent to at least 4 vertices of A, and finally, let $D = V - A - B - C$. If $x \in D$ then $8 \leq \deg x \leq \sqrt{n}$.

The proof of the following lemma is straightforward (and very similar to that of Lemma 2 in [**5**]).

Lemma 2.

$$2e(G_0) \geq 8n - 2|B| - 128\sqrt{n}\,.$$

Let now E be the set of vertices of degree at least $n/12$. By (3) we have

$$|E| < 192\,. \tag{5}$$

According to (1) every vertex of B is adjacent to 3 vertices of E. Let abc be the set of vertices of B adjacent to vertices a, b, $c \in E$ (a set of the form abc will be called a triple-set). B consists of at most $\binom{192}{3}$ triple-sets. Let F be the union of triple-sets from B with cardinalities at least $4\sqrt{n}$. Then

$$|B| < |F| + 4\binom{192}{3}\sqrt{n}\,. \tag{6}$$

Denote by T the system of neighbours of triple-sets in F and by W the set of vertices occurring in T. Now we shall state some lemmas.

Lemma 3. *Any two triples of T have a common element.*

The proof is straightforward and very similar to that of Lemma 3 in [**5**].

Lemma 4. *Let J be the set of vertices x having the following property: there is some triple-set abc in F such that x is not adjacent to any of the vertices a, b, c. Then $|J| \le 4\binom{192}{3}\sqrt{n}$.*

The proof follows from the obvious fact that to any triple-set abc there exist at most $4\sqrt{n}$ vertices adjacent to neither a, b, c (see also the proof of Lemma 4 in [**5**]).

Lemma 5. *T is covered by two vertices.*

Proof. Assume the opposite. Then by Lemma 4 each of at least $n - 4\binom{192}{3}\sqrt{n}$ vertices must be adjacent to at least 3 vertices of W. However, by Lemma 1 and Lemma 2 we get $|W| \le 7$, a contradiction with (1). □

In what follows we shall use the notation $\lfloor pn \rfloor = k$.
First of all, from (3), (6) and Lemma 2 follows

$$(7)\quad 8n - 2|F| - 2I\sqrt{n} \le 16n - 22k - 16I\sqrt{n}, \quad \text{thus } |F| \ge 11k - 4n + 7I\sqrt{n}\,.$$

Let now $M = V - A - J - H$ where H is a set of cardinality less than $\sqrt{n}$ which will be specified later. From (4) and Lemma 4 we have

$$(8)\qquad m = |M| > n - I\sqrt{n}\,.$$

Thus (7) can be rewritten in the form

$$(9)\qquad |F| \ge 11k - 4m + 3I\sqrt{n}\,.$$

Further considerations will be restricted to the vertices of M. The following two lemmas will be of some use later.

Lemma 6. *If a, b, c_1, $\dots$, c_r are distinct vertices of W, and there exist p vertices of M adjacent to both a and b then*

$$|abc_1 \cup \dots \cup abc_r| + rp \le r(3k - m)\,.$$

Proof. For every $i \in \{1, \dots, r\}$ the number of edges incident to vertices a, b, c_i is at least $m + p + abc_i$ $(\le 3k)$, and thus the assertion follows. □

From the obvious inequality $|abc_1| \le p$ we get, by taking $r = 1$:

Corollary.

$$|abc_i| \le (3k - m)/2\,.$$

Lemma 7. *There exist less than* $(5m - 11k)/2$ *vertices of* M *having at least 5 neighbours in* $V - M$.

Proof. If it is not the case then $e(G_0) \geq 3m + 2(5m - 11k)/2 = 8m - 11k$, a contradiction with (3). $\square$

Now we shall prove that (3) leads to a contradiction. First of all, if all triples of T contain a fixed vertex then we have $|F| \leq k$, a contradiction with (9) for $k \geq 2n/5$ (see (1)). Assume now that T is covered by two vertices x, y (see Lemma 5). Denote by X, Y and Z, respectively, the union of all triple-sets of F adjacent only to x, only to y, and to both x and y, respectively.

We have to distinguish several cases depending on the form of the sets X, Y and Z.

Case 1. xuv, $xwt \subset X$ and yuw, $yvt \subset Y$ (or yuv, $ywt \subset Y$, or yut, $yvw \subset Y$) where all included vertices are distinct.

Let $K = \{x, y\}$, $L = \{u, v, w, t\}$. Further let S, Q, and R, respectively, be the set of vertices of M adjacent to exactly one vertex, exactly two vertices, and no vertex, respectively, of K. We can easily derive the following inequalities:

$$2m + |S| \leq 6k, \qquad \text{i.e. } |S| \leq 6k - 2m; \tag{10}$$

$$2|Q| + |S| \leq 2k, \qquad \text{i.e. } |Q| \leq m - 2k; \tag{11}$$

$$|R| = m - |S| - |Q| \geq 2m - 4k\,. \tag{12}$$

Suppose $Z = xyz_1 \cup \dots \cup xyz_i$ and let (xyx_1) $[(xyx_2)]$ be the set of all vertices of M adjacent to x, y, x_1 (to x, y, x_2, respectively). The vertices of R are adjacent to every z_j, hence by (12) we get

$$(xyz_j) + 2m - 4k \leq k, \qquad \text{i.e. } (xyz_j) \leq 5k - 2m\,. \tag{13}$$

Now we need to consider several subcases.

Case 1a. If there exist at least 3 vertices $z_j \notin L$ then a vertex of R is adjacent to 2 vertices of L, to at least three vertices z_j but this is by (12) a contradiction to Lemma 7.

Case 1b. There exists at most one $z_j \notin L$. Because all the remaining triple-sets of F contain 3 vertices of $L \cup K$, we have $|F - xyz_i| \leq 6k - 2m$, and by (13) we get a contradiction with (9). (In this case for $|F| = 11k - 4n - 3$ we obtain the extremal graph – see Section 3.)

Case 1c. Now we have the most complicated case when Z contains exactly 2 triple-sets xyz_1 and xyz_2 such that z_1, $z_2 \notin K \cup L$. Denote by N, O, and P, respectively, the set of all vertices of M adjacent to at least two vertices, to one vertex, and to no vertex, respectively, of L. Then

$$\begin{aligned} 2|N| + |O| &\leq 4k\,, \\ |O| + |P| &\leq |Q|\,, \\ -2(|N| + |O| + |P|) &= -2m. \end{aligned}$$

Adding these inequalities, and taking into account (11), we have

$$|P| \geq 2m - 4k - |Q| \geq m - 2k\,.$$

Hence, by (13)

$$|P| - |(xyz_1)| - |(xyz_2)| \geq 5m - 12k\,.$$

Now put $p = 5/12 - \varepsilon$, $0 < \varepsilon < 1/60$ (see (1)). Then, from (8), $5m - 12k \geq 5(n - I\sqrt{n}) - 12(5/12 - \varepsilon)n = 12\varepsilon n - 5I\sqrt{n}$ which for sufficiently large n is greater than $10\varepsilon n$. Hence there exist at least $10\varepsilon n$ vertices of M not adjacent to any vertex of L, thus adjacent to both x and y, but not adjacent to z_1, z_2. By (3), among these vertices there exists a vertex b of degree less than $16/(10\varepsilon)$. Let $N(b)$ be the set of neighbours of b, and put $H = N(b)$ (H is the above-mentioned set.) Now, each vertex of R is adjacent to at least two vertices of L, to the vertices z_1, z_2 and to some vertex of H. Thus each vertex of R is adjacent to at least 5 vertices of $V - M$ which by (1) and (12) contradicts Lemma 7.

Thus if the conditions xuv, $xwt \subset X$, yuw, $yvt \subset Y$ (or $yuv, ywt \subset Y$, or yut, $yvw \subset Y$) are satisfied, then we always get a contradiction. Assume now that these conditions are not satisfied. Say, X does not contain triple-sets of the form xuv, xwt. Then the following cases can occur:

(a) $X = xuv \cup xuw \cup xvw$,
(b) all the triple-sets of X are adjacent to a further fixed vertex of W,
(c) $X = xuv$.

Now consider these cases.

Case 2. Suppose $X = xuv \cup xuw \cup xvw$. The vertices of M not adjacent to x are adjacent to at least two vertices of the set $\{u, v, w\}$. Hence $2(m-k) + 2|X| \leq 3k$, i.e. $|X| \leq (5k - 2m)/2$, a contradiction with (9), because all the remaining vertices of F are adjacent to y.

Case 3. Suppose $X = xtx_1 \cup \cdots \cup xtx_q$, $q \geq 2$. Again we have to distinguish several subcases.

Case 3a. $yx_ix_j \subset Y$ for some $i, j \in \{1, \ldots, q\}$. Then obviously $q = 2$. Let $U = xtx_1 \cup xtx_2 \cup yx_1x_2$. Then every vertex of M is adjacent to at least two vertices of the set $\{x, t, y, x_1, x_2\} = W_0$ and every vertex of U is adjacent to 3 vertices of W_0. Thus $2m + |U| \leq 5k$, i.e. $|U| \leq 5k - 2m$ which is a contradiction to (9) because all remaining vertices of F are adjacent to Y.

Case 3b. Suppose $Y = yty_1 \cup \cdots \cup yty_r$ and put $Z = xyz_1 \cup \cdots \cup xyz_s$. Assume that among the vertices $x_1, \ldots, x_p, y_1, \ldots, y_r, z_1, \ldots, z_s$ there exist u distinct vertices $w_1, \ldots, w_u$ different from x, y, t, and put $W_1 = \{x, y, t, w_1, \ldots, w_u\}$. Since F is not covered by a single vertex, every vertex of M is adjacent to at least two vertices of W_1 and the vertices of F to 3 vertices of W_1. Thus $2m + |F| \leq (3 + u)k$ which contradicts (9) if $u \leq 3$. Hence assume $u > 3$.

Denote by L the set of all vertices of M not adjacent to any vertex of the set $\{x, y, t\}$. Every vertex of F is adjacent to two vertices of this set, hence by (9) we have

$$3k + |L| \geq m + |F|, \quad \text{i.e. } |L| \geq m - 3k + |F| \geq 8k - 3m\,. \tag{14}$$

The vertices of L are adjacent to all vertices w_i, thus for $u = 4$ we have

$$2m + |F| + 2(m - 3k + |F|) \leq 7k, \quad \text{i.e. } |F| \leq (13k - 4m)/3$$

which is, by (1), a contradiction to (9).

For $u \geq 5$ all vertices of L are adjacent to at least 5 vertices of W_1 which by (14) contradicts Lemma 7.

Case 4. The last case is $X = xuv$. If $Z = \emptyset$ then F is covered also by y and u, and we may proceed as in the case 3a. Similarly, if all vertices of F are adjacent to u (or v) then we get again the case 3b. Hence we may assume

$$F = xuv \cup xyx_1 \cup \cdots \cup xyx_i \cup uyu_1 \cup \cdots \cup uyu_j \cup vyv_1 \cup \cdots \cup vyv_q$$

where i, j, q are different from 0. Now we show that all the vertices x_a, u_b, v_c are distinct. Indeed, suppose, for example, that $x_1 = u_1$ and consider $U = xuv \cup xyx_1 \cup uyx_1$. Every vertex of M is adjacent to two vertices of $W_3 = \{x, y, u, v, x_1\}$ and every vertex of U is adjacent to 3 vertices of W_3, thus $|U| \leq 5k - 2m$. However, all remaining vertices of F are adjacent to y, and so we get a contradiction with (9). Now, if $i = j = q = 1$ then by Corollary of Lemma 6 we get a contradiction to (9). Hence suppose $i > 1$ (as F is covered also by the couples u, y and v, y, we may proceed in the remaining cases similarly). Distinguish now two subcases.

Case 4a. Suppose $j = q = 1$. Consider first the case $i \geq 3$. By (9) and by Corollary to Lemma 6 we have

$$|xyx_1 \cup \cdots \cup xyx_i| \geq 11k - 4m - (9k - 3m)/2\,.$$

Thus the number of vertices adjacent neither to x nor to y is at least $m - 2k + [11k - 4m - (9k - 3m)/2] = (9k - 3m)/2$. However, each such vertex is adjacent to at least 5 vertices of $V - M$, a contradiction to Lemma 7 (see (1)).

Assume now $i = 2$. Let s be the number of vertices adjacent to y but not to x_1 nor to x_2. Each such vertex is adjacent to at least one of vertices x, u, v. So let s_1, s_2, and s_3, respectively, be the number of such vertices adjacent to x, u, v, respectively. Obviously,

$$s_1 + s_2 + s_3 \geq s\,. \tag{15}$$

Now according to Lemma 6 we have

$$|uyu_1| \leq (3k - m - s_2)/2, \qquad |vyv_1| \leq (3k - m - s_3)/2\,. \tag{16}$$

Consider now the number of vertices adjacent to x. By (16),

$$|xuv| + |xyx_1| + |xyx_2| \geq |F| - 3k + m + (s_2 + s_3)/2\,. \tag{17}$$

Each vertex of $M - F$ must be adjacent to at least one of vertices x, y, x_1. However, there exist at most $k - |xyx_1|$ such vertices adjacent to x_1 and $s - s_1$ such vertices adjacent to y but not to x. So, by (17), the total number of vertices adjacent to x is at least

$$[|F| - 3k + m + (s_2 + s_3)/2] + [m - |F| - k + |xyx_1| - s + s_1] \leq k\,.$$

Thus by (15) we get

$$|xyx_1| \leq 5k - 2m + (s_1 + s_2)/2\,.$$

Hence, according to (16), we get

$$|uyu_1| + |vyv_1| + |xyx_1| \leq 8k - 3m\,,$$

thus, by Corollary of Lemma 6, $|F| \leq 11k - 4m$, a contradiction to (9).

Case 4b. Assume that at least two of the numbers i, j, q are greater than 1, say, $i \geq 2$, $j \geq 2$ (in the remaining cases we may proceed similarly). Now we introduce some notation. Let $W_3 = \{x, u, v, y\}$, and let X_1, U_1, and V_1 be the set of all vertices of M adjacent only to x, only to u, and only to v (and to no other vertices of W_3), respectively. Let further XU, XV, and UV be the set of vertices adjacent only to x and u, only to x and v, and only to u and v (and to no other vertices of W_3), respectively. Finally, let XUV be the set of vertices adjacent to x, u, and v but not to y. Then

$$|X_1| + |U_1| + |V_1| + |XU| + |XV| + |UV| + |XUV| \geq m - k\,. \tag{18}$$

The number of vertices adjacent to x is

$$k \geq |XUV| + |XU| + |XV| + |X_1| + |xyx_1| + \cdots + |xyx_i|\,. \tag{19}$$

The number of vertices adjacent to u is

$$k \geq |XUV| + |XU| + |UV| + |U_1| + |uyu_1| + \cdots + |uyu_j|\,. \tag{20}$$

Adding (18), (19) and (20) and using Lemma 6 gives

$$\begin{aligned} |V_1| &\geq m - 3k + |XU| + |F| - (|vyv_1| + \cdots + |vyv_p|) \\ &\geq m - 3k + |F| - (3k - m) = 2m - 6k + |F|\,. \end{aligned}$$

However, all vertices of V_1 are adjacent to vertices v, x_1, x_2, u_1, u_2, thus $e(G) \geq 4m - |F| + (2m - 6k + |F|) = 6m - 6k$ which, by (1), contradicts (3).

We have seen that (3) leads to a contradiction in all cases, and so for any graph $G \in H_2(n, \lfloor pn \rfloor)$ we have

$$e(G) \geq 8n - 11\lfloor pn \rfloor - 8I\sqrt{n}\,. \tag{21}$$

3. Proof of the Theorem

Consider the graph G_1 consisting of

(a) nine vertices $a, b, c, d, e, f, g, h, i$ of high degrees;
(b) the edges $ab, ag, af, bg, bd, ce, cg, ch, df, di, eg, eh, fi, gi, gh$;
(c) triple-sets acd, aef, abg, bcf, bed;
(d) groups $abih$, $dfgh$, $ceig$ of vertices of degree 4 adjacent to vertices involved in these 4-tuples.

In case $3k - n$ is even, the cardinalities of these triple-sets and groups are (in other case proceed similarly):

$$\begin{aligned} |acd| &= |aef| = |bcf| = |bed| = (3k-n)/2, \\ |abg| &= 5k - 2n - 3, \\ |abih| &= 3n - 7k, \\ |ceig| &= |dfgh| = n - 2k - 3\,. \end{aligned}$$

Then the vertices a, b, c, d, e, f are of degree k, g is of degree $k-3$ and deg $h =$ deg $i = 4n - 9k$ which is less than k for $p > 2/5$ and n sufficiently large. It is easy to check that $G_1 \in H_2(n, \lfloor pn \rfloor)$ for such p and n and $e(G_1) = 8n - 11k - 18$. Hence, by (21), if $G \in H_2(n, \lfloor pn \rfloor)$ where p satisfies (1) and n is sufficiently large, we get

$$8n - 11\lfloor pn \rfloor - 8I\sqrt{n} \le F_2(n, \lfloor pn \rfloor) \le 8n - 11\lfloor pn \rfloor - 18\,,$$

and the assertion of Theorem follows.

Remark. The structure of extremal graphs in a similar problem for graphs of diameter 3 was determined in [**3**]. The author hopes to find a characterization of extremal graphs in general for our case in a future paper. The "kernel system" of the extremal graph is a uniquely determined hypergraph in general.

Acknowledgement. This paper was written while the author was visiting the Department of Mathematics and Statistics of McMaster University to whom he would like to express his gratitude for its hospitality.

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