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REFINEMENT OF AN INEQUALITY OF E. LANDAU

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Abstract. We prove: Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial with $n \geq 1$ and $a_0 a_n \neq 0$. If z is a zero of P, then we have for all real numbers $t > 0$:

$$|z| > \frac{|a_0| t}{|a_0| + K_n(t)} \tag{*}$$

with

$$K_n(t) = \frac{1}{1 - \alpha_n(t)^n} \min_{1 \leq m \leq n} \Big[(\alpha_n(t)^m - \alpha_n(t)^n) \max_{m \leq p \leq n} A_p(t) + (1 - \alpha_n(t)^m) \max_{1 \leq p \leq n} A_p(t) \Big],$$

$$\alpha_n(t) = \frac{|a_0|}{|a_0| + \max_{1 \leq p \leq n} A_p(t)},$$

$$A_p(t) = \frac{1}{p} \sum_{k=1}^{p} |a_k| t^k .$$

Inequality (*) sharpens a result of E. Landau.

In this paper we denote by P the polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$, where z and the a_k's are complex numbers. Moreover, we assume $n \geq 1$ and $a_0 a_n \neq 0$. In 1914 E. Landau [**2**] presented the following lower bound for the moduli of the zeros of P:

Theorem A. *If z is a zero of P and t is any positive real number, then*

$$|z| \geq \frac{|a_0| t}{|a_0| + \max_{1 \leq p \leq n} |a_p| t^p} . \tag{1}$$

The same result was also obtained by J. Karamata [**1**] and D. Markovitch [**3**]. In 1967 D. M. Simeunović [**5**] proved an interesting refinement of inequality (1) (see also [**4,** pp. 222–223]):

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Theorem B. *If z is a zero of P and t is any positive real number, then*

$$|z| \geq \frac{|a_0|t}{|a_0| + \max\limits_{1 \leq p \leq n} A_p(t)} \tag{2}$$

with

$$A_p(t) = \frac{1}{p} \sum_{k=1}^{p} |a_k| t^k \,.$$

It is the aim of this note to establish an inequality which sharpens (1) as well as (2).

Theorem. *Let z be a zero of P. Then we have for all real numbers $t > 0$:*

$$|z| > \frac{|a_0|t}{|a_0| + K_n(t)} \tag{3}$$

with

$$K_n(t) = \frac{1}{1 - \alpha_n(t)^n} \min_{1 \leq m \leq n} \Big[(\alpha_n(t)^m - \alpha_n(t)^n) \max_{m \leq p \leq n} A_p(t) + (1 - \alpha_n(t)^m) \max_{1 \leq p \leq n} A_p(t) \Big]$$

and

$$\alpha_n(t) = \frac{|a_0|}{|a_0| + \max\limits_{1 \leq p \leq n} A_p(t)} \,.$$

Remark. Since $\alpha_n(t) < 1$ we have

$$K_n(t) \leq \max_{1 \leq p \leq n} A_p(t) \,. \tag{4}$$

This implies that inequality (3) improves the results of Landau and Simeunović. It is easy to find conditions such that inequality (4) is strict. For example, if $n > 2$ and $\frac{1}{p-1} \sum_{k=2}^{p} |a_k| t^{k-1} < |a_1|$ for all $p \in \{2, \dots, n\}$, then (4) holds with "$<$" instead of "$\leq$".

Proof of the Theorem. Let $|z| = r$, $R = \frac{|a_0|t}{|a_0| + K_n(t)}$ and $D = \{w \in \mathbb{C} \mid |w| \leq R\}$. We assume that z lies on the disk D. Because of

$$|a_0| = \left| \sum_{k=1}^{n} a_k z^k \right| \leq \sum_{k=1}^{n} |a_k| r^k$$

and $0 < \frac{r}{t} \leq \frac{R}{t} < 1$ we obtain

$$\frac{\sum_{k=1}^{n} |a_k| t^k}{\sum_{k=1}^{n} (r/t)^k} \geq \frac{|a_0|}{\sum_{k=1}^{n} (r/t)^k} > \frac{t-r}{r}|a_0| \geq \frac{t-R}{R}|a_0| = K_n(t)\,. \tag{5}$$

Let $m \in \{1, \ldots, n\}$; using Abel's identity we get

$$\begin{aligned}
\sum_{k=1}^{n} |a_k| r^k &= \sum_{k=1}^{n} |a_k| t^k (r/t)^k \\
&= \sum_{k=1}^{m-1} kA_k(t)[(r/t)^k - (r/t)^{k+1}] + \sum_{k=m}^{n-1} kA_k(t)[(r/t)^k - (r/t)^{k+1}] \\
&\quad + nA_n(t)(r/t)^n \\
&\leq \max_{1\leq p\leq n} A_p(t) \sum_{k=1}^{m-1} k[(r/t)^k - (r/t)^{k+1}] \\
&\quad + \max_{m\leq p\leq n} A_p(t) \sum_{k=m}^{n-1} k[(r/t)^k - (r/t)^{k+1}] + \max_{m\leq p\leq n} A_p(t) n (r/t)^n \\
&= \max_{m\leq p\leq n} A_p(t) \sum_{k=1}^{n} (r/t)^k \\
&\quad + \left(\max_{1\leq p\leq n} A_p(t) - \max_{m\leq p\leq n} A_p(t)\right)\left(\sum_{k=1}^{m} (r/t)^k - m(r/t)^m\right) \\
&\leq \max_{m\leq p\leq n} A_p(t) \sum_{k=1}^{n} (r/t)^k + \left(\max_{1\leq p\leq n} A_p(t) - \max_{m\leq p\leq n} A_p(t)\right) \sum_{k=1}^{m} (r/t)^k\,.
\end{aligned}$$

This implies

$$\frac{\sum_{k=1}^{n} |a_k| r^k}{\sum_{k=1}^{n} (r/t)^k} \leq \max_{m\leq p\leq n} A_p(t) + \left(\max_{1\leq p\leq n} A_p(t) - \max_{m\leq p\leq n} A_p(t)\right) \frac{1-(r/t)^m}{1-(r/t)^n}\,. \tag{6}$$

From inequality (2) we obtain $\frac{r}{t} \geq \alpha_n(t)$. Since the function $x \mapsto \frac{1-x^m}{1-x^n}$ is decreasing on $(0,1)$ we conclude from (6):

$$\frac{\sum_{k=1}^{n}|a_k|r^k}{\sum_{k=1}^{n}(r/t)^k} \le \min_{1\le m\le n}\left[\max_{m\le p\le n} A_p(t) + \Big(\max_{1\le p\le n} A_p(t) - \max_{m\le p\le n} A_p(t)\Big)\frac{1-\alpha_n(t)^m)}{1-\alpha_n(t)^n}\right]$$
$$= K_n(t)\,,$$

which contradicts inequality (5). Therefore, z lies outside the disk D. □

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