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THE FRÖLICHER–NIJENHUIS BRACKET IN NON COMMUTATIVE DIFFERENTIAL GEOMETRY

A. CAP, A. KRIEGL, P. W. MICHOR AND J. VANŽURA

INTRODUCTION

There seems to be an emerging theory of non-commutative differential geometry. In the beginning the ideas of non-commutative geometry and of non-commutative topology were intended as tools for attacking problems in topology, in particular the Novikov conjecture and, more generally, the Baum-Connes conjecture. Later on, often motivated by physics, one tended to consider 'non-commutative spaces' as basic structures and to study them in their own right. This is also the point of view we adopt in this paper. We carry over to a quite general non-commutative setting some of the basic tools of differential geometry. From the very beginning we use the setting of convenient vector spaces developed by Frölicher and Kriegl. The reasons for this are the following: If the non-commutative theory should contain some version of differential geometry, a manifold M should be represented by the algebra $C^\infty(M,\mathbb{R})$ of smooth functions on it. The simplest considerations of groups (and quantum groups begin to play an important role now) need products, and $C^\infty(M\times N,\mathbb{R})$ is a certain completion of the algebraic tensor product $C^\infty(M,\mathbb{R})\otimes C^\infty(N,\mathbb{R})$. Now the setting of convenient vector spaces offers in its multilinear version a monoidally closed category, i.e. there is an appropriate tensor product which has all the usual (algebraic) properties with respect to bounded multilinear mappings. So multilinear algebra is carried into this kind of functional analysis without loss. Moreover convenient spaces are the best realm for differentiation which we need in Section 6 to treat a non-commutative version of principal bundles.

We note that all results of this paper also hold in a purely algebraic setting: Just equip each vector space with the finest locally convex topology, then all linear mappings are bounded. They even remain valid if we take a commutative ring of characteristic $\neq 2,3$ instead of the ground field.

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In the first section we give a short description of the setting of convenient spaces elaborating those aspects which we will need later. Then we repeat the usual construction of non-commutative differential forms for convenient algebras in the second section. There we consider triples (A, Ω_*^A, d), where (Ω_*^A, d) is a graded differential algebra with $\Omega_0^A = A$ and $\Omega_n^A = 0$ for negative n. Such a triple is called a **quasi resolution** of A in the book [**Karoubi, 1987**]. See in particular [**Dubois-Violette, 1988**] who studies the action of the Lie algebra of all derivations on Ω_*^A. We will call (Ω_*^A, d) a differential algebra for A. A universal construction of such an algebra Ω_*^A for a commutative algebra A is described in [**Kunz, 1986**], where it is called the algebra of Kähler differentials, since apparently this notion was proposed for the first time by [**Kähler, 1953**]. The first ones to subsume the theory of Kähler differentials over a regular affine variety under standard homological algebra were [**Hochschild, Kostant, Rosenberg, 1962**]. We present below a non-commutative version of the construction of Kunz, since we will need more information. This is the construction of [**Karoubi, 1982, 1983**] which is also used in [**Connes, 1985**]. Connes' contributions started the general interest in non-commutative differential geometry. He described the Chern character in K-homology coming from Fredholm modules and used the unversal differential forms as a tool for describing the cyclic cohomology of an algebra.

Next we show that the bimodule $\Omega_n(A)$ represents the functor of the normalized Hochschild n-cocyles; this is in principle contained in [**Connes, 1985**]. In the third section we introduce the non-commutative version of the Frölicher-Nijenhuis bracket by investigating all bounded graded derivations of the algebra of differential forms. This bracket is then used to formulate the concept of integrability and involutiveness for distributions and to indicate a route towards a theorem of Frobenius (the central result of usual differential geometry, if there is one). This is then used to discuss bundles and connections in the non-commutative setting and to go some steps towards a non-commutative Chern-Weil homomorphism. In the final section we give a brief description of the non-commutative version of the Schouten-Nijenhuis bracket and describe Poisson structures.

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1. Convenient Vector Spaces

1.1. The traditional differential calculus works well for Banach spaces. For more general locally convex spaces a whole flock of different theories were developed, each of them rather complicated and none really convincing. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology. This was the original motivation for the development of a whole new field within general topology, convergence spaces.

Then in 1982, Alfred Frölicher and Andreas Kriegl presented independently the solution to the quest for the right differential calculus in infinite dimensions. They joined forces in the further development of the theory and the (up to now) final outcome is the book [**Frölicher, Kriegl, 1988**].

The appropriate spaces for this differential calculus are the convenient vector spaces mentioned above. In addition to their importance for differential calculus these spaces form a category with very nice properties.

In this section we will sketch the basic definitions and the most important results concerning convenient vector spaces and Frölicher-Kriegl calculus. All locally convex spaces will be assumed to be Hausdorff.

1.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c:\mathbb{R}\to E$ is called **smooth** or C^∞ if all derivatives exist (and are continuous) - this is a concept without problems. Let $C^\infty(\mathbb{R},E)$ be the space of smooth curves. It can be shown that $C^\infty(\mathbb{R},E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).

The final topologies with respect to the following sets of mappings into E coincide:

(i) $C^\infty(\mathbb{R},E)$.
(ii) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s}:t\neq s\}$ is bounded in E).
(iii) $\{E_B\to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x):=\inf\{\lambda>0 : x\in\lambda B\}$.
(iv) Mackey-convergent sequences $x_n\to x$ (there exists a sequence $0<\lambda_n\nearrow\infty$ with $\lambda_n(x_n-x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology; it is not a vector space topology, since addition is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than the c^∞-topology is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E=E$.

1.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a **convenient vector space** if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(i) Any Mackey-Cauchy-sequence (so that (x_n-x_m) is Mackey convergent to 0) converges.
(ii) If B is bounded closed absolutely convex, then E_B is a Banach space.
(iii) Any Lipschitz curve in E is locally Riemann integrable.
(iv) For any $c_1\in C^\infty(\mathbb{R},E)$ there is $c_2\in C^\infty(\mathbb{R},E)$ with $c_1=c_2'$ (existence of antiderivative).

Obviously c^∞-completeness is weaker than sequential completeness so any sequentially complete locally convex vector space is convenient. From 1.2.4 one easily sees that c^∞-closed linear subspaces of convenient vector spaces are again convenient. We always assume that a convenient vector space is equipped with its bornological topology.

1.4. Lemma. *Let E be a locally convex space. Then the following properties are equivalent:*

(i) *E is c^∞-complete.*
(ii) *If $f:\mathbb{R}\to E$ is scalarwise Lip^k, then f is Lip^k, for $k>1$.*
(iii) *If $f:\mathbb{R}\to E$ is scalarwise C^∞ then f is differentiable at 0.*
(iv) *If $f:\mathbb{R}\to E$ is scalarwise C^∞ then f is C^∞.*

Here a mapping $f:\mathbb{R}\to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. f scalarwise C^∞ means that $\lambda\circ f$ is C^∞ for all continuous linear functionals on E.

This lemma says that on a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals.

1.5. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f:E\to F$ is called **smooth** or C^∞, if $f\circ c\in C^\infty(\mathbb{R},F)$ for all $c\in C^\infty(\mathbb{R},E)$; so $f_*:C^\infty(\mathbb{R},E)\to C^\infty(\mathbb{R},F)$ makes sense. Let $C^\infty(E,F)$ denote the space of all smooth mappings from E to F.

For E and F finite dimensional this gives the usual notion of smooth mappings: this has been first proved in [**Boman, 1967**]. Constant mappings are smooth. Multilinear mappings are smooth if and only if they are bounded. Therefore we denote by $L(E,F)$ the space of all bounded linear mappings from E to F.

1.6. Lemma. *For any locally convex space E there is a convenient vector space $\tilde E$ called the completion of E and a bornological embedding $i:E\to\tilde E$, which is characterized by the property that any bounded linear map from E into an arbitrary convenient vector space extends to $\tilde E$.*

1.7. As we will need it later on we describe the completion in a special situation: Let E be a locally convex space with completion $i:E\to\tilde E$, $f:E\to E$ a bounded projection and $\tilde f:\tilde E\to\tilde E$ the prolongation of $i\circ f$. Then $\tilde f$ is also a projection and $\tilde f(\tilde E)=\ker(Id-\tilde f)$ is a c^∞-closed and thus convenient linear subspace of $\tilde E$. Using that $f(E)$ is a direct summand in E one easily shows that $\tilde f(\tilde E)$ is the completion of $f(E)$. This argument applied to $Id-f$ shows that $\ker(\tilde f)$ is the completion of $\ker(f)$.

1.8. Structure on $C^\infty(E,F)$. We equip the space $C^\infty(\mathbb{R},E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E,F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E,F) \to C^\infty(\mathbb{R},F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R},E)$.

1.9. Lemma. *For locally convex spaces E and F we have:*

(i) *If F is convenient, then also $C^\infty(E,F)$ is convenient, for any E. The space $L(E,F)$ is a closed linear subspace of $C^\infty(E,F)$, so it is convenient also.*
(ii) *If E is convenient, then a curve $c : \mathbb{R} \to L(E,F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*

1.10. Theorem. *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F,G)),$$

which is even a diffeomorphism.

Of course this statement is also true for c^∞-open subsets of convenient vector spaces.

1.11. Corollary. *Let all spaces be convenient vector spaces. Then the following canonical mappings are smooth.*

$$\begin{aligned}
&\operatorname{ev} : C^\infty(E,F) \times E \to F, \quad \operatorname{ev}(f,x) = f(x).\\
&\operatorname{ins} : E \to C^\infty(F, E \times F), \quad \operatorname{ins}(x)(y) = (x,y).\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G), \quad \hat f(x,y) = f(x)(y).\\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G)), \quad \check g(x)(y) = g(x,y).\\
&\operatorname{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F') \times C^\infty(E',E) \to C^\infty(C^\infty(E,F), C^\infty(E',F'))\\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

1.12. Theorem. *Let E and F be convenient vector spaces. Then the differential operator*

$$\begin{aligned}
&d : C^\infty(E,F) \to C^\infty(E, L(E,F)),\\
&df(x)v := \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t},
\end{aligned}$$

exists and is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x)v = df(g(x))dg(x)v.$$

1.13. The category of convenient vector spaces and bounded linear maps is complete and cocomplete, so all categorical limits and colimits can be formed. In particular we can form products and direct sums of convenient vector spaces.

For convenient vector spaces $E_1, \dots ,E_n$ and F we can now consider the space of all bounded n-linear maps, $L(E_1, \dots, E_n; F)$, which is a closed linear subspace of $C^\infty(\prod_{i=1}^n E_i, F)$ and thus again convenient. It can be shown that multilinear maps are bounded if and only if they are partially bounded, i.e. bounded in each coordinate and that there is a natural isomorphism (of convenient vector spaces) $L(E_1, \dots, E_n; F) \cong L(E_1, \dots, E_k; L(E_{k+1}, \dots, E_n; F))$

1.14. Theorem. *On the category of convenient vector spaces there is a unique tensor product $\tilde\otimes$ which makes the category symmetric monoidally closed, i.e. there are natural isomorphisms of convenient vector spaces $L(E_1; L(E_2, E_3)) \cong L(E_1 \tilde\otimes E_2, E_3)$, $E_1 \tilde\otimes E_2 \cong E_2 \tilde\otimes E_1$, $E_1 \tilde\otimes (E_2 \tilde\otimes E_3) \cong (E_1 \tilde\otimes E_2) \tilde\otimes E_3$ and $E \tilde\otimes \mathbb{R} \cong E$.*

The tensor product can be constructed as follows: On the algebraic tensor product put the finest locally convex topology such that the canonical bilinear map from the product into the tensor product is bounded and then take the completion of this space.

1.15. Remarks. Note that the conclusion of Theorem 1.10 is the starting point of the classical calculus of variations, where a smooth curve in a space of functions was assumed to be just a smooth function in one variable more.

If one wants Theorem 1.10 to be true and assumes some other obvious properties, then the calculus of smooth functions is already uniquely determined.

There are, however, smooth mappings which are not continuous. This is unavoidable and not so horrible as it might appear at first sight. For example the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous if and only if E is normable, but it is always smooth. Clearly smooth mappings are continuous for the c^∞-topology.

For Fréchet spaces smoothness in the sense described here coincides with the notion C_c^∞ of [**Keller, 1974**]. This is the differential calculus used by [**Michor, 1980**], [**Milnor, 1984**], and [**Pressley, Segal, 1986**].

2. Non-commutative Differential Forms

2.1. Axiomatic setting for the algebra of differential forms. Throughout this section we assume that A is a convenient algebra, i.e. A is a convenient vector space together with a bounded bilinear associative multiplication $A \times A \to A$. Moreover we assume that A has a unit 1. We consider now a graded associative convenient algebra $\Omega_*^A = \bigoplus_{p0} \Omega_p^A$ where $\Omega_0^A = A$ and each Ω_p^A is a convenient vector space, with a bounded bilinear product : $\Omega_p^A \times \Omega_q^A \to \Omega_{p+q}^A$, such that there is a bounded linear mapping $d = d_p : \Omega_p^A \to \Omega_{p+1}^A$ with $d^2 = 0$ and $d(\omega_p\omega_q) = d\omega_p\omega_q + (-1)^p\omega_p d\omega_q$ for all $\omega_p \in \Omega_p^A$ and $\omega_q \in \Omega_q^A$. This mapping is called the

differential of Ω^A_*. Note that we do not assume that the product is graded commutative: $\omega_p\omega_q \neq (-1)^{pq}\omega_q\omega_p$ in general.

Let $\overline{[\Omega^A_*,\Omega^A_*]_r}$ be the locally convex closure of the subspace generated by all graded commutators $[\omega_p,\omega_q] := \omega_p\omega_q - (-1)^{pq}\omega_q\omega_p$ with $p+q=r$. We put $\bar\Omega^A_r := \Omega^A_r/\overline{[\Omega^A_*,\Omega^A_*]_r}$ and we let $T:\Omega^A_r \to \bar\Omega^A_r$ be the projection which will be called the **graded trace** of Ω^A_*.

Since we have $d([\omega_p,\omega_q]) = [d\omega_p,\omega_q] + (-1)^p[\omega_p,d\omega_q]$, the differential passes to $\bar\Omega^A_*$ and still satisfies $d^2=0$. The separated homology of this quotient complex is called the **non-commutative De Rham homology** of Ω^A_* or of A, if Ω^A_* is clear. We denote it by

$$H\bar\Omega^A_p = \bar H^A_p = \ker(d:\bar\Omega^A_p \to \bar\Omega^A_{p+1})/\overline{\operatorname{im}(d:\bar\Omega^A_{p-1}\to\bar\Omega^A_p)}.$$

2.2. Derivations. Let M be a convenient bimodule over the convenient algebra A, i.e. M is a convenient vector space together with two bounded homomorphisms of unital algebras $\lambda: A\to L(M,M)$ and $\rho: A^{op}\to L(M,M)$, where A^{op} denotes the opposite algebra to A, such that for $a,b\in A$ we have $\lambda(a)\circ\rho(b) = \rho(b)\circ\lambda(a)$. We will write am for $\lambda(a)(m)$ and ma for $\rho(a)(m)$. This definition is equivalent to having bounded bilinear maps $\lambda: A\times M\to M$ and $\rho: M\times A\to M$, which satisfy the usual axioms. A (bounded) **derivation** of A in M is a bounded linear mapping $D:A\to M$ such that $D(ab) = D(a)b + aD(b)$ for all a, $b\in A$. We denote by $\operatorname{Der}(A;M)$ the vector space of all derivations of A into M. This is obviously a closed linear subspace of $L(A,M)$ and thus a convenient vector space. If A is commutative, then $\operatorname{Der}(A;M)$ is again an A-module.

The vector space $\operatorname{Der}(A;A)$ is a convenient Lie algebra where the bracket is the commutator. It is an A-module if and only if A is commutative.

2.3. The algebra of dual numbers. of a convenient algebra A with respect to a convenient A-bimodule M is the semidirect product $A \circledS M$, i.e. the convenient vector space $A\times M$ with the bounded bilinear multiplication $(a_1,m_1)(a_2,m_2) := (a_1a_2, a_1m_2 + m_1a_2)$. This is an associative convenient algebra with unit $(1,0)$.

2.4. Lemma. *The bounded derivations from A into the A-bimodule M correspond exactly to the bounded algebra homomorphisms $\varphi: A\to A\circledS M$ satisfying $pr_1\circ\varphi = Id_A$.*

2.5. Universal derivations. A bounded derivation $D: A\to M$ into a bimodule M is called **universal** if the following holds:

> For any bounded derivation $D': A\to N$ into a convenient A-bimodule N there is a unique bounded A-bimodule homomorphism $\Phi: M\to N$ such that $D' = \Phi\circ D$.

Of course for any two universal derivations $D_1: A\to M_1$ and $D_2: A\to M_2$ there is a unique A-bimodule isomorphism $\Phi: M_1\to M_2$ such that $D_2 = \Phi\circ D_1$. So a universal derivation is unique up to canonical isomorphism.

Lemma. *For every convenient algebra A there exists a universal derivation which we denote by $d : A \to \Omega_1(A)$.*

Proof. First we define an A-bimodule structure on $A \tilde{\otimes} A$ as follows: Let $(a, b) \mapsto a \otimes b : A \times A \to A \tilde{\otimes} A$ be the canonical bilinear map. Now consider the map $\bar{\lambda} : A \to L(A \times A, A \tilde{\otimes} A)$ defined by $\bar{\lambda}(a)(b, c) := ab \otimes c$. Obviously the map $\bar{\lambda}$ has values in the space $L(A, A; A \tilde{\otimes} A)$ of bilinear maps and thus we can compose it with the isomorphisms of 1.13 and 1.14 to get $\lambda : A \to L(A \tilde{\otimes} A, A \tilde{\otimes} A)$ which is easily seen to be an algebra homomorphism. Similarly we define $\rho : A \to L(A \tilde{\otimes} A, A \tilde{\otimes} A)$ using $\bar{\rho}(a)(b, c) := b \otimes ca$.

The multiplication on A induces a bounded linear map $\mu : A \tilde{\otimes} A \to A$ which is an A-bimodule homomorphism by associativity. Thus $\Omega_1(A) := \ker(\mu)$ is a convenient A-bimodule.

Next we define $d : A \to \Omega_1(A)$ by $d(a) := 1 \otimes a - a \otimes 1$. Obviously d is a bounded derivation.

To see that this derivation is universal let $D : A \to M$ be a bounded derivation from A into a convenient A-bimodule M. Let $\bar{\Phi} : A \times A \to M$ be the map defined by $\bar{\Phi}(a, b) := aD(b)$. Then $\bar{\Phi}$ is obviously bilinear and bounded and thus it induces a bounded linear map $\Phi : A \tilde{\otimes} A \to M$, whose restriction to $\Omega_1(A)$ we also denote by Φ. As any derivation vanishes on 1 we get:

$$(\Phi \circ d)(a) = \Phi(1 \otimes a - a \otimes 1) = 1D(a) - aD(1) = D(a)$$

So it remains to show that Φ is a bimodule homomorphism. For $a, b, c \in A$ we get: $(\Phi \circ \lambda(a))(b \otimes c) = \Phi(ab \otimes c) = abD(c) = a(\Phi(b \otimes c))$ and thus $\Phi : A \tilde{\otimes} A \to M$ is a homomorphism of left modules.

On the other hand $(\Phi \circ \rho(a))(b \otimes c) = bD(ca) = (bD(c))a + bcD(a)$ and thus we get the identity $(\Phi \circ \rho(a))(x) = (\Phi(x))a + \mu(x)D(a)$ for all $x \in A \tilde{\otimes} A$ and so $\Phi : \Omega_1(A) \to M$ is a homomorphism of right modules, too. □

2.6. Corollary. *For an A-bimodule M the canonical linear mapping*

$$d^* : \operatorname{Hom}_A^A(\Omega_1(A), M) \to \operatorname{Der}(A; M)$$
$$\varphi \mapsto \varphi \circ d$$

is an isomorphism of convenient vector spaces, where $\operatorname{Der}(A; M)$ *carries the structure described in* 2.2*, while the space* $\operatorname{Hom}_A^A(\Omega_1(A), M)$ *of bounded bimodule homomorphisms is considered as a closed linear subspace of* $L(\Omega_1(A), M)$*. In particular we have* $\operatorname{Hom}_A^A(\Omega_1(A), A) \cong \operatorname{Der}(A; A)$*.*

Proof. Since d is bounded and linear so is d^*. In the proof of the lemma above we saw that the inverse to d^* is given by mapping D to the prolongation of $\ell \circ (Id \times D)$, where ℓ denotes the left action of A on M and this map is easily seen to be bounded. □

2.7. Lemma. *Let A be a convenient algebra, M a convenient right A-module and N a convenient left A-module.*

(i) *There is a convenient vector space $M\tilde{\otimes}_A N$ and a bounded bilinear map $b : M \times N \to M\tilde{\otimes}_A N$, $(m,n) \mapsto m \otimes_A n$ such that $b(ma,n) = b(m,an)$ for all $a \in A$, $m \in M$ and $n \in N$ which has the following universal property: If E is a convenient vector space and $f : M \times N \to E$ is a bounded bilinear map such that $f(ma,n) = f(m,an)$ then there is a unique bounded linear map $\tilde{f} : M\tilde{\otimes}_A N \to E$ with $\tilde{f} \circ b = f$.*

(ii) *Let $L^A(M,N;E)$ denote the space of all bilinear bounded maps $f : M \times N \to E$ having the above property, which is a closed linear subspace of $L(M,N;E)$. Then we have an isomorphism of convenient vector spaces $L^A(M,N;E) \cong L(M\tilde{\otimes}_A N, E)$.*

(iii) *If B is another convenient algebra such that N is a convenient right B-module and such that the actions of A and B on N commute, then $M\tilde{\otimes}_A N$ is in a canonical way a convenient right B-module.*

(iv) *If in addition P is a convenient left B-module then there is a natural isomorphism of convenient vector spaces*

$$M\tilde{\otimes}_A(N\tilde{\otimes}_B P) \cong (M\tilde{\otimes}_A N)\tilde{\otimes}_B P$$

Proof. We construct $M\tilde{\otimes}_A N$ as follows: Let $M \otimes N$ be the algebraic tensor product of M and N equipped with the (bornological) topology mentioned in 1.14 and let V be the locally convex closure of the subspace generated by all elements of the form $ma \otimes n - m \otimes an$ and define $M\tilde{\otimes}_A N$ to be the completion of $M \otimes_A N := (M \otimes N)/V$. As $M \otimes N$ has the universal property that bounded bilinear maps from $M \times N$ into arbitrary locally convex spaces induce bounded and hence continuous linear maps on $M \otimes N$, (1) is clear.
(2): By (1) the bounded linear map $b^* : L(M\tilde{\otimes}_A N, E) \to L^A(M,N;E)$ is a bijection. Thus it suffices to show that its inverse is bounded, too. From 1.14 we get a bounded linear map $\varphi : L(M,N;E) \to L(M \otimes N, E)$ which is inverse to the map induced by the canonical bilinear map. Now let $L^{\text{ann } V}(M \otimes N, E)$ be the closed linear subspace of $L(M \otimes N, E)$ consisting of all maps which annihilate V. Restricting φ to $L^A(M,N;E)$ we get a bounded linear map $\varphi : L^A(M,N;E) \to L^{\text{ann } V}(M \otimes N, E)$.

Let $\psi : M \otimes N \to M \otimes_A N \to M\tilde{\otimes}_A N$ be the composition of the canonical projection with the inclusion into the completion. Then ψ induces a well defined linear map $\hat{\psi} : L^{\text{ann } V}(M \otimes N, E) \to L(M\tilde{\otimes}_A N, E)$ and $\hat{\psi} \circ \varphi$ is inverse to b^*. So it suffices to show that $\hat{\psi}$ is bounded.

This is the case if and only if the associated map $L^{\text{ann } V}(M \otimes N, E) \times (M\tilde{\otimes}_A N) \to E$ is bounded. This in turn is equivalent to boundedness of the associated map $M\tilde{\otimes}_A N \to L(L^{\text{ann } V}(M \otimes N, E), E)$. But this is just the prolongation to the

completion of the map $M \otimes_A N \to L(L^{\text{ann } V}(M \otimes N, E), E)$ which sends x to the evaluation at x and this map is clearly bounded.
(3): Let $\rho : B^{op} \to L(N, N)$ be the right action of B on N and let $\Phi : L^A(M \times N, M\tilde{\otimes}_A N) \cong L(M\tilde{\otimes}_A N, M\tilde{\otimes}_A N)$ be the isomorphism constructed in (2). We define the right module structure on $M\tilde{\otimes}_A N$ as:

$$B^{op} \xrightarrow{\rho} L(N,N) \xrightarrow{Id\times .} L(M \times N, M \times N) \xrightarrow{b_*}$$
$$\to L^A(M, N; M\tilde{\otimes}_A N) \xrightarrow{\Phi} L(M\tilde{\otimes}_A N, M\tilde{\otimes}_A N)$$

This map is obviously bounded and easily seen to be an algebra homomorphism.
(4): Straightforward computations show that both spaces have the following universal property: For a convenient vector space E and a trilinear map $f : M \times N \times P \to E$ which satisfies $f(ma, n, p) = f(m, an, p)$ and $f(m, nb, p) = f(m, n, bp)$ there is a unique linear map prolonging f. □

2.8. Homomorphisms of differential algebras. Let $\varphi : A \to B$ be a homomorphism of convenient algebras, let (Ω^A, d^A) be a differential algebra for A in the sense of 2.1, and let (Ω^B, d^B) be one for B.

By a φ-homomorphism $\Phi : \Omega^A \to \Omega^B$ we mean a bounded homomorphism of graded differential algebras such that $\Phi_0 = \varphi : \Omega_0^A = A \to B = \Omega_0^B$.

2.9. Theorem. Existence of the universal graded differential algebra. *For each convenient algebra A there is a convenient graded differential algebra $(\Omega(A), d)$ for A with the following property:*

> *For any bounded homomorphism $\varphi : A \to B$ of convenient algebras and for any convenient graded differential algebra (Ω^B, d^B) for B there exists a unique φ-homomorphism $\Omega(A) \to \Omega^B$.*

Proof. Put $\Omega_0(A) = A$ and $\Omega_k(A) := \Omega_1(A)\tilde{\otimes}_A \dots \tilde{\otimes}_A \Omega_1(A)$ (k factors). Then each $\Omega_k(A)$ is a convenient A-bimodule by 2.7.3, which also defines the multiplication with elements of $\Omega_0(A)$. For $k, \ell > 0$ we define the multiplication as the canonical bilinear map

$$\Omega_k(A) \times \Omega_\ell(A) \to \Omega_k(A)\tilde{\otimes}_A \Omega_\ell(A) \cong \Omega_{k+\ell}(A)$$

Thus $\Omega(A) = \bigoplus_k \Omega_k(A)$ is a convenient graded algebra.

Claim. There is an isomorphism $\Omega_1(A) \cong A\tilde{\otimes}(A/\mathbb{R})$ of convenient vector spaces.
Consider the embedding $i : \mathbb{R} \to A$ and the projection $p : A \to A/\mathbb{R}$, denoted also by $p(a) =: \bar{a}$. We consider the following diagram, where the horizontal and the

vertical sequences are exact:

$$\begin{tikzcd}
 & & 0 & & \\
 & A \arrow[r, equal] & A \arrow[u] & & \\
0 \arrow[r] & A\tilde{\otimes}\mathbb{R} \arrow[u, equal] \arrow[r, "Id\tilde{\otimes}i"] & A\tilde{\otimes}A \arrow[u, "\mu"] \arrow[r, "Id\tilde{\otimes}p"] & A\tilde{\otimes}(A/\mathbb{R}) \arrow[r] & 0 \\
 & & \Omega_1(A) \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

The vertical sequence is splitting: $a \mapsto a \otimes 1$ is a section for μ and the prolongation of $(a, b) \mapsto a\,d(b)$ is a retraction onto $\Omega_1(A)$ which even factors over $Id\tilde{\otimes}p$, since by 1.7 the space $\Omega_1(A)$ is the completion of the kernel of the prolongation of the multiplication map to $A \otimes A$. So we may invert all arrows of the vertical sequence and the two sequences are isomorphic as required.

Claim. There is an isomorphism of convenient vector spaces

$$A\tilde{\otimes}\overbrace{A/\mathbb{R}\tilde{\otimes}\cdots\tilde{\otimes}A/\mathbb{R}}^{k\text{-times}} \to \Omega_k(A)$$

which is induced by the map $(a_0, \bar{a}_1, \dots, \bar{a}_k) \mapsto a_0 da_1 \otimes_A da_2 \otimes_A \cdots \otimes_A da_k$. This is a direct consequence of the last claim and Lemma 2.7.

We now define $d : \Omega_k(A) \to \Omega_{k+1}(A)$ by $d(a) = 1 \otimes a - a \otimes 1$ for $a \in \Omega_0(A) = A$ and for $k > 0$ as the mapping defined on $\Omega_k(A) \cong A\tilde{\otimes}A/\mathbb{R}\tilde{\otimes}\dots\tilde{\otimes}A/\mathbb{R}$ which is associated to:

$$\begin{aligned}
(a_0, \bar{a}_1, \dots, \bar{a}_k) &\mapsto 1 \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_k \\
A \times (A/\mathbb{R})^k &\to A\tilde{\otimes}A/\mathbb{R}\tilde{\otimes}\dots\tilde{\otimes}A/\mathbb{R} \cong \Omega_{k+1}(A)
\end{aligned}$$

Let us show now that d is a graded derivation: We have to show that for $\omega_k \in \Omega_k(A)$ and $\omega_\ell \in \Omega_\ell(A)$ we have $d(\omega_k\omega_\ell) = d(\omega_k)\omega_\ell + (-1)^k\omega_k d(\omega_\ell)$. We proceed by induction on k. By the claim above it suffices to check the identity for elements $A \times (A/R)^i$. For $k = 0$ we have $a(b_0, \bar{b}_1, \dots, \bar{b}_\ell) = (ab_0, \bar{b}_1, \dots, \bar{b}_\ell)$ which is mapped by d to the element $(1, \overline{ab_0}, \bar{b}_1, \dots, \bar{b}_\ell)$ which under the isomorphism with $\Omega_\ell(A)$ goes to $d(ab_0) \otimes_A db_1 \otimes_A \cdots \otimes_A db_\ell$ so the result follows from the derivation property of $d : A \to \Omega_1(A)$.

In the general case we first see that using this derivation property again, the product of $(a_0, \bar{a}_1, \dots, \bar{a}_k)$ and $(b_0, \bar{b}_1, \dots, \bar{b}_\ell)$ in $\Omega_{k+\ell}(A)$ can be written as

$$a_0 da_1 \otimes_A \dots \otimes_A da_{k-1} \otimes_A d(a_k b_0) \otimes_A db_1 \otimes_A \dots \otimes_A db_\ell - \\ - (a_0 da_1 \otimes_A \dots \otimes_A da_{k-1})(a_k db_0 \otimes_A db_1 \otimes_A \dots \otimes_A db_\ell)$$

and from this the result follows easily using the induction hypothesis.

So let us turn to the universal property. Let B be a convenient algebra, (Ω^B, d^B) a convenient differential algebra for B and $\varphi : A \to B$ a bounded homomorphism of algebras. Via φ and the multiplication of Ω^B all spaces Ω_i^B are convenient A-bimodules.

As d^B is a graded derivation the map $d^B \circ \varphi : A \to \Omega_1^B$ is a derivation. Thus by the universal property of $\Omega_1(A)$ we get a unique bounded bimodule homomorphism $\varphi_1 : \Omega_1(A) \to \Omega_1^B$. Thus for $a \in A$ and $\omega \in \Omega_1(A)$ we have $\varphi_1(a\omega) = \varphi(a)\varphi_1(\omega)$ and $\varphi_1(\omega a) = \varphi_1(\omega)\varphi(a)$. Consider the map $f : (\Omega_1(A))^k \to \Omega_k^B$ defined by $f(\omega_1, \omega_2, \dots, \omega_k) := \varphi_1(\omega_1)\varphi_1(\omega_2)\dots\varphi_1(\omega_k)$ which is obviously bounded and k-linear. Moreover as φ_1 is a bimodule homomorphism we get $f(\dots, \omega_i a, \omega_{i+1}, \dots) = f(\dots, \omega_i, a\omega_{i+1}, \dots)$. Thus there is a unique prolongation of f to $\Omega_k(A)$ which we define to be φ_k. From this definition it is obvious that the maps φ_i form a bounded homomorphism of graded algebras.

The composition:

$$A \times A/\mathbb{R} \times \dots \times A/\mathbb{R} \to A \tilde{\otimes} A/\mathbb{R} \dots \tilde{\otimes} A/\mathbb{R} \cong \Omega_k(A) \xrightarrow{\varphi_k} \Omega_k^B$$

is given by

$$(a_0, \bar{a}_1, \dots, \bar{a}_k) \mapsto a_0 da_1 \otimes_A da_2 \otimes_A \dots \otimes_A da_k \mapsto \\ \varphi(a_0)\varphi_1(da_1)\dots\varphi_1(da_k) = \varphi(a_0) d^B(\varphi(a_1))\dots d^B(\varphi(a_k))$$

and this element is mapped by d^B to $d^B(\varphi(a_0)) d^B(\varphi(a_1)) \dots d^B(\varphi(a_k))$. This shows that $\varphi_{k+1} \circ d = d^B \circ \varphi_k$ □

2.10. Corollary. *The construction $A \mapsto \Omega_*(A)$ defines a covariant functor from the category of convenient algebras with unit to the category of convenient graded differential algebras.*

So for a bounded algebra homomorphism $f : A \to B$ we denote by $\Omega_*(f) : \Omega_*(A) \to \Omega_*(B)$ its universal prolongation.

3. Some Related Questions

In the following we treat two questions which arise naturally in the context of Section 2 but which are not relevant for the developments afterwards.

3.1. The kernel of the multiplication $\mu : A\tilde{\otimes}A \to A$ is the very important space $\Omega_1(A)$. What about the analogue with more factors?

Proposition. *Let A be a convenient algebra with unit. Then the kernel of the n-ary multiplication $\mu^n : A^{\tilde{\otimes}n} \to A$ is the subspace*

$$\sum_{i=0}^{n-2} A^{\tilde{\otimes}i}\tilde{\otimes}\Omega_1(A)\tilde{\otimes}A^{\tilde{\otimes}(n-2-i)} \subset A^{\tilde{\otimes}n}.$$

Proof. Note that $\mu^2 = \mu : A\tilde{\otimes}A \to A$. We prove the assertion by induction on n. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A^{\tilde{\otimes}(n-1)}\tilde{\otimes}\Omega_1(A) \arrow[r] \arrow[d] & \Omega_1(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left(\sum_{i=0}^{n-2} A^{\tilde{\otimes}i}\tilde{\otimes}\Omega_1(A)\tilde{\otimes}A^{\tilde{\otimes}(n-2-i)}\right)\tilde{\otimes}A \arrow[r] \arrow[d] & A^{\tilde{\otimes}(n+1)} \arrow[r, "\mu^n\tilde{\otimes}A"] \arrow[d, "A^{\tilde{\otimes}(n-1)}\tilde{\otimes}\mu"'] \arrow[dr, "\mu^{n+1}"] & A\tilde{\otimes}A \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \sum_{i=0}^{n-2} A^{\tilde{\otimes}i}\tilde{\otimes}\Omega_1(A)\tilde{\otimes}A^{\tilde{\otimes}(n-2-i)} \arrow[r] \arrow[d] & A^{\tilde{\otimes}n} \arrow[r, "\mu^n"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The right hand column is the defining sequence for $\Omega_1(A)$ and it is splitting. The middle column being the right hand one tensored with $A^{\tilde{\otimes}(n-1)}$ from the left is then again splitting and thus exact. The bottom row is exact by the induction hypothesis and is also splitting since μ^n admits many obvious sections. The middle row is the bottom one tensored with A from the right and it is again splitting and thus exact. The left hand side vertical arrow is multiplication from the right. The top horizontal arrow is total multiplication onto the left of $\Omega_1(A)$.

Let us now take an element $x \in A^{\tilde{\otimes}(n+1)}$ which is in the kernel of μ^{n+1}. Then a simple diagram chasing shows that x is in the sum of the two subspaces of $A^{\tilde{\otimes}(n+1)}$ which are above and to the left. The converse is trivial, so the result follows. □

3.2. We have seen in 2.6 that $\Omega_1(A)$ is the representing object for the functor $\mathrm{Der}(A, \quad)$ on the category of A-bimodules. Which functor is represented by $\Omega_n(A)$?

Recall that $\Omega_n(A) = \Omega_1(A)\tilde{\otimes}_A \dots \tilde{\otimes}_A \Omega_1(A)$ (n times). We consider the n-linear mapping

$$\begin{aligned} d^n &: A^n \to (A/\mathbb{R})^n \to \Omega_n(A), \\ d^n(a_1, \dots a_n) &:= da_1 \otimes_A \dots \otimes_A da_n. \end{aligned}$$

We view it as a Hochschild cochain which is bounded as a multilinear mapping and normalized, i. e. it factors to $(A/\mathbb{R})^n$. It is well known that the normalized Hochschild complex leads to the usual Hochschild cohomology, see [**Cartan, Eilenberg, 1956, p. 176**].

Lemma. *The mapping d^n is a normalized and bounded Hochschild cocycle with values in the A-bimodule $\Omega_n(A)$.*

Proof. By definition of the right A-module structure on $\Omega_n(A)$ we have

$$\begin{aligned} d^n(a_1, \dots, a_n)a_{n+1} &= (da_1 \otimes_A \dots \otimes_A da_n)a_{n+1} \\ &= da_1 \otimes_A \dots \otimes_A d(a_na_{n+1}) - (da_1 \otimes_A \dots \otimes_A da_{n-1})a_n \otimes_A da_{n+1} \\ &= d^n(a_1, \dots, a_na_{n+1}) - d^n(a_1, \dots, a_{n-1}a_n, a_{n+1}) \\ &\quad + (da_1 \otimes_A \dots \otimes_A da_{n-2})a_{n-1} \otimes_A (da_n \otimes_A da_{n+1}) \\ &= \dots \\ &= \sum_{i=1}^{n} (-1)^{n-i} d^n(a_1, \dots, a_ia_{i+1}, \dots, a_{n+1}) + (-1)^n a_1 d^n(a_2, \dots, a_{n+1}), \end{aligned}$$

and thus as required

$$\begin{aligned} 0 &= a_1 d^n(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i d^n(a_1, \dots, a_ia_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} d^n(a_1, \dots, a_n)a_{n+1} \\ &=: (\delta d^n)(a_1, \dots, a_{n+1}), \end{aligned}$$

where δ denotes the usual Hochschild coboundary operator. □

3.3. Proposition. *Let M be an A-bimodule. Then the mapping*

$$(d^n)^* : \operatorname{Hom}_A^A(\Omega_n(A), M) \to \bar{Z}^n(A, M)$$

is an isomorphism onto the space of all normalized and bounded Hochschild cocycles with values in M.

Proof. Clearly for any bimodule homomorphism $\Phi : \Omega_n(A) \to M$ the n-linear mapping $\Phi \circ d^n : \bar{A}^n \to M$ is a normalized and bounded Hochschild cocycle. Let us assume conversely that $c : A^n \to M$ is a normalized bounded cocycle. In the proof of 2.9 we got a natural isomorphism of convenient vector spaces

$$A\tilde{\otimes}\overbrace{A/\mathbb{R}\tilde{\otimes}\cdots\tilde{\otimes}A/\mathbb{R}}^{k\text{-times}} \to \Omega_k(A)$$

which is given by $a_0\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_k \mapsto a_0da_1\otimes_A da_2\otimes_A\cdots\otimes_A da_k$. Using this we define $\Phi_c : \Omega_n(A) \to M$ by $\Phi_c(a_0da_1\dots da_n) := a_0c(a_1,\dots,a_n)$. Then clearly $\Phi\circ d^n = c$. Obviously Φ_c is a homomorphism of left A-modules and from the definition of the right A-module structure on $\Omega_n(A)$ we see that $\delta c = 0$ translates into Φ_c being a right module homomorphism, by a computation which is completely analogous to the one in the proof of 3.2. Obviously both constructions are bounded. $\square$

3.4. Is it possible to recognize the Hochschild coboundaries in the description $\bar{Z}^n(A,M) \cong \operatorname{Hom}^A_A(\Omega_n(A),M)$?

In order to answer this question we consider the canonical normalized mapping, where $a \mapsto \bar{a}$ is the quotient mapping $A \to A/\mathbb{R}$:

$$\begin{aligned}\varphi : A^{n-1} &\to A\tilde{\otimes}\overbrace{(A/\mathbb{R})\tilde{\otimes}\dots\tilde{\otimes}(A/\mathbb{R})}^{n-1}\tilde{\otimes}A\\ \varphi(a_1,\dots,a_{n-1}) &:= 1\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_{n-1}\otimes 1\end{aligned}$$

Then $\partial\varphi \in \bar{Z}^n(A;A\tilde{\otimes}(A/\mathbb{R})^{\tilde{\otimes}(n-1)}\tilde{\otimes}A)$ is given by

$$\begin{aligned}\partial\varphi(a_1,\dots,a_n) &= a_1\varphi(a_2,\dots,a_n) + \sum_{i=1}^{n-1}(-1)^i\varphi(a_1,\dots,a_ia_{i+1},\dots,a_n)\\ &\quad + (-1)^n\varphi(a_1,\dots,a_{n-1})a_n\\ &= a_1\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_n\otimes 1\\ &\quad + \sum_{i=1}^{n-1}(-1)^i 1\otimes\bar{a}_1\otimes\cdots\otimes\overline{a_ia_{i+1}}\otimes\cdots\otimes\bar{a}_n\otimes 1\\ &\quad + (-1)^n 1\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_{n-1}\otimes a_n.\end{aligned}$$

By Proposition 3.3 there exists a unique bimodule homomorphism $I : \Omega_n(A) \to A\tilde{\otimes}(A/\mathbb{R})^{\tilde{\otimes}(n-1)}\tilde{\otimes}A$ such that $\partial\varphi = I\circ d^n$.

A short computation (again essentially the same as in the proof of Lemma 3.2) shows that this bimodule homomorphism I coincides with the following composition of canonical mappings:

$$\begin{aligned}&\Omega_n(A) = \Omega_1(A)\tilde{\otimes}_A\dots\tilde{\otimes}_A\Omega_1(A) \xrightarrow{i\otimes\cdots\otimes i}\\ &\xrightarrow{i\otimes\cdots\otimes i} (A\tilde{\otimes}A)\tilde{\otimes}_A\dots\tilde{\otimes}_A(A\tilde{\otimes}A) \cong \overbrace{A\tilde{\otimes}\dots\tilde{\otimes}A}^{n+1} \to A\tilde{\otimes}\overbrace{(A/\mathbb{R})\tilde{\otimes}\dots\tilde{\otimes}(A/\mathbb{R})}^{n-1}\tilde{\otimes}A,\end{aligned}$$

where i is the injection $\Omega_1(A) = \ker\mu \to A\tilde{\otimes}A$.

3.5. Proposition. *Let $\Phi : \Omega_n(A) \to M$ be a bimodule homomorphism. Then the corresponding normalized Hochschild cocycle $\Phi \circ d^n$ is a coboundary if and only if Φ factors over I to a bimodule homomorphism $\tilde{\Phi} : A\tilde{\otimes}(A/\mathbb{R})^{\tilde{\otimes}(n-1)}\tilde{\otimes}A \to M$, so that $\Phi = \tilde{\Phi} \circ I$.*

In more details: for any bimodule homomorphism $\Psi : A\tilde{\otimes}(A/\mathbb{R})^{\tilde{\otimes}(n-1)}\tilde{\otimes}A \to M$ we have $\Psi \circ I \circ d^n = \partial\psi$ where the normalized bounded cochain $\psi : A^{n-1} \to M$ is given by

$$\psi(a_1, \dots, a_{n-1}) = \Psi(1 \otimes \bar{a}_1 \otimes \dots \otimes \bar{a}_{n-1} \otimes 1).$$

Proof. Let $\Phi \circ d^n$ be a coboundary. Then there is an $(n-1)$-linear mapping $c : A^{n-1} \to M$ such that $\partial c = \Phi \circ d^n$. This mapping c induces a unique bimodule homomorphism

$$\begin{gathered}\tilde{\Phi} : A\tilde{\otimes}(A/\mathbb{R})^{\tilde{\otimes}(n-1)}\tilde{\otimes}A \to M,\\ \tilde{\Phi}(a_0 \otimes \bar{a}_1, \dots, \bar{a}_n, a_{n+1}) = a_0 \cdot c(a_1, \dots, a_n) \cdot a_{n+1}.\end{gathered}$$

and we have $\tilde{\Phi} \circ I \circ d^n = \tilde{\Phi} \circ \partial\varphi$, and moreover

$$\begin{aligned}(\tilde{\Phi} \circ \partial\varphi)(a_1, \dots, a_n) &= \tilde{\Phi}(a_1 \otimes \bar{a}_2 \otimes \dots \otimes \bar{a}_n \otimes 1)\\ &\quad + \sum_{i=1}^{n-1} (-1)^i \tilde{\Phi}(1 \otimes \bar{a}_1 \otimes \dots \otimes \overline{a_i a_{i+1}} \otimes \dots \otimes \bar{a}_n \otimes 1)\\ &\quad + (-1)^n \tilde{\Phi}(1 \otimes \bar{a}_1 \otimes \dots \otimes \bar{a}_{n-1} \otimes a_n)\\ &= \partial c(a_1, \dots, a_n).\end{aligned}$$

So we get $\Phi \circ d^n = \partial c = \tilde{\Phi} \circ I \circ d^n$ and the result follows from 3.3.

The second assertion of the proposition follows also from the last computation.□

3.6. Corollary. *For a convenient algebra A and a convenient bimodule M over A we have*

$$H^n(A, M) \cong \frac{\operatorname{Hom}_A^A(\Omega_n(A), M)}{I^*(\operatorname{Hom}_A^A(A\tilde{\otimes}\bar{A}^{\tilde{\otimes}(n-1)}\tilde{\otimes}A, M))}.$$

4. The Calculus of Frölicher and Nijenhuis

4.1. In this section let A be a convenient algebra with unit and let $\Omega(A) = \Omega_*(A)$ be the universal graded differential algebra for A. The space $\operatorname{Der}_k \Omega(A)$ consists of all bounded **(graded) derivations** of degree k, i.e. all bounded linear mappings $D : \Omega(A) \to \Omega(A)$ with $D(\Omega_\ell(A)) \subset \Omega_{k+\ell}(A)$ and $D(\varphi\psi) = D(\varphi)\psi + (-1)^{k\ell}\varphi D(\psi)$ for $\varphi \in \Omega_\ell(A)$. Obviously $\operatorname{Der}_k \Omega(A)$ is a closed linear subspace of $L(\Omega(A), \Omega(A))$ and thus a convenient vector space.

Lemma. *The space* $\operatorname{Der}\Omega(A) = \bigoplus_k \operatorname{Der}_k \Omega(A)$ *is a convenient graded Lie algebra with the graded commutator* $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ *as bracket. This means that the bracket is graded anticommutative,* $[D_1, D_2] = -(-1)^{k_1 k_2}[D_2, D_1]$, *and satisfies the graded Jacobi identity*

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2}[D_2, [D_1, D_3]]$$

(*so that* $ad(D_1) = [D_1, \quad]$ *is itself a derivation*).

Proof. Plug in the definition of the graded commutator and compute. The boundedness of the bracket follows from 1.11. □

4.2. Fields. Recall from 2.6 that $d^* : \operatorname{Hom}_A^A(\Omega_1(A), A) \to \operatorname{Der}(A; A)$ is an isomorphism, which we will also denote by $\mathcal{L}$. We denote the space $\operatorname{Hom}_A^A(\Omega_1(A), A)$ by $\mathfrak{X}(A)$ and call it the space of **fields** for the algebra A. Then $\mathcal{L} : \mathfrak{X}(A) \to \operatorname{Der}(A; A)$ is an isomorphism of convenient vector spaces. The space of derivations $\operatorname{Der}(A; A)$ is a convenient Lie algebra with the commutator $[\quad,\quad]$ as bracket, and so we have an induced Lie bracket on $\mathfrak{X}(A) = \operatorname{Hom}_A^A(\Omega_1(A), A)$ which is given by $\mathcal{L}([X, Y])a = [\mathcal{L}_X, \mathcal{L}_Y]a = \mathcal{L}_X\mathcal{L}_Y a - \mathcal{L}_Y\mathcal{L}_X a$. It will be referred to as the Lie bracket of fields.

4.3. Lemma. *Each field* $X \in \mathfrak{X}(A) = \operatorname{Hom}_A^A(\Omega_1(A), A)$ *is by definition a bounded* A*-bimodule homomorphism* $\Omega_1(A) \to A$. *It prolongs uniquely to a graded derivation* $j(X) = j_X : \Omega(A) \to \Omega(A)$ *of degree* -1 *by*

$$\begin{aligned}
j_X(a) &= 0 \quad \text{for } a \in A = \Omega_0(A),\\
j_X(\omega) &= X(\omega) \quad \text{for } \omega \in \Omega_1(A)\\
j_X(\omega_1 \otimes_A \cdots \otimes_A \omega_k) &= \\
= \sum_{i=1}^{k-1} &(-1)^{i-1}\omega_1 \otimes_A \cdots \otimes_A \omega_{i-1} \otimes_A X(\omega_i)\omega_{i+1} \otimes_A \cdots \otimes_A \omega_k\\
&+ (-1)^{k-1}\omega_1 \otimes_A \cdots \otimes_A \omega_{k-1} X(\omega_k)
\end{aligned}$$

for $\omega_i \in \Omega_1(A)$. *The derivation* j_X *is called the* **contraction operator** *of the field* X.

Proof. This is an easy computation □

With some abuse of notation we write also $\omega(X) = X(\omega) = j_X(\omega)$ for $\omega \in \Omega_1(A)$ and $X \in \mathfrak{X}(A) = \operatorname{Hom}_A^A(\Omega_1(A), A)$.

4.4. A derivation $D \in \operatorname{Der}_k \Omega(A)$ is called **algebraic** if $D \mid \Omega_0(A) = 0$. Then $D(a\omega) = aD(\omega)$ and $D(\omega a) = D(\omega)a$ for $a \in A$, so D restricts to a bounded bimodule homomorphism, an element of $\operatorname{Hom}_A^A(\Omega_l(A), \Omega_{l+k}(A))$. Since we have $\Omega_l(A) = \Omega_1(A)\tilde{\otimes}_A \ldots \tilde{\otimes}_A \Omega_1(A)$ and since for a product of one forms we have

$D(\omega_1 \otimes_A \cdots \otimes_A \omega_l) = \sum_{i=1}^l (-1)^{ik} \omega_1 \otimes_A \cdots \otimes_A D(\omega_i) \otimes_A \cdots \otimes_A \omega_l$, the derivation D is uniquely determined by its restriction

$$K := D|\Omega_1(A) \in \operatorname{Hom}_A^A(\Omega_1(A), \Omega_{k+1}(A));$$

we write $D = j(K) = j_K$ to express this dependence. Note the defining equation $j_K(\omega) = K(\omega)$ for $\omega \in \Omega_1(A)$. Since it will be very important in the sequel we will use the notation

$$\begin{aligned}\Omega_k^1 = \Omega_k^1(A) : &= \operatorname{Hom}_A^A(\Omega_1(A), \Omega_k(A))\\ \Omega_*^1 = \Omega_*^1(A) &= \bigoplus_{k=0}^{\infty} \Omega_k^1(A).\end{aligned}$$

Elements of the space Ω_k^1 will be called **field valued k-forms**, those of Ω_*^1 will be called just **field valued forms**.

4.5. In 4.3 we have already met some algebraic graded derivations: for a field $X \in \mathfrak{X}(A)$ the derivation j_X is of degree -1. The basic derivation d is of degree 1. Note also that $\mathcal{L}_X := d\,j_X + j_X\,d$ translates to $\mathcal{L}_X = [j_X, d]$ and that this extends $\mathcal{L}_X$ from a derivation A to a derivation of degree 0 of $\Omega_*(A)$.

4.6 Theorem. (1) *For $K \in \Omega_{k+1}^1(A)$ and $\omega_i \in \Omega_1(A)$ the formula*

$$j_K(\omega_0 \otimes_A \cdots \otimes_A \omega_\ell) = \sum_{i=0}^{\ell} (-1)^{ik} \omega_0 \otimes_A \cdots \otimes_A K(\omega_i) \otimes_A \cdots \otimes_A \omega_k$$

defines an algebraic graded derivation $i_K \in \operatorname{Der}_k \Omega(A)$ and any algebraic derivation is of this form.

(2) *The map*

$$j : \Omega_{k+1}^1 = \operatorname{Hom}_A^A(\Omega_1(A), \Omega_{k+1}(A)) \to \operatorname{Der}_k^{alg} \Omega(A)$$

where $\operatorname{Der}_k^{alg} \Omega(A)$ denotes the closed linear subspace of $\operatorname{Der}_k \Omega(A)$ consisting of all algebraic derivations is an isomorphism of convenient vector spaces.

(3) *By $j([K, L]^\Delta) := [j_K, j_L]$ we get a bracket $[\quad,\quad]^\Delta$ on the space Ω_{*-1}^1 which defines a convenient graded Lie algebra structure with the grading as indicated, and for $K \in \Omega_{k+1}^1$, and $L \in \Omega_{\ell+1}^1$ we have*

$$[K, L]^\Delta = j_K \circ L - (-1)^{k\ell} j_L \circ K.$$

$[\quad,\quad]^\Delta$ is called the **algebraic bracket** or also the **abstract De Wilde, Lecomte bracket** see [**DeWilde, Lecomte, 1988**].

Proof. The first assertion is clear from the definition.

Clearly the map $D \mapsto D|\Omega_1(A)$ is bounded. To show that j is bounded recall that $\operatorname{Der}_d \Omega(A)$ is a closed subspace of $L(\Omega(A), \Omega(A)) \cong \prod_k L(\Omega_k(A), \Omega(A))$. By 2.7.2 it suffices to show that j is bounded as a map to $L^A(\Omega_1(A), \dots, \Omega_1(A); \Omega(A))$ and by the linear uniform boundedness principle 1.9.2 it is enough to show that for all $\omega_i \in \Omega_1(A)$ the map $K \mapsto j_K(\omega_1 \otimes_A \cdots \otimes_A \omega_k)$ is bounded. But this is clear by (1).

For the third assertion it suffices to evaluate $[j_K, j_L]$ at some $\omega \in \Omega_1(A)$. □

4.7. The exterior derivative d is an element of $\operatorname{Der}_1 \Omega(A)$. In view of the formula $\mathcal{L}_X = [j_X, d] = j_X\, d + d\, j_X$ for fields X, we define for $K \in \Omega^1_k$ the **Lie derivation** $\mathcal{L}_K = \mathcal{L}(K) \in \operatorname{Der}_k \Omega(A)$ by $\mathcal{L}_K := [j_K, d]$.

Then the mapping $\mathcal{L} : \Omega^1_* \to \operatorname{Der} \Omega(A)$ is obviously bounded and it is injective by the universal property of $\Omega_1(A)$, since $\mathcal{L}_K a = j_K da = K(da)$ for $a \in A$.

Theorem. *For any graded derivation $D \in \operatorname{Der}_k \Omega(A)$ there are unique homomorphisms $K \in \Omega^1_k$ and $L \in \Omega^1_{k+1}$ such that*

$$D = \mathcal{L}_K + j_L.$$

We have $L = 0$ if and only if $[D, d] = 0$. D is algebraic if and only if $K = 0$.

Proof. $D|A : a \mapsto Da$ is a derivation $A \to \Omega_d(A)$, so by 2.5 it is of the form $D|A = K \circ d$ for a unique $K \in \Omega^1_k$.

The defining equation for K is $Da = j_K da = \mathcal{L}_K a$ for $a \in A$. Thus $D - \mathcal{L}_K$ is an algebraic derivation, so $D - \mathcal{L}_K = j_L$ by 4.4 for unique $L \in \Omega^1_{k+1}$.

Since we have $[d, d] = 2d^2 = 0$, by the graded Jacobi identity we obtain $0 = [j_K, [d, d]] = [[j_K, d], d] + (-1)^{k-1}[d, [j_K, d]] = 2[\mathcal{L}_K, d]$. The mapping $L \mapsto [j_L, d] = \mathcal{L}_L$ is injective, so the last assertion follows. □

4.8. The Frölicher-Nijenhuis bracket. Note that $j(Id_{\Omega_1(A)})\omega = k\omega$ for $\omega \in \Omega_k(A)$. Therefore we have $\mathcal{L}(Id_{\Omega_1(A)})\omega = j(Id_{\Omega_1(A)})d\omega - d\,j(Id_{\Omega_1(A)})\omega = (k+1)d\omega - kd\omega = d\omega$. Thus $\mathcal{L}(Id_{\Omega_1(A)}) = d$.

4.9. Let $K \in \Omega^1_k$ and $L \in \Omega^1_\ell$. Then obviously $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L])$$

for a uniquely defined $[K, L] \in \Omega^1_{k+\ell}$. This vector valued form $[K, L]$ is called the **abstract Frölicher-Nijenhuis bracket** of K and L.

Theorem. *The space $\Omega^1_* = \bigoplus_k \Omega^1_k$ with its usual grading and the Frölicher-Nijenhuis bracket is a convenient graded Lie algebra. $Id_{\Omega_1(A)} \in \Omega^1_1$ is in the center, i.e. $[K, Id_{\Omega_1(A)}] = 0$ for all K.*

$\mathcal{L} : (\Omega^1_*, [\quad,\quad]) \to \operatorname{Der} \Omega(A)$ is a bounded injective homomorphism of graded Lie algebras. For fields in $\operatorname{Hom}^A_A(\Omega_1(A), A)$, i.e. bounded derivations of A, the Frölicher-Nijenhuis bracket coincides with the bracket defined in 4.2.

Proof. Boundedness of the bracket follows from the fact that the map $\mathcal{L}_K \mapsto K$ is bounded as it is just the composition of the restriction to A with the bounded inverse to d^* constructed in 2.6.

For $X, Y \in \operatorname{Hom}_A^A(\Omega_1(A), A)$ we have $j([X,Y])da = \mathcal{L}([X,Y])a = [\mathcal{L}_X, \mathcal{L}_Y]a$. The rest is clear. □

4.10. Lemma. *For homomorphisms $K \in \Omega^1_k$ and $L \in \Omega^1_{\ell+1}$ we have*

$$[\mathcal{L}_K, j_L] = j([K,L]) - (-1)^{k\ell}\mathcal{L}(j_L \circ K), \text{ or}$$
$$[j_L, \mathcal{L}_K] = \mathcal{L}(j_L \circ K) - (-1)^k j([L,K]).$$

Proof. For $a \in A$ we have $[j_L, \mathcal{L}_K]a = j_L j_K\, da - 0 = j_L(K(da)) = (j_L \circ K)(da) = \mathcal{L}(j_L \circ K)a$. So $[j_L, \mathcal{L}_K] - \mathcal{L}(j_L \circ K)$ is an algebraic derivation.

$$\begin{aligned}[][[j_L, \mathcal{L}_K], d] &= [j_L, [\mathcal{L}_K, d]] - (-1)^{k\ell}[\mathcal{L}_K, [j_L, d]] = \\ &= 0 - (-1)^{k\ell}\mathcal{L}([K,L]) = (-1)^k [j([L,K]), d]).\end{aligned}$$

Since $[\quad, d]$ kills the '$\mathcal{L}$'s' and is injective on the 'j's', the algebraic part of $[j_L, \mathcal{L}_K]$ is $(-1)^k j([L,K])$. □

4.11. Theorem. *For homomorphisms $K_i \in \Omega^1_{k_i}$ and $L_i \in \Omega^1_{k_i+1}$ we have*

$$\begin{aligned}(1)\qquad [\mathcal{L}_{K_1} + j_{L_1}, \mathcal{L}_{K_2} + j_{L_2}] &= \\ = \mathcal{L}&\left([K_1, K_2] + j_{L_1} \circ K_2 - (-1)^{k_1 k_2} j_{L_2} \circ K_1\right) \\ + i&\left([L_1, L_2]^\Delta + [K_1, L_2] - (-1)^{k_1 k_2}[K_2, L_1]\right).\end{aligned}$$

Each summand of this formula looks like a semidirect product of graded Lie algebras, but the mappings

$$\begin{aligned} j &: \Omega^1_{*-1} \to \operatorname{End}_{\mathbb{K}}(\Omega^1_*, [\quad,\quad]) \\ \operatorname{ad} &: \Omega^1_* \to \operatorname{End}_{\mathbb{K}}(\Omega^1_{*-1}, [\quad,\quad]^\Delta), \quad \operatorname{ad}_K L = [K, L],\end{aligned}$$

do not take values in the subspaces of graded derivations. We have instead for homomorphisms $K \in \Omega^1_k$ and $L \in \Omega^1_{\ell+1}$ the following relations:

$$\begin{aligned}(2)\qquad j_L \circ [K_1, K_2] &= [j_L \circ K_1, K_2] + (-1)^{k_1 \ell}[K_1, j_L \circ K_2] \\ &- \left((-1)^{k_1\ell} j(\operatorname{ad}_{K_1} L) \circ K_2 - (-1)^{(k_1+\ell)k_2} j(\operatorname{ad}_{K_2} L) \circ K_1\right)\end{aligned}$$

$$\begin{aligned}(3)\qquad \operatorname{ad}_K [L_1, L_2]^\Delta &= [\operatorname{ad}_K L_1, L_2]^\Delta + (-1)^{kk_1}[L_1, \operatorname{ad}_K L_2]^\Delta - \\ &- \left((-1)^{kk_1} \operatorname{ad}(j(L_1) \circ K)L_2 - (-1)^{(k+k_1)k_2} \operatorname{ad}(j(L_2) \circ K)L_1\right)\end{aligned}$$

The algebraic meaning of the relations of this theorem and its consequences in group theory have been investigated in [**Michor, 1990**]. The corresponding product of groups is well known to algebraists under the name 'Zappa-Szep'-product.

Proof. Equation (1) is an immediate consequence of 4.10. Equations (2) and (3) follow from (1) by writing out the graded Jacobi identity, or as follows: Consider $\mathcal{L}(j_L \circ [K_1, K_2])$ and use 4.10 repeatedly to obtain $\mathcal{L}$ of the right hand side of (2). Then consider $j([K, [L_1, L_2]^\Delta])$ and use again 4.10 several times to obtain i of the right hand side of (3). □

4.12. Naturality of the Frölicher-Nijenhuis bracket. Let $f : A \to B$ be a bounded algebra homomorphism. Two forms $K \in \Omega^1_k(A) = \mathrm{Hom}^A_A(\Omega_1(A), \Omega_k(A))$ and $K' \in \Omega^1_k(B) = \mathrm{Hom}^B_B(\Omega_1(B), \Omega_k(B))$ are called **f-related** or **f-dependent**, if we have

$$K' \circ \Omega_1(f) = \Omega_k(f) \circ K : \Omega_1(A) \to \Omega_k(B), \tag{1}$$

where $\Omega_*(f)$ is described in 2.10.

Theorem.

(ii) *If K and K' as above are f-related then $j_{K'} \circ \Omega(f) = \Omega(f) \circ j_K : \Omega(A) \to \Omega(B)$.*

(iii) *If $j_{K'} \circ \Omega(f)|d(A) = \Omega(f) \circ j_K|d(A)$, then K and K' are f-related, where $d(A) \subset \Omega_1(A)$ denotes the space of exact 1-forms.*

(iv) *If K_j and K'_j are f-related for $j = 1, 2$, then $j_{K_1} \circ K_2$ and $j_{K'_1} \circ K'_2$ are f-related, and also $[K_1, K_2]^\Delta$ and $[K'_1, K'_2]^\Delta$ are f-related.*

(v) *If K and K' are f-related then $\mathcal{L}_{K'} \circ \Omega(f) = \Omega(f) \circ \mathcal{L}_K : \Omega(A) \to \Omega(B)$.*

(vi) *If $\mathcal{L}_{K'} \circ \Omega(f) \mid \Omega_0(A) = \Omega(f) \circ \mathcal{L}_K \mid \Omega_0(A)$, then K and K' are f-related.*

(vii) *If K_j and K'_j are f-related for $j = 1, 2$, then their Frölicher-Nijenhuis brackets $[K_1, K_2]$ and $[K'_1, K'_2]$ are also f-related.*

Proof. (2). Since both sides are graded derivations over $\Omega(f)$ it suffices to check this for a 1-form $\omega \in \Omega_1(A)$. By 4.6 and 2.10 we have $\Omega_k(f) j_K(\omega) = \Omega_k(f) K(\omega) = K'(\Omega_1(f)\omega) = j_{K'} \Omega_1(f)(\omega)$.

(3) follows from the universal property of $\Omega_1(A)$ because $K' \circ \Omega_1(f) \circ d$ and $\Omega_k(f) \circ K \circ d$ are both derivations from A into $\Omega_k(B)$ which is an A-bimodule via f and the multiplication in $\Omega(B)$.

(4) is obvious; the result for the bracket then follows from 4.6.3.

(5) The algebra homomorphism $\Omega(f)$ intertwines the operators j_K and $j_{K'}$ by (2), and $\Omega(f)$ commutes with the exterior derivative d. Thus $\Omega(f)$ intertwines the commutators $[j_K, d] = \mathcal{L}_K$ and $[j_{K'}, d] = \mathcal{L}_{K'}$.

(6) For an element $g \in \Omega_0(A)$ we have $\mathcal{L}_K\, \Omega(f)\, g = j_K\, d\, \Omega(f)\, g = j_K\, \Omega(f)\, dg$ and $\Omega(f)\, \mathcal{L}_{K'}\, g = \Omega(f)\, j_{K'}\, dg$. By (3) the result follows.

(7) The algebra homomorphism $\Omega(f)$ intertwines $\mathcal{L}_{K_j}$ and $\mathcal{L}_{K'_j}$, so also their graded commutators which equal $\mathcal{L}([K_1, K_2])$ and $\mathcal{L}([K'_1, K'_2])$, respectively. Now use (6) . □

5. Distributions and Integrability

5.1. Distributions. By a **distribution** in a convenient algebra A we mean a c^∞-closed sub-A-bimodule $\mathcal{D}$ of $\Omega_1(A)$.

The distribution $\mathcal{D}$ is called **globally integrable** if there exists a c^∞-closed subalgebra B of A such that $\mathcal{D}$ is the c^∞-closure in $\Omega_1(A)$ of the subspace generated by $A(d(B))$ and $d(B)A$.

The distribution $\mathcal{D}$ is called **splitting** if there exists a bounded projection $P \in \Omega_1^1(A) = \operatorname{Hom}_A^A(\Omega_1(A), \Omega_1(A))$ onto $\mathcal{D}$, i.e. $P \circ P = P$ and $\mathcal{D} = P(\Omega_1(A))$. Then there is a complementary submodule $\ker P \subset \Omega_1(A)$.

The distribution $\mathcal{D}$ is called **involutive** if the c^∞-closed ideal $(\mathcal{D})_{\Omega_*(A)}$ generated by $\mathcal{D}$ in the graded algebra $\Omega_*(A)$ is stable under d, i.e. if $d(\mathcal{D}) \subset (\mathcal{D})_{\Omega_*(A)}$.

5.2. Comments. One should think of this as follows: In differential geometry, where we have $A = C^\infty(M, \mathbb{R})$ for a manifold M, a distribution is usually given as a sub vector bundle E of the tangent bundle TM. Then $\mathcal{D}$ is the A-bimodule of those 1-forms which annihilate the subbundle E of TM. Global integrability then means that it is integrable and that the space of functions which are constant along the leaves of the foliation generates those forms. This is a strong condition: There are foliations where this space of functions consists only of the constants, and this can be embedded into any manifold. So in $C^\infty(M, \mathbb{R})$ there are always involutive distributions which are not globally integrable. To prove some Frobenius theorem a notion of local integrability would be necessary.

5.3 Curvature and cocurvature. Let $P \in \Omega_1^1(A) = \operatorname{Hom}_A^A(\Omega_1(A), \Omega_1(A))$ be a projection, then the image $P(\Omega_1(A))$ is a splitting distribution, called the **vertical distribution** of P and the complement $\ker P$ is also a splitting distribution, called the **horizontal** one. $\bar{P} := Id_{\Omega_1(A)} - P$ is a projection onto the horizontal distribution.

We consider now the Frölicher-Nijenhuis bracket $[P, P]$ of P and define

$$\begin{aligned} R = R_P &= [P, P] \circ P && \text{the } \textbf{curvature,} \\ \bar{R} = \bar{R}_P &= [P, P] \circ \bar{P} && \text{the } \textbf{cocurvature.} \end{aligned}$$

The curvature and the cocurvature are elements of $\Omega_2^1(A) = \operatorname{Hom}_A^A(\Omega_1(A), \Omega_2(A))$. The curvature kills elements of the horizontal distribution, so it is **vertical**. The cocurvature kills elements of the vertical distribution.

Since the identity $Id \in \Omega_1^1(A)$ lies in the center of the Frölicher-Nijenhuis algebra we get $[\bar{P}, \bar{P}] = [Id - P, Id - P] = [P, P]$ and hence $\bar{R}_P = R_{\bar{P}}$. We shall also need the homomorphisms of graded algebras $\Omega(P), \Omega(\bar{P}) : \Omega(A) \to \Omega(A)$ with $\Omega_0(P) = \Omega_0(\bar{P}) = Id_A$ which are induced by the bimodule homomorphisms $P, \bar{P} : \Omega_1(A) \to \Omega_1(A)$.

5.4. Lemma. *In the setting of* 5.3 *the following assertions hold:*
1. *For* $\omega \in \Omega_1(A)$ *we have*

$$R_P(\omega) = [P,P](P(\omega)) = -2(\Omega(\bar{P}) \circ d \circ P)(\omega)$$
$$\bar{R}_P(\omega) = [P,P](\bar{P}(\omega)) = -2(\Omega(P) \circ d \circ \bar{P})(\omega).$$

2. *For the* c^∞*-closed ideals generated by the distributions* $\ker P$ *and* $P(\Omega_1(A))$ *we have* $(\ker P)_{\Omega_*(A)} = \ker \Omega(P)$ *and* $(P(\Omega_1(A)))_{\Omega_*(A)} = \ker \Omega(\bar{P})$.

3. *The curvature* $R = [P,P] \circ P$ *is zero if and only if the horizontal distribution is involutive. The cocurvature* $\bar{R} = [P,P] \circ (Id - P)$ *is zero if and only if the vertical distribution* $P(\Omega_1(A))$ *is involutive.*

Proof. (1) It suffices to show the first equation. For $\omega \in \Omega_1(A)$ we have:

$$\begin{aligned}
[P,P](\omega) &= [\bar{P},\bar{P}](\omega) = j([\bar{P},\bar{P}])(\omega) \\
&= [\mathcal{L}_{\bar{P}}, j_{\bar{P}}](\omega) + \mathcal{L}(j_{\bar{P}}\bar{P})(\omega) \quad \text{ by 4.10} \\
&= \mathcal{L}_{\bar{P}} j_{\bar{P}}(\omega) - j_{\bar{P}}\mathcal{L}_{\bar{P}}(\omega) + \mathcal{L}_{\bar{P}}(\omega) \quad \text{ since } j_{\bar{P}}\bar{P} = \bar{P}^2 = \bar{P} \\
&= 2(j_{\bar{P}} d\bar{P}(\omega) - d\bar{P}(\omega)) - j_{\bar{P}} j_{\bar{P}} d(\omega) + j_{\bar{P}} d(\omega).
\end{aligned}$$

For ω, $\varphi \in \Omega_1(A)$ we have

$$\begin{aligned}
j_{\bar{P}} j_{\bar{P}}(\omega \otimes_A \varphi) &= j_{\bar{P}}(\bar{P}(\omega) \otimes_A \varphi + \omega \otimes_A \bar{P}(\varphi)) \\
&= \bar{P}(\omega) \otimes_A \varphi + 2\bar{P}(\omega) \otimes_A \bar{P}(\varphi) + \omega \otimes_A \bar{P}(\varphi) \\
&= (2\Omega(\bar{P}) + j_{\bar{P}})(\omega \otimes_A \varphi), \quad \text{ thus} \\
j_{\bar{P}} j_{\bar{P}} | \Omega_2(A) &= (2\Omega(\bar{P}) + j_{\bar{P}}) | \Omega_2(A).
\end{aligned}$$

So we have

$$\begin{aligned}
[P,P](\omega) &= 2(j_{\bar{P}} d\bar{P}(\omega) - d\bar{P}(\omega) - \Omega(\bar{P})(d(\omega))) \\
R_P(\omega) &= [P,P](P(\omega)) = -2\Omega(\bar{P}) dP(\omega)
\end{aligned}$$

as required.

(2) The kernel of the bounded algebra homomorphism $\Omega(P)$ is a c^∞-closed ideal and contains $\ker P$. On the other hand any $\omega \in \Omega_1(A) \otimes_A \cdots \otimes_A \Omega_1(A) \cap \ker \Omega(P)$ (non-completed tensor product) may be written as a finite sum $\omega = \sum_i \omega_{1,i} \otimes_A \cdots \otimes_A \omega_{k,i}$ with the property that $\sum_i P(\omega_{1,i}) \otimes_A \cdots \otimes_A P(\omega_{k,i}) = 0$. Since $P + \bar{P} = Id_{\Omega_1(A)}$ we have $\omega_{j,i} = P(\omega_{j,i}) + \bar{P}(\omega_{j,i})$ for all j and i. Thus each summand of ω splits into a sum of products of $P(\omega_{j,i})$ and $\bar{P}(\omega_{j,i})$ and the sum of those products containing only $P(\omega_{j,i})$ vanishes. So at least one $\bar{P}(\omega_{j,i})$ appears in each summand and the whole sum is in the ideal generated by $\ker \Omega_1(P) = \bar{P}(\Omega_1(A))$.

By 1.7 $\Omega_k(A) \cap \ker(\Omega(P))$ is the completion of $\Omega_1(A) \otimes_A \cdots \otimes_A \Omega_1(A) \cap \ker \Omega(P)$ so it must be the c^∞-closure in $\Omega_k(A)$ of this space and hence must also be contained in the c^∞-closed ideal.

The second assertion follows by symmetry.

(3) We have to prove only the first assertion. The distribution $\ker \bar{P}$ is involutive if and only if for all $\omega \in \Omega_1(A)$ we have $dP\omega \in (\ker P)_{\Omega_*(A)} = \ker \Omega(P)$. By (2) this is equivalent to $R(\omega) = -2\Omega(\bar{P})(dP(\omega)) = 0$ for all $\omega \in \Omega_1(A)$. □

5.5. Lemma (Bianchi identity). *If $P \in \Omega_1^1(A)$ is a projection with curvature R and cocurvature $\bar{R}$, then we have*

$$[P, R + \bar{R}] = 0$$
$$2[R, P] = j_R \bar{R} + j_{\bar{R}} R.$$

Proof. We have $[P, P] = R + \bar{R}$ by 5.3 and $[P, [P, P]] = 0$ by the graded Jacobi identity. So the first formula follows. We have $R = [P, P] \circ P = j_{[P,P]} \circ P$. By 4.11.2 we get $j_{[P,P]} \circ [P, P] = 2[j_{[P,P]} \circ P, P] - 0 = 2[R, P]$. Therefore $2[R, P] = j_{[P,P]} \circ [P, P] = j(R + \bar{R}) \circ (R + \bar{R}) = j_R \circ \bar{R} + j_{\bar{R}} \circ R$ since R has vertical values and kills vertical vectors, so $j_R \circ R = 0$; likewise for $\bar{R}$. □

6. Bundles and Connections

Let G be a Lie group in the usual sense. We want to carry over to non-commutative differential geometry the concepts of principal bundles, characteristic classes, and Chern-Weil homomorphism. The last two concepts still make difficulties, since we do not know how to express local triviality and only some of the usual properties hold in the general setup we use.

6.1. Definition. By a bundle in non-commutative differential geometry we mean a convenient algebra A together with a closed subalgebra $B \hookrightarrow A$.

The bundle is said to have a finite dimensional Lie group G as **structure group** if we have an injective homomorphism $\lambda : G \to \mathrm{Aut}(A)$, such that $\lambda : G \to L(A, A)$ is smooth and $B = A^G$, the subalgebra of all elements fixed by the G-action.

We remark that for the notion of a principal bundle one should add requirements like quantum transitiveness on the fiber, compare with [**Narnhofer, Thirring, Wicklicky, 1988**], but this is still not enough to get the Chern-Weil homomorphism, see also 6.9.

If $p : P \to M$ is a smooth principal bundle in the usual sense, we put $A = C^\infty(P, \mathbb{R})$ and $B = C^\infty(M, \mathbb{R})$, which is embedded into A via p^*. Then clearly all requirements are satisfied.

6.2. Lemma. *For each $g \in G$ the algebra automorphism $\lambda_g : A \to A$ extends to an automorphism of the algebra of differential forms as follows:*

$$\begin{array}{ccc} A & \longrightarrow & \Omega(A) \\ {\scriptstyle\lambda_g}\downarrow & & {\scriptstyle\lambda_g}\downarrow \\ A & \longrightarrow & \Omega(A). \end{array}$$

Proof. This follows from the universal property 2.9. □

6.3. Horizontal forms. Recall, that on a classical bundle the horizontal forms are exactly those which annihilate vertical vectors. Guided by this we define the space of **horizontal 1-forms** $\Omega_1^{\mathrm{hor}}(A)$ as the closed A-bimodule generated by $\Omega_1(B)$ in $\Omega_1(A)$, in the bornological topology. Likewise we define the algebra $\Omega^{\mathrm{hor}}(A)$ of all **horizontal forms** as the closed subalgebra of $\Omega(A)$ generated by $A + \Omega(B)$.

So $\Omega_1^{\mathrm{hor}}(A)$ is the closed linear subspace generated by all elements of the form $a(db)a'$ for a, $a' \in A$ and $b \in B$. Since in $\Omega_1(A) \subset A\tilde{\otimes}A$ we have $a(db)a' = a(1\otimes b - b\otimes 1)a' = a\otimes ba' - ab\otimes a'$, we get $A\tilde{\otimes}A/\Omega_1^{\mathrm{hor}}(A) = A\tilde{\otimes}_B A$ where A is viewed as a B-bimodule. The situation is explained in the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_1^{\mathrm{hor}}(A) \arrow[r, equal] \arrow[d] & \Omega_1^{\mathrm{hor}}(A) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_1(A) \arrow[r] \arrow[d] & A\tilde{\otimes}A \arrow[r, "\mu"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_1(A)/\Omega_1^{\mathrm{hor}}(A) \arrow[r] \arrow[d] & A\tilde{\otimes}_B A \arrow[r, "\mu"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which has exact columns and also rows since the middle row is splitting.

6.4. Principal connections. We have a good description of horizontal forms, whereas vertical vector fields do not exist in sufficient supply, thus we describe connections in the form of horizontal projections. So a **connection** on a bundle $B \hookrightarrow A$ is an element $\chi \in \Omega_1^1(A) = \operatorname{Hom}_A^A(\Omega_1(A), \Omega_1(A))$ which satisfies $\chi\circ\chi = \chi$ (equivalently $j_\chi \circ \chi = \chi$), such that the image of χ is $\Omega_1^{\mathrm{hor}}(A)$, the space of horizontal 1-forms of the bundle.

Note that a connection $\chi : \Omega_1(A) \to \Omega_1^{\mathrm{hor}}(A)$ has a unique extension as an A-bimodule homomorphism

$$\Omega_k(A) = \Omega_1(A)\otimes_A \cdots \otimes_A \Omega_1(A) \xrightarrow{\Omega(\chi)} \Omega_1^{\mathrm{hor}}(A)\otimes_A \cdots \otimes_A \Omega_1^{\mathrm{hor}}(A)$$
$$\omega_1\otimes\cdots\otimes\omega_k \mapsto \chi(\omega_1)\otimes\cdots\otimes\chi(\omega_k).$$

A connection χ on a bundle with structure group G is called a **principal connection** if it is G-equivariant: $\chi\circ\lambda_g = \lambda_g\circ\chi$ for all $g \in G$.

For a usual principal bundle this corresponds to the projection of forms onto horizontal forms, which describe the vertical distribution. This explains our choice of names here and in 5.3.

PROBLEM: What means 'locally trivial' for a bundle? Does it imply the existence of connections?

6.5. Curvature. Let χ be a connection on a non-commutative /n bundle $B \hookrightarrow A$. The **curvature** $R = R(\chi)$ of the connection is given by

$$R = [\chi, \chi] \in \Omega^1_2(A) = \operatorname{Hom}^A_A(\Omega_1(A), \Omega_2(A)),$$

the abstract Frölicher-Nijenhuis bracket of χ with itself.

6.6. Lemma. *The curvature of a connection satisfies*

$$R \in \operatorname{Hom}^A_A\left(\Omega_1(A)/\Omega^{\text{hor}}_1(A), \Omega^{\text{hor}}_2(A)\right).$$

If the connection is principal then also R is G-equivariant.

Proof. By definition $\Omega^{\text{hor}}_1(A) = \chi(\Omega_1(A))$ is globally integrable, thus $R_\chi = [\chi, \chi] \circ \chi = 0$ and we have

$$\begin{aligned} R := [\chi, \chi] = \bar{R}_\chi &= [\chi, \chi] \circ (Id - \chi) && \text{by 5.3} \\ &= -2\Omega(\chi) \circ d \circ (Id - \chi) && \text{by 5.4.1.} \end{aligned}$$

The last expression implies the first assertion. If χ is a principal connection it is G-equivariant and by 4.12 also $R = [\chi, \chi]$ is G-equivariant. □

6.7. Steps towards the Chern-Weil homomorphism. Let $B \subset A$ be a non-commutative bundle with structure group G. Let $\mathfrak{g}$ denote the Lie algebra of G. We differentiate the action $\lambda : G \to \operatorname{Aut}(A)$ and get bounded linear mappings

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "T_e\lambda"] \arrow[d, equal] & \operatorname{Der}(A; A) \\ \mathfrak{g} \arrow[r, "\lambda'"] & \operatorname{Hom}^A_A(\Omega_1(A), A). \arrow[u, "d^*"'] \end{tikzcd}$$

Using this we define a mapping

$$\begin{gathered} \alpha : \Omega_1(A)/\Omega^{\text{hor}}_1(A) \to A \otimes \mathfrak{g}^* \\ (Id_A \otimes \operatorname{ev}_X)\alpha(\omega) := \lambda'(X)(\omega) \text{ for } X \in \mathfrak{g}, \omega \in \Omega_1(A). \end{gathered}$$

6.8. Lemma. *This mapping α is well defined, an A-bimodule homomorphism, and is G-equivariant for the action $\lambda_g \otimes \mathrm{Ad}(g^{-1})^*$ on the right hand side.*

Proof. For $X \in \mathfrak{g}$, $a, a' \in A$, and $\omega \in \Omega_1(A)$ we have

$$\begin{aligned}(A \otimes \mathrm{ev}_X)\alpha(a\omega a') &= \lambda'(X)(a\omega a')\\ &= a\lambda'(X)(\omega)a' \quad \text{since } \lambda'(X) \in \mathrm{Hom}_A^A(\Omega_1(A), A)\\ &= (A \otimes \mathrm{ev}_X)(a\alpha(\omega)a'),\end{aligned}$$

so α is a bimodule homomorphism. For $b \in B$ we have

$$\begin{aligned}(A \otimes \mathrm{ev}_X)\alpha(a(db)a') &= a\lambda'(X)(db)a'\\ &= a(T_e\lambda.X)(b)a' = 0 \quad \text{since } \lambda_g(b) = b.\end{aligned}$$

So α annihilates horizontal forms and is thus well defined. In order to prove that α is G-equivariant we begin with the following computation, where $g \in G$:

$$\begin{aligned}\lambda_g(T_e\lambda.X)(a) &= \lambda_g \tfrac{d}{dt}|_0 \lambda_{\exp tX}(a)\\ &= \tfrac{d}{dt}|_0 \lambda_g \lambda_{\exp tX}(a) \quad \text{since } \lambda_g \text{ is linear and bounded}\\ &= \tfrac{d}{dt}|_0 \lambda_{g\exp(tX)\,g^{-1}}(\lambda_g(a))\\ &= T_e\lambda(\mathrm{Ad}(g)X)(\lambda_g(a)).\end{aligned}$$

By the universality of d we have $\Omega_1(\lambda_g) \circ d = d \circ \lambda_g$ and thus we get

$$\begin{aligned}\lambda_g(\lambda'(X)(a\,da')) = \lambda_g(a\lambda'(X)(da')) &= \lambda_g(a)\lambda_g(\lambda'(X)da')\\ &= \lambda_g(a)\,\lambda_g((T_e\lambda.X)(a'))\\ &= \lambda_g(a)\,(T_e\lambda.\,\mathrm{Ad}(g)X)(\lambda_g(a'))\\ &= \lambda_g(a)\,\lambda'(\mathrm{Ad}(g)X))(d\lambda_g(a'))\\ &= \lambda'(\mathrm{Ad}(g)X))(\lambda_g(a\,da')).\end{aligned}$$

So finally we have

$$\begin{aligned}(A \otimes \mathrm{ev}_X)\alpha(\Omega_1(\lambda_g)\omega) &= \lambda'(X)(\Omega_1(\lambda_g)\omega)\\ &= \lambda_g(\lambda'(\mathrm{Ad}(g^{-1})X)\omega)\\ &= (\lambda_g \otimes \mathrm{ev}_{\mathrm{Ad}(g^{-1})X})\alpha(\omega)\\ &= (A \otimes \mathrm{ev}_X)(\lambda_g \otimes \mathrm{Ad}(g^{-1}))\alpha(\omega),\end{aligned}$$

so $\alpha \circ \Omega_1(\lambda_g) = (\lambda_g \otimes \mathrm{Ad}(g^{-1})) \circ \alpha$ as required. □

6.9. Remarks. We stop our development here and add just some remarks about the Chern-Weil homomorphism. To continue from this point one should add requirements to the bundle A which imply that α is invertible (the inverse then describes the fundamental vector field mapping) and that the extension of the inverse to invariant polynomials on $\mathfrak{g}$ factors to the $\bar{\Omega}(B)$.

A good model for the Chern-Weil homomorphism is described in the paper [**Lecomte, 1985**] where the following construction is given:

Let $P \to M$ be a smooth principal fiber bundle with structure group G. Then we have the following exact sequence of vector bundles over M:

$$0 \to P[\mathfrak{g}, Ad] \to TP/G \xrightarrow{Tp} TM \to 0.$$

The smooth sections of these bundles give rise to the following exact sequence of Lie algebras:

$$0 \to \mathfrak{X}_{\text{vert}}(P)^G \to \mathfrak{X}_{\text{proj}}(P)^G \to \mathfrak{X}(M) \to 0,$$

namely first all vertical G-equivariant vector fields (the Lie algebra of the gauge group), second the all projectable G-equivariant vector fields on P (the infinitesimal principal bundle automorphisms), third all vector fields on the base. The 'dual' of this sequence of Lie algebras is

$$0 \leftarrow (\Omega_*(A)/\Omega^{\text{hor}}(A))^G \leftarrow \Omega_*(A)^G \leftarrow \Omega_*(B) \leftarrow 0,$$

where $A = C^\infty(P, \mathbb{R})$ and $B = C^\infty(M, \mathbb{R})$. For general algebras this sequence is not exact. For any short exact sequence of Lie algebras [**Lecomte, 1985**] has described a generalization of the Chern-Weil homomorphism in purely algebraic terms, using Chevalley cohomology of the Lie algebras in question. This should be the starting point of the Chern-Weil homomorphism in non-commutative differential geometry.

7. Polyderivations and the Schouten-Nijenhuis Bracket

In this section we describe the analogue of the Schouten-Nijenhuis bracket in the setting of non-commutative differential geometry. It turns out that one has to require skew symmetry in the construction in order to get a meaningful theory. In the end we obtain the Poisson structures for convenient algebras. The results in this section are also a generalization for non-commutative algebras of the results in [**Krasil'shchik, 1988**], which were the original motivation for the developments here, but our approach is different: we first show that the Nijenhuis-Richardson bracket (c.f. [**Nijenhuis, Richardson, 1967**] and [**Lecomte, Michor, Schicketanz**]) passes to the convenient setting and then by restricting it to a suitable space of polyderivations (the non-commutative analog of multi vector fields) we derive a generalization of the Schouten-Nijenhuis bracket.

7.1. It has been noticed in [**De Wilde, Lecomte, 1985**] that for any smooth manifold M the Schouten-Nijenhuis bracket on the space $C^\infty(\Lambda TM)$ of all multivector fields imbeds as a graded sub Lie algebra into the space $\Lambda^{*+1}(C^\infty(M,\mathbb{R}); C^\infty(M,\mathbb{R}))$ with the Nijenhuis-Richardson bracket (see 7.2 for a description of this space). Lecomte told us, that a very elegant proof of this fact can be given in the following way: The space $C^\infty(M,\mathbb{R})$ of smooth functions is the degree -1 part of the Schouten-Nijenhuis algebra. By the universal property of the Nijenhuis-Richardson algebra $(\Lambda^{*+1}(C^\infty(M,\mathbb{R}); C^\infty(M,\mathbb{R})), [\quad,\quad]^\wedge)$ described in [**Lecomte, Michor, Schicketanz**] the identity on $C^\infty(M,\mathbb{R})$ prolongs to a unique homomorphism Φ of graded Lie algebras from the Schouten-Nijenhuis algebra into the Nijenhuis-Richardson algebra, and a simple computation described in [**Lecomte, Melotte, Roger, 1989**] shows that $\Phi(T) = d^*(T) = T\circ(d\times\dots\times d)$, where d is the exterior differential.

This shows that the Schouten-Nijenhuis bracket which we will construct below boils down to the usual one in the commutative case $A = C^\infty(M,\mathbb{R})$.

7.2 The Nijenhuis-Richardson bracket in the convenient setting. Let V be a convenient vector space. We consider the space $\Lambda^k(V)$ of all bounded k-linear skew symmetric functionals $V\times\dots\times V\to\mathbb{R}$, where $\Lambda^0(V)=\mathbb{R}$. Then $\Lambda(V)=\bigoplus_{k\ge 0}\Lambda^k(V)$ is a graded commutative convenient algebra with the usual wedge product

$$(1)\quad (\varphi\wedge\psi)(v_1,\dots,v_{k+\ell}) = \\ = \tfrac{1}{k!\ell!}\sum_\sigma \operatorname{sign}\sigma\;\varphi(v_{\sigma 1},\dots,v_{\sigma k})\psi(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}),$$

where the sum is over all permutation of $k+\ell$ symbols.

Now let W be another convenient vector space. We need the space $\Lambda^k(V;W)$ of all bounded k-linear mappings $V\times\dots\times V\to W$. Then $\Lambda(V;W)=\bigoplus_{k\ge 0}\Lambda^k(V,W)$ is a graded convenient vector space and a graded convenient module over the graded commutative algebra $\Lambda(V)$ with the wedge product (1) from above. If A is a convenient algebra then $\Lambda(V;A)$ is an associative graded convenient algebra with the (formally) same wedge product.

Now for $K\in\Lambda^{k+1}(V;V)$ and $\Phi\in\Lambda^p(V;W)$ we define

$$(2)\quad (i_K\Phi)(v_1,\dots,v_{k+p}) = \\ = \tfrac{1}{(k+1)!(p-1)!}\sum_\sigma \operatorname{sign}\sigma\;\Phi(K(v_{\sigma 1},\dots,v_{\sigma(k+1)}),v_{\sigma(k+2)},\dots,v_{\sigma(k+p)}).$$

Then the following results hold; for proofs see [**Nijenhuis, Richardson, 1967**], [**Michor, 1987**], and [**Lecomte, Michor, Schicketanz**] for multigraded versions; the extension to the convenient setting does not offer any difficulties.

(iii) For $K\in\Lambda^{k+1}(V;V)$, $\varphi\in\Lambda^p(V)$, and $\Phi\in\Lambda(V;W)$ we have $i_K(\varphi\wedge\Phi) = i_K\varphi\wedge\Phi + (-1)^{kp}\varphi\wedge i_K\Phi$ so i_K is a graded derivation of degree k of the $\Lambda(V)$-module $\Lambda(V;W)$ and any derivation is of that form.

(iv) The space of graded derivations of the graded $\Lambda(V)$-module $\Lambda(V;W)$ is a graded Lie algebra with bracket the graded commutator $[D_1, D_2] = D_1D_2 - (-1)^{d_1 d_2} D_2 D_1$, see 3.1.
(v) For $K \in \Lambda^{k+1}(V)$ and $L \in \Lambda^{\ell+1}(V)$ we have $[i_K, i_L] = i([K,L]^\wedge)$ where $[K,L]^\wedge = i_K L - (-1)^{k\ell} i_L K$. So by (4) we get a graded Lie algebra $(\Lambda^{*+1}(V;V), [\quad,\quad]^\wedge)$, called the Nijenhuis-Richardson algebra.
(vi) If $\mu \in \Lambda^2(V;V)$, i. e. $\mu : V \times V \to V$ is bounded skew symmetric bilinear, then $[\mu,\mu]^\wedge = 2i_\mu \mu = 0$ if and only if (V,μ) is a convenient Lie algebra.

7.3. Polyderivations. Let A be a convenient algebra and let $L^k(A) \subset \Lambda^{k+1}(A;A)$ be the space of all maps K such that for any $a_1, \dots a_k \in A$ the linear map $a \mapsto K(a, a_1, \dots, a_k)$ is a derivation of A. Obviously this is a closed linear subspace and thus each $L^k(A)$ is a convenient vector space. We call $L(A) := \bigoplus_{k0} L^k(A)$ the space of all **skew symmetric polyderivations** of A. Obviously $L^k(A)$ is not an A submodule of $\Lambda^{k+1}(A;A)$ in general.

7.4. Theorem. *Let A be a convenient algebra. Then $(L(A), [\quad,\quad]^\wedge)$ is a graded Lie subalgebra of the Nijenhuis-Richardson algebra $(\Lambda^{*+1}(A;A), [\quad,\quad]^\wedge)$. So $(L(A), [\quad,\quad]^\wedge)$ is a convenient graded Lie algebra called the* **Schouten-Nijenhuis algebra** *of A.*

Proof. It suffices to show that for $K_i \in L^{k_i}(A)$ the bracket $[K_1, K_2]^\wedge$ again lies in $L(A)$. This means that we have to show that for arbitrary elements $a, b \in A$ we have:

$$i_{ab}[K_1,K_2]^\wedge = (i_a[K_1,K_2]^\wedge)b + a(i_b[K_1,K_2]^\wedge)$$

From 7.2.(5) we see that for $a \in A = \Lambda^0(A;A)$ and $K \in \Lambda^{k+1}(A)$ we have

$$i_a i_K - (-1)^k i_K i_a = i([a,K]^\wedge) = i(i_a K). \tag{1}$$

If furthermore $L \in L^\ell$ we obviously have from the polyderivation property of L:

$$i(K \wedge a)L = i_K L \wedge a + K \wedge i_a L, \tag{2}$$
$$i(a \wedge K)L = a \wedge i_K L + (-1)^{(k+1)\ell} i_a L \wedge K. \tag{3}$$

Using this we may compute as follows, where we delete $\wedge$ if one of the factors is in the algebra A:

$$
\begin{aligned}
i_{ab}[K_1,K_2]^\wedge &= i_{ab}(i(K_1)K_2) - (-1)^{k_1k_2} i_{ab}(i(K_2)K_1) = \\
&= i(i_{ab}K_1)K_2 + (-1)^{k_1} i(K_1)(i_{ab}K_2) - \\
&\quad - (-1)^{k_1k_2} i(i_{ab}K_2)K_1 - (-1)^{(k_1+1)k_2} i(K_2)(i_{ab}K_1) = \\
&= i(i_aK_1b)K_2) + i(a(i_bK_1)K_2) + \\
&\quad + (-1)^{k_1} i(K_1)(i_aK_2b) + (-1)^{k_1} i(K_1)(ai_bK_2) - \\
&\quad - (-1)^{k_1k_2} i(i_aK_2b)K_1 - (-1)^{k_1k_2} i(ai_bK_2)K_1 - \\
&\quad - (-1)^{(k_1+1)k_2} i(K_2)(i_aK_1b) - (-1)^{(k_1+1)k_2} i(K_2)(ai_bK_1) = \\
&= (i(i_aK_1)K_2)b + (i_aK_1)\wedge(i_bK_2) + \\
&\quad + a(i(i_bK_1)K_2) + (-1)^{k_1k_2}(i_aK_2)\wedge(i_bK_1) + \\
&\quad + (-1)^{k_1}(i(K_1)(i_aK_2))b + (-1)^{k_1} a(i(K_1)(i_bK_2)) - \\
&\quad - (-1)^{k_1k_2}(i(i_aK_2)K_1)b - (-1)^{k_1k_2}(i_aK_2)\wedge(i_bK_1) - \\
&\quad - (-1)^{k_1k_2} a(i(i_bK_2)K_1) - (i_aK_1)\wedge(i_bK_2) - \\
&\quad - (-1)^{(k_1+1)k_2}(i(K_2)(i_aK_1))b - (-1)^{(k_1+1)k_2} a(i(K_2)(i_bK_1)) = \\
&= (i_a(i(K_1)K_2))b - (-1)^{k_1k_2}(i_a(i(K_2)K_1))b + \\
&\quad + a(i_b(i(K_1)K_2)) - (-1)^{k_1k_2} a(i_b(i(K_2)K_1)) = \\
&= (i_a[K_1,K_2]^\wedge)b + a(i_b[K_1,K_2]^\wedge)
\end{aligned}
$$

□

7.5. Definition. Let A be an algebra. A 2-derivation $\mu \in L^1(A)$ is called a **Poisson structure** on A if $[\mu,\mu]^\wedge = 0$.

7.6. Theorem. *Let μ be a Poisson structure for the algebra A. Then $\mu : A \times A \to A$ is a Lie algebra structure. Furthermore we have*

$$
\begin{aligned}
\mu(ab,c) &= a\mu(b,c) + \mu(a,c)b, \\
\mu(a,bc) &= b\mu(a,c) + \mu(a,b)c.
\end{aligned}
$$

The mapping $\check{\mu} : A \to \mathrm{Der}(A), a \mapsto \mu(a, \quad)$ is a homomorphism of Lie algebras $(A,\mu) \to (\mathrm{Der}(A), [\quad,\quad])$, where the second bracket is the Lie bracket (commutator), see 4.2.

This is the non-commutative generalization of the Poisson bracket of differential geometry.

Proof. 7.2.(6) implies that μ is a Lie algebra structure. The other assertion is just the property of a polyderivation. □

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A. Cap, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

A. Kriegl, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

A. Michor, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria, *e-mail:* michor@awirap.bitnet

J. Vanžura, Mathematical Institute of the ČSAV, Department Brno, Mendelovo nám. 1, 662 82 Brno, Czechoslovakia, *e-mail:* mathmu@cspuni12.bitnet