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PRECOLORING EXTENSION. II. GRAPHS CLASSES RELATED TO BIPARTITE GRAPHS

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Abstract. We continue the study of the following general problem on vertex colorings of graphs. Suppose that some vertices of a graph G are assigned to some colors. Can this "precoloring" be extended to a proper coloring of G with at most k colors (for some given k)? Here we investigate the complexity status of precoloring extendibility on some graph classes which are related to bipartite graphs, giving a complete solution for graphs with "co-chromatic number" 2, i.e., admitting a partition $V = V_1 \cup V_2$ of the vertex set V such that each V_i induces a complete subgraph or an independent set. On one hand, we show that our problem is closely related to the bipartite maximum matching problem that leads to a polynomial solution for split graphs and for the complements of bipartite graphs. On the other hand, the problem turns out to be NP-complete on bipartite graphs.

1. Introduction

We consider finite undirected graphs $G = (V, E)$ with vertex set V and edge set E. The **clique number** or **maximum clique size** and the **chromatic number** of G are denoted by $\omega(G)$ and $\chi(G)$, respectively. For any vertex set $W \subseteq V$, G_W denotes the subgraph induced by W. By definition, for a given integer $k \geq 2$, a (proper) k-**coloring** is a function $f\colon V \to \{1, 2, \dots, k\}$ such that $uv \in E$ implies $f(u) \neq f(v)$.

The problem we investigate in this paper was initiated in [**2**] and is called the PRECOLORING EXTENSION problem, or PrExt in short. PrExt is more general than the usual CHROMATIC NUMBER problem and less general than LIST-COLORING [**22**]. PrExt can be formulated as follows.

Instance. An integer $k \geq 2$, a graph $G = (V, E)$ with $|V| \geq k$, a vertex subset $W \subseteq V$, and a proper k-coloring φ of G_W.

Question. Can φ be extended to a proper k-coloring of the entire graph G?

PrExt is closely related to many interesting concepts of combinatorics, including partial Latin squares, integer-valued multicommodity flows, bipartite matchings, perfect graphs, etc. Those connections were partly discussed in [**2**] and will also be explored here and in our forthcoming papers [**17, 18**]. We note that PrExt

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has also motivated some graph-coloring games [**15**]. Some practical applications of PrExt have already been sketched in [**2, 3**]. In this paper we first show that PrExt is NP-complete on bipartite graphs. On the other hand, for some important related graph classes we prove that PrExt is polynomially solvable.

Basic notions.

For an instance of PrExt we say that the number k is the **color bound**, and G is a **precolored** or a **partially k-colored** graph. The vertices of W and $V - W$ are called **precolored** and **precolorless**, respectively. The **precolored classes** are the sets $C_i = \{x \in W : f(x) = i\}$, $i = 1, 2, \ldots, k$. In words, the i^{th} precolored class consists of all precolored vertices assigned to color i, $i = 1, 2, \ldots, k$.

Given a nonnegative integer d, the subproblem d-PrExt is defined as the problem in which the instances of PrExt are restricted to those partially k-colored graphs where the size of each precolored class is at most d. Note that 0-PrExt is the usual **chromatic number** problem, i.e., "Is $\chi(G) \leq k$?".

Since the problem of finding $\chi(G)$ is a subproblem of PrExt, and it is NP-hard even in rather restricted cases (see [**19, 10, 11**]), we have

Theorem 1.1. *For any fixed color bound $k \geq 3$,* PrExt *is* NP*-complete. It remains* NP*-complete even for partially* 3*-colored planar graphs of maximum degree* 4.

On the other hand, a simple argument will show that for $k = 2$, PrExt is polynomially solvable (cf. Section 4).

2. NP-Complete Results

The main result of this section is that PrExt is NP-complete on bipartite graphs. We shall also see that the same holds for the line graphs of bipartite graphs. The former result will be proved in the following stronger form.

Theorem 2.1. 1-PrExt *is* NP*-complete on bipartite graphs.*

This result will be proved by showing that a polynomial algorithm for 1-PrExt on bipartite graphs would yield a polynomial algorithm that decides if any given graph is properly 3-colorable. In order to make the proof more transparent, we decompose the reduction into three steps. We consider the following decision problems:

BIPARTITE PrExt,	or	B-PrExt;
BIPARTITE 1-PrExt,	or	B-1-PrExt;
BIPARTITE LIST COLORING,	or	B-LC;
3-COLORATION,	or	3C.

Here **bipartite PrExt** and **bipartite 1-PrExt** mean PrExt and 1-PrExt on bipartite graphs, respectively. In case of **bipartite list coloring**, the input is a bipartite graph $G = (V, E)$ on n vertices, and a list $L(v) \subseteq \{1, 2, \ldots, n\}$ of colors for each vertex $v \in V$. The question is whether or not there exists a proper coloring $f\colon V \to \{1, 2, \ldots, n\}$ for which $f(v) \in L(v)$ holds for each $v \in V$. If the answer is affirmative, then G is said to be list-colorable. Finally, the 3-coloration problem simply is the question "Is $\chi(G) \leq 3$?" for an arbitrary graph G.

Given two decision problems, P and Q, we use the notation $P \propto Q$ if there exists a polynomial reduction of P to Q in the usual sense (cf. [**10**]). We prove the following three lemmas.

Lemma 2.2. B-PrExt $\propto$ B-1-PrExt.

Lemma 2.3. B-LC $\propto$ B-PrExt.

Lemma 2.4. 3C $\propto$ B-LC.

We mention that further complexity aspects of LIST COLORING will be investigated in the forthcoming paper [**20**].

Proof of Lemma 2.2. Consider an instance of B-PrExt with color bound k, i.e., a partially k-colored bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$. If $x, x' \in X$ are precolored with the same color, then identifying x and x' changes neither the bipartite status of G nor the answer for PrExt. Therefore, we may assume that G is precolored in such a way that each color occurs at most once in X. The same may be assumed for Y. Now, let $X' = \{x'_1, x'_2, \ldots, x'_k\}$ and $Y' = \{y'_1, y'_2, \ldots, y'_k\}$ be k-element sets for which X, Y, X', and Y' are pairwise disjoint. We add $X' \cup Y'$ as a set of new vertices to G yielding a bipartition $(X \cup X') \cup (Y \cup Y')$ of the new graph G' such that for any $x \in X$, $y \in Y$ and $i, j \in \{1, 2, \ldots, k\}$, the following new adjacency relations hold:

x'_i is adjacent to y'_j if and only if $i \neq j$;

x'_i is adjacent to y if and only if y is precolored with a color distinct from i;

y'_j is adjacent to x if and only if x is precolored with a color distinct from j.

We consider G' as an instance of B-1-PrExt where x'_i is precolored with i, $i = 1, 2, \ldots, k$. Now the answer to this instance of 1-PrExt is "yes" if and only if the original precoloring of G is extendible. Moreover, G' can be obtained from G in polynomial time. This completes the proof. $\square$

Proof of Lemma 2.3. Consider an instance of B-LC, i.e., assume that a bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$ is given, and for each $v \in V$, we have a list $L(v) \subseteq \{1, 2, \ldots, n\}$ of the possible colors of v. Let $k = |\cup_{v \in V} L(v)|$. Similarly to the previous proof, we add two k-element sets of vertices, $X'' = \{x''_1, x''_2, \ldots, x''_k\}$ and $Y'' = \{y''_1, y''_2, \ldots, y''_k\}$, to G such that X, Y, X'', and Y'' are pairwise disjoint. Furthermore, we take a bipartition $(X \cup X'') \cup (Y \cup Y'')$ of the new graph G'', and for any $x \in X$, $y \in Y$, and $i, j \in \{1, 2, \ldots, k\}$, define the following new adjacency relations:

$X'' \cup Y''$ is an independent set;

x_i'' is adjacent to y if and only if $i \notin L(y)$.

y_i'' is adjacent to x if and only if $i \notin L(x)$.

We precolor x_i'' and y_i'' with color i in G'', $i = 1, 2, \ldots, k$, leaving the vertices of G precolorless. Now the answer for this instance of bipartite PrExt is "yes" if and only if G is list-colorable. Moreover, G'' and its precoloring can be derived from G (and from the lists $L(v)$, $v \in V$) in polynomial time. This completes the proof. □

Proof of Lemma 2.4. Consider an instance of $3C$, i.e., let an arbitrary graph $G = (V, E)$ be given with $V = \{v^1, \ldots, v^n\}$ and $E = \{e^1, \ldots, e^m\}$. Let E_0, E_1, E_2 be pairwise disjoint sets, each disjoint from V, $E_q = \{e_q^1, \ldots, e_q^m\}$, $q = 0, 1, 2$. We define a bipartite graph G^L with bipartition $V \cup U$ of its vertex set where $U = E_0 \cup E_1 \cup E_2$. Two vertices v^i and e_q^j, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $q = 0, 1, 2$, will be adjacent in G^L if and only if v^i is incident to e^j in G. Clearly, G^L can be obtained from G in polynomial time.

We assign a list $L(x)$ to each $x \in V \cup U$ as follows: For $i, i' = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $q = 0, 1, 2$, let $L(v^i) = \{i, n+i, 2n+i\}$ and $L(e_q^j) = \{qn+i, qn+i'\}$ if $e^j = v^i v^{i'}$ in G. Now we prove that G^L is list-colorable if and only if G is 3-colorable. First assume that G admits a proper 3-coloring; the color of v^i is denoted by $q^i + 1$, where $q^i \in \{0, 1, 2\}$. In G^L, the colors $f(x)$, $x \in X$, will be defined as follows: For $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $q = 0, 1, 2$, let $f(v^i) = q^i n + i$ and let $f(e_q^j)$ be any element of the color set $\{qn+i, qn+i'\} - \{q^i n + i, q^{i'} n + i'\}$ where $e^j = v^i v^{i'}$ in G. (Such an element always exists since $q^i \neq q^{i'}$.)

Observe that f, which can be obtained from G and from the given proper 3-coloring of it in polynomial time, defines a proper list coloring of G^L.

Next, having fixed the above color lists $L(x)$, $x \in V \cup U$, from any proper list-coloring f' of G^L, we construct a proper 3-coloring of G. Recall that $f'(v^i) \in \{i, n+i, 2n+i\}$, $i = 1, 2, \ldots, n$ and $f'(e_q^j) \in \{qn+i, qn+i'\}$ if $e^j = v^i v^{i'}$ in G, $j = 1, 2, \ldots, m$, $q = 0, 1, 2$. In G, we assign vertex v^i to color $q^i + 1$ where $q^i = (f'(v^i) - i)/n$. We claim that this is a proper 3-coloring of G. Indeed, if $e^j = v^i v^{i'}$ is an arbitrary edge in G, then by the adjacencies in G^L, we have $f'(v^i) = q^i n + i$, $f'(e_{q^i}^j) = q^i n + i'$; hence $q^{i'} n + i' = f'(v^{i'}) \neq q^i n + i'$ implying $q^{i'} \neq q^i$. Observe that the colors $q^i + 1$, $i = 1, 2, \ldots, n$, can be computed in polynomial time from f'. This completes the proof. □

Proof of Theorem 2.1. Combine Lemmas 2.2–2.4 with the NP-completeness of 3-colorability [**19, 10, 11**]. □

The following problem is still open.

Problem 2.5. *Is* PrExt NP-*complete on bipartite graphs if the color bound is fixed but greater than two?*

Line graphs.

R. Häggkwist kindly called our attention to the fact that known results on partial Latin squares imply the intractability of PrExt on **line graphs** of bipartite graphs. A **partial Latin square** is an $n \times n$ matrix M with entries from the set $\{0, 1, \ldots, n\}$ such that no column or row contains any repeated entry other than 0. The question of the extendibility problem is this: Can M be extended to a (full) Latin square, i.e., can we replace each zero entry in M by an element of $\{1, 2, \ldots, n\}$ in such a way that no row or column contains a repeated entry? Colbourn [**4**] showed that the partial Latin square extendibility problem is NP-complete.

To show the explicit connection between this problem and PrExt, let L_n denote the line graph of the complete bipartite graph $K_{n,n}$, $n = 2, 3, \ldots$. This L_n is a $(2n-2)$-regular perfect graph with maximum clique size n, and the number of proper complete n-colorings of L_n is a very fast increasing function of n.

Theorem 2.6. PrExt *is* NP-*complete on* $\{L_n;\ n = 2, 3, \ldots\}$.

Proof. Consider an arbitrary instance of the partial Latin square extendibility problem. From such an $n \times n$ matrix M we construct a partial n-coloring of L_n as follows. The vertices e_{ij} of L_n (corresponding to the edges $r_i c_j$ of the complete bipartite graph on the rows r_i and columns c_j as vertices) can be identified with the entries m_{ij} of M. In L_n, e_{ij} is precolorless or precolored with color m_{ij} if and only if $m_{ij} = 0$ or $m_{ij} \in \{1, 2, \ldots, n\}$, respectively. Thus there is a one-to-one correspondence between the partial Latin squares and the precolorings of L_n with color bound n. Therefore, any extension of a partial n-coloring is actually an extension of a partial Latin square, and vice versa. Of course, a complete n-coloring corresponds to a Latin square. Thus Colbourn's theorem completes the proof. □

We close this section noting that 2-PrExt is also NP-complete on the class of interval graphs (see [**2**]).

3. Some Polynomial Reductions

Given a partially k-colored graph $G = (V, E)$, a vertex $v \in V$ is said to be

irrelevant if the answer for PrExt is the same on G and $G_{V-\{v\}}$.

Assume that we can find some irrelevant vertex in polynomial time (together with a proof that it is in fact irrelevant). Then the polynomial or NP-complete status of PrExt does not change if we delete this irrelevant vertex from G. Therefore, we can simplify an instance of PrExt as follows.

First, we seek an irrelevant vertex. Second, we delete it from the partially k-colored input graph. Then we repeat these two steps. If finally no precolorless vertex is left or in each connected component the number of vertices is at most k, we have obtained a "yes" answer to the original instance of PrExt in polynomial time. Of course, this case is not the typical one since PrExt is NP-complete in general. Now we define some special kinds of irrelevant vertices. In each case, the irrelevance can be proved in polynomial time.

For any vertex $v \in V$, let $N(v)$ denote its (open) neighborhood, i.e., the set of vertices u for which $uv \in E$. The subset $N_0(v) \subseteq N(v)$ denotes the set of precolorless neighbors, and $K(v)$ denotes the set of distinct colors occurring on $N(v) - N_0(v)$. Let $\delta(v) = |N_0(v)|$ and $\rho(v) = |K(v)|$. Now we distinguish some vertices of the precolored graph as follows:

A **precolored** vertex v is called

redundant if $N_0(v) = \emptyset$, or there exists another precolored vertex u colored with the same color such that $\emptyset \neq N_0(v) \subseteq N_0(u)$.

A **precolorless** vertex x is called

redundant if there exists another vertex $u \in (V - \{v\}) - N(v)$ (no matter whether u is precolored or precolorless) such that $N_0(v) \subseteq N_0(u)$ and $K(v) \subseteq K(u)$.

A **precolorless** vertex v is called

preventive if $\rho(v) = k$;
compelled if $\rho(v) = k - 1$;
negligible if $\rho(v) \leq k - 2$ and $\rho(v) + \delta(v) < k$.

The proofs of the following lemmas can be deduced easily from the definitions.

Lemma 3.1. *Any redundant or negligible vertex is irrelevant, and the irrelevance can be proved in polynomial time.*

Lemma 3.2. *The existence of any preventive vertex implies that the answer for* PrExt *is "no". Furthermore, this case can be checked in polynomial time.*

Lemma 3.3. *Assume that $v \in V$ is a compelled vertex in a partially k-colored graph $G = (V, E)$. The existence of such a vertex can be checked in polynomial time. Then coloring v with the unique color it can properly get, the answer for* PrExt *does not change.*

We say that a partially k-colored graph $G = (V, E)$ is **reduced** if each connected component of it has more than k vertices, and G contains neither preventive, nor redundant, nor compelled, nor negligible vertices. According to the above observations, PrExt can either be solved in polynomial time, or from the original instance, a reduced instance can be obtained in polynomial time such that it is sufficient to solve PrExt only on the reduced instance. This fact makes the reduced partially k-colored graphs very important. The following lemma is immediate by definition.

Lemma 3.4. *A partially k-colored graph $G = (V, E)$ is reduced if and only if each of its connected components is reduced.*

Next we study the connected, reduced, partially k-colored graphs. Such a graph is the triangle on three precolorless vertices with color bound $k = 2$. The following lemma shows that this is the simplest example from several points of view.

Lemma 3.5. *If $G = (V, E)$ is a reduced, partially k-colored connected graph, then it has at least 3 precolorless vertices; moreover, each precolorless vertex has at least 2 precolorless neighbors.*

Proof. If no precolorless vertex exists, then each vertex is redundant. If v is a precolorless vertex with $\delta(v) \leq 1$, then either v is preventive (if $\rho(v) = k$), or v is compelled (if $\rho(v) = k - 1$), or v is negligible (if $\rho(v) \leq k - 2$). Thus $\delta(v) \geq 2$ for each precolorless vertex v, and therefore at least three precolorless vertices exist.□

4. Polynomial Algorithms

First we prove that PrExt can be hard only if the color bound is at least three.

Proposition 4.1. *If the color bound is* 2, *then* PrExt *can be solved in polynomial time.*

Proof. Let $G = (V, E)$ be a partially 2-colored graph. First we add two new adjacent vertices, u_1 and u_2 to G such that any $v \in V$ will be adjacent to u_p, $p = 1, 2$, if and only if v is precolored with color p. Observe that the answer for PrExt on the precolored graph G with color bound 2 is "yes" if and only if the new graph G^+ is bipartite. The well-known fact that bipartite graphs can be recognized in polynomial time completes the proof since G^+ can be obtained from G in polynomial time. □

Next we investigate PrExt on forests. The essential observation is

Proposition 4.2. *No forest is reduced.*

Proof. Consider a partially k-colored tree $G = (V, E)$ on $n \geq k + 1$ vertices. Let t be maximal such that G contains t precolorless vertices inducing a path, and denote one of its endpoints by v. If G were reduced, Lemma 3.5 would imply $t \geq 3$ and $\delta(v) \geq 2$. However, from the maximality of t, $\delta(v) = 1$. This contradiction and Lemma 3.4 complete the proof. □

This proposition proves the following result.

Theorem 4.3. PrExt *is polynomially solvable on forests.*

Next we study PrExt on **split graphs**, which can be considered as relatives of bipartite graphs since by definition, split graphs are graphs whose vertex set is the union of a complete subgraph and an independent set. Split graphs were introduced by Földes and Hammer [**9**]; however the basic characterization of split graphs had already been found by Gyárfás and Lehel [**13**]: The class of split graphs is exactly the largest self-complementary class of graphs containing neither a chordless 4-cycle nor a chordless 5-cycle. In other words, these graphs can be characterized by the property that they are chordal and their complements are also chordal. Thus split graphs are perfect, and the class is closed under induced subgraphs.

Most of the usual problems of algorithmic graph theory are known to be polynomially solvable on split graphs (see e.g. [**12**]). However, the k-DOMINATING SET problem is NP-complete on split graphs (cf. [**7**]) showing that their structure is not so simple as it seems. Further problems have been shown to be hard on this class of graphs in [**5**]. We note that among the randomly generated graphs which contain no 4-cycle as an induced subgraph, almost all graphs are split (see [**21**]).

Knowing that PrExt is NP-complete on bipartite graphs (Theorem 2.1), on the line graphs of bipartite graphs (Theorem 2.6), and on interval graphs [**2**], the following theorem is worth some attention.

Theorem 4.4. PrExt *is polynomially solvable on split graphs.*

Proof. We use the observations of the previous section and we apply induction on the color bound. If the color bound is 2, we are home by Proposition 4.1. For $k \geq 3$, we may restrict ourselves to those partially k-colored split graphs $G = (V, E)$ which are reduced. By definition, there exists a maximum clique $C \subseteq V$ such that G_{V-C} is edgeless. Actually, such a C can be found in polynomial time. Let $n = |V|$ and $m = |C| = \omega(G)$.

The answer for PrExt is trivially "no" if $m > k$. Therefore, without loss of generality, we may assume that $m \leq k$. Since G is reduced, $n > k$; hence $V - C \neq \emptyset$. Observe that $\rho(y) + \delta(y) \leq m$ holds for each $y \in V - C$; hence $V - C$ is completely precolored because G is reduced. Moreover, no color of a $y \in V - C$ occurs on C, since otherwise y were redundant. Similarly, no color of a precolored vertex $x \in C$ occurs on $V - \{x\}$.

Observe that if $x \in C$ is precolored, then the answer for PrExt on G with color bound k is the same as on $G_{N(x)}$ with color bound $k-1$. Hence in this case we are home by induction. Therefore, we may assume that C is completely precolorless.

Let $C = \{x^1, x^2, \ldots, x^m\}$, and let $Z = \{z^1, z^2, \ldots, z^k\}$ be an arbitrary set of size k such that $C \cap Z = \emptyset$. We define a bipartite graph G^* with bipartition $V^* = C \cup Z$ of its vertex set V^* such that any $x \in C$ will be adjacent to a $z^i \in Z$ if and only if there is no $y \in N(x) \cap (V - C)$ precolored with i in G. Observe that if $x^j z^{\nu(j)}$, $j = 1, 2, \ldots, m$, are the edges of a matching in G^*, then by coloring each x^j with $\nu(j)$, we obtain a proper k-coloring extension of the given precoloring of G. On the other hand, any proper precoloring extension on G defines a matching $x^j z^{\nu(j)}$, $j = 1, 2, \ldots, m$, in G^*, where color j is assigned to $x^{\nu^{-1}(j)}$. Consequently, the answer for PrExt on G is "yes" if and only if G^* has a matching of size m. It is well-known that a maximum matching of G^* can be found in polynomial time, e.g. by applying the famous "Hungarian method", or by the algorithm of Hopcroft and Karp [**16**]. This fact completes the proof since G^* can be constructed in polynomial time from the original instance of PrExt. □

As a matter of fact, the proof of Theorem 4.4 shows that the time complexity of PrExt on reduced split graphs is the same as that of the maximum matching problem on bipartite graphs.

Next we study PrExt on the complements of bipartite graphs. Interestingly enough, also in this case PrExt is related to maximum matchings.

Theorem 4.5. PrExt *is polynomially solvable on the complements of bipartite graphs.*

Proof. We apply induction on the color bound, k. If $k = 2$, we are home by Proposition 4.1. Assume that $k \geq 3$, and consider a partially k-colored graph $G = (V, E)$ where G is a complement of a bipartite graph with vertex bipartition $V = X \cup Y$. If there are two precolored vertices $x \in X$ and $y \in Y$ of the same color i, then i cannot be assigned to any further vertex of G, so that the answer for PrExt with color bound k on G is "yes" if and only if it is "yes" on $G_{V-\{x,y\}}$ with color bound $k-1$. Hence in this case the assertion follows by induction. Therefore, we may assume that we have an instance of 1-PrExt, i.e., no color occurs more than once in the partial k-coloring of G.

Make a new graph $\tilde{G} = (V, \tilde{E})$ from G by connecting any pair of precolored vertices of distinct colors. Observe that $\tilde{G}$ is also a complement of a bipartite graph, and that $\tilde{G}$ is k-colorable if and only if the given partial k-coloring of G is extendible. On the other hand, $\tilde{G}$ is k-colorable if and only if the complement of $\tilde{G}$ (which is a bipartite graph) has a matching of size at least $|V| - k$. Since one can check in polynomial time if such a matching exists, PrExt can be solved in polynomial time. □

Finally we prove that in spite of the NP-completeness of PrExt on general bipartite graphs, it can be solved in polynomial time on a subclass. A graph G is said to be P_t-free or C_t-free if it has no induced subgraph isomorphic to P_t or to C_t, the path or the cycle on t vertices, respectively. G is $2K_2$-free if its complement is C_4-free.

Theorem 4.6. PrExt *is polynomially solvable on the* P_5*-free bipartite graphs.*

Proof. Let $G = (V, E)$ be connected, bipartite, P_5-free, with bipartition $V = X \cup Y$, with a given precoloring, and suppose that G is reduced. Take an $xy \in E$ with $N(x) = X$ and $N(y) = Y$. The existence of such an edge follows from a whole bunch of recent results [**1, 6, 8, 14**], by observing that the following conditions are equivalent on the class of connected bipartite graphs: P_5-free; P_5-free and

C_5-free; $2K_2$-free. Since G has no redundant vertices, each $z \in V - \{x, y\}$ should be precolored, contradicting to Lemma 3.5 and proving the theorem. □

By the previous proof it also follows that any reduced, partially k-colored, P_5-free graph contains a cycle with 3, 4 or 5 vertices. A simple reduced bipartite graph is C_6, the cycle on 6 vertices, with color bound $k = 2$ and with all vertices precolorless.

Problem 4.7. *Is there any fixed integer $t \geq 6$ such that* PrExt *is* NP*-complete on the P_t-free bipartite graphs?*

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Note added in proof. In his recent paper, J. Kratochvíl (*Precoloring extension with fixed color bound*, to appear) solved Problems 2.5 and 4.7. He proved that PrExt is NP-complete on bipartite graphs with color bound 3, as well as on P_{14}-free bipartite graphs. The former result is also proved in (H. L. Bodlaender, K. Jansen, and G. J. Woeginger, *Scheduling with incompatible jobs*, manuscript, 1992).

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