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DISCRETIZATION AND SOME QUALITATIVE PROPERTIES OF ORDINARY DIFFERENTIAL EQUATIONS ABOUT EQUILIBRIA

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Abstract. Discretizations and Grobman-Hartman Lemma, discretizations and the hierarchy of invariant manifolds about equilibria are considered. For one-step methods, it is proved that the linearizing conjugacy for ordinary differential equations in Grobman-Hartman Lemma is, with decreasing stepsize, the limit of the linearizing conjugacies of the discrete systems obtained via time-discretizations. Similar results are proved for all types of invariant manifolds about equilibria. The estimates are given in terms of the degree of smoothness of the original ordinary differential equation as well as in terms of the stepsize and of the order of the discretization method chosen. The results sharpen and unify those of Beyn [**6**], Beyn and Lorenz [**7**] and Fečkan [**17**], [**19**].

0. Introduction

In recent years, several papers were devoted to studying the qualitative properties of discrete-time dynamical systems obtained via discretization methods. **The basic question** was/is whether the qualitative properties of continuous-time systems are preserved under discretization. Various concepts of differentiable dynamics were investigated. Without claiming completeness, we mention here results on stability and attraction properties [**17**], [**23**], bifurcations [**8**], periodic orbits [**5**], [**15**], [**29**], [**36**], invariant tori [**13**], [**14**], the saddle-point structure about equilibria [**6**], [**1**], invariant manifolds about equilibria [**6**], [**7**], [**17**], algebraic-topological invariants [**23**], [**27**], averaging [**19**]. Most of these results relate one-step methods for ordinary differential equations. Several authors remark that their results are also true for multistep methods (but the details are usually not presented. Such remarks do not seem to be entirely justified. The starting point is a result for the Euler method. Though somewhat more technical skill is required, generalizations for other one-step methods are, from the conceptual point of view, easily given.

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However, in order to carry over the result for multistep methods (as it is demonstrated by the example of the saddle-point structure in [**6**]) some new ideas and lots of computations are also needed. An abstract framework for doing this task was created in [**16**].)

The typical result states that, in nondegenerate cases and for sufficiently small stepsize, qualitative properties are preserved under time-discretization. In other words, (a significant portion of) the phase portrait is correctly reproduced by numerical methods. With a slight abuse of language, one can say that, from the qualitative point of view, the discretized system and the original system are the same. Even estimates are given, in terms of the stepsize and of the order of the discretization method chosen.

The typical proof consists of three steps:

A. to show that the qualitative properties of the original continuous-time dynamical/differential system and of its time-h-map are, in some well-defined sense, the same

B. to point out that the h-discretized system (i.e. the discrete-time dynamical system obtained via time-discretization with stepsize h) is nothing else but a small perturbation of the time-h-map of the original system

C. to apply perturbation results/methods from nonlinear analysis and related fields (differential equations, nonlinear semigroup theory, differential topology, algebraic topology).

However, the above A-B-C subdivision is not always transparent. Most papers on this area seem to start, understandably, from the problem of explaining numerical experience and not from the classical qualitative theory. It is worth to mention here that the stepsize need not approach zero but can be considered as a bifurcation parameter as well [**8**]. Large stepsize may easily lead to chaos.

The aim of the present paper is to investigate the link between one-step methods [**35**], [**10**] and some local properties of ordinary differential equations about equilibria [**21**], [**32**]. Our results generalize those of [**6**], [**7**], [**17**], [**19**]. Throughout this paper, results on discretizations are termed as Corollaries. The proof of most Corollaries is subdivided into parts A-B-C. Part C is always the same: an easy application of the parametrized contraction mapping principle (recalled in proving Corollary 2.4).

The next section is of preparatory character.

Sections 2 and 3 are devoted to hyperbolic equilibria and to equilibria with pseudo-hyperbolic splittings, respectively. The main result in Section 2 is the observation that, in the vicinity of hyperbolic equilibria, discretizations, while slightly perturbing the linearizing conjugacy in Grobman-Hartman Lemma, can be considered as co-ordinate transformations. As a corollary, it follows that stable/unstable manifolds of the discretizations converge, with decreasing stepzise, to the stable/unstable manifold of the original differential equation. Section 3 extends the latter convergence result to all types (center etc.) of invariant manifolds about equilibria. In various norms, the distance between the original counterparts is estimated in terms of the stepsize and of the order of the one-step method chosen.

In a forthcoming paper, we point out that normally hyperbolic invariant manifolds persist under discretizations. Also the estimates derived in Section 3 for the equilibrium case remain valid.

Remark. My interest in the qualitative analysis of numerical methods originated in a paper of M. Fečkan [**17**]. The first version of the present paper ended with a detailed proof (similar to those in the last section of the present paper) of a slight generalization of Corollary 2.7 for the Euler method. Before submitting it, I learned of the results of W. J. Beyn and J. Lorenz [**6**], [**7**]. This led me to consider higher order methods as well as non-hyperbolic equilibria. Immediately before his visit to Budapest, November '92, M. Fečkan sent us a copy of [**19**] and I learned that, in case of the Euler method, Corollary 2.3 is identical to the main result in [**19**].

1. One-step Discretization Methods and the "Local from Globalized" Principle

Throughout this paper, we use standard notation and terminology. In the following K_1, K_2 etc. will denote positive constants. The positive constants $K_1(\varepsilon)$, $K_2(\varepsilon)$ etc. will depend on some parameter ε. The constants $K_1, K_2(\varepsilon)$ etc. will not necessarily be the same at different appearances. Given a Banach space $\mathcal{Z}$, $C^j = C^j(\mathcal{Z},\mathcal{Z})$ denotes, with norm

$$|f|_j = \max\left\{\sup\left\{|f^{(m)}(z)| \mid z \in \mathcal{Z}\right\},\ \ m = 0,1,2,\dots,j\right\},\ \ j = 0,1,2,\dots,$$

the Banach space of all j times continuously differentiable functions from $\mathcal{Z}$ to $\mathcal{Z}$, with bounded derivatives. For brevity, we write $|f|_0 = |f|$. Partial derivatives are denoted by s'_x, s'_y, φ'_h, φ''_h, $\varphi_h^{(m)}$ etc. The Banach space of bounded linear operators from $\mathcal{Z}$ to $\mathcal{Z}$ is denoted by $L(\mathcal{Z},\mathcal{Z})$. The spectrum of $C \in L(\mathcal{Z},\mathcal{Z})$ is denoted by $\sigma(C)$. Though single bars denote norms in different spaces, no confusion should arise. In product spaces, the norm is defined by $\max\{|x|,|y|\}$. Lipschitz constants are denoted by Lip (.).

Let $f\colon \mathcal{Z} \to \mathcal{Z}$ be (globally) Lipschitzian and consider the ordinary differential equation

$$\dot z = f(z). \tag{1}$$

By its h-**discretized equation** we mean equation

$$Z = \varphi(h,z), \qquad (z, Z \in \mathcal{Z},\ \ h > 0) \tag{2}$$

the shortened form of the recursive system $z_{k+1} = \varphi(h, z_k)$, $k = 0, 1, 2, \dots$ where φ is a fixed one-step method with stepsize h. We assume that φ is of order p for some integer $p \geq 1$ i.e. there exist a constant h_0 and a constant $K_1 = K_1(f)$ depending on f such that

$$(3) \qquad |\varphi(h, z) - \Phi(h, z)| \leq K_1 h^{p+1} \quad \text{for all} \quad h \in (0, h_0],\ z \in \mathcal{Z}$$

where $\Phi(h, \cdot)\colon \mathcal{Z} \to \mathcal{Z}$ is the time-h-map of the induced solution flow of (1). We assume also the existence of a continuous function $\omega = \omega(\cdot, f)\colon (0, h_0] \to \mathbf{R}^+$ satisfying $\omega(h) \to 0$ as $h \to 0$ such that

$$\varphi(h, \cdot) - \Phi(h, \cdot)\colon \mathcal{Z} \to \mathcal{Z}$$

is Lipschitzian and

$$(4) \qquad \operatorname{Lip}\big(\varphi(h, \cdot) - \Phi(h, \cdot)\big) \leq h\omega(h) \quad \text{for all} \quad h \in (0, h_0].$$

To motivate (4), assume that (3) is satisfied and that $f, \varphi \in C^{p+n+1}$, $n \in \mathbf{N}$. It follows easily that $\varphi_h^{(m)}(0, z) = \Phi_h^{(m)}(0, z)$, $m = 0, 1, 2, \dots, p$ and hence, by Taylor's expansion formula with the remainder in integral form, there holds

$$(\varphi(h, z) - \Phi(h, z)) = (p!)^{-1} \int_0^1 (1 - \tau)^p \big(\varphi_h^{(p+1)}(\tau h, z) - \Phi_h^{(p+1)}(\tau h, z)\big) d\tau h^{p+1}$$

for all $h > 0$, $z \in \mathcal{Z}$. The conclusion is the existence of a constant $K_2 = K_2(f, n)$ for which

$$(5) \qquad |\varphi_z^{(m)}(h, z) - \Phi_z^{(m)}(h, z)| \leq K_2 h^{p+1}, \qquad m = 0, 1, 2, \dots, n$$

and, argueing similarly,

$$(6) \qquad |\varphi_z^{(n+m)}(h, z) - \Phi_z^{(n+m)}(h, z)| \leq K_2 h^{p+1-m}, \qquad m = 0, 1, 2, \dots, p + 1$$

for all $h \in (0, h_0]$ (actually, for all $h > 0$) and $z \in \mathcal{Z}$.

To simplify the technicalities, conditions on φ are/were formulated **on the entire space** $\mathcal{Z}$. What we actually need is that (3), (4), (5) and/or (6) are satisfied on a fixed ball in $\mathcal{Z}$. In this sense, with $p = 1$, $h_0 = 1$, (3) is satisfied provided that $f \in C^1$ and $\varphi(h, z) = z + hf(z)$, the classical Euler method. If, in addition, f' is uniformly continuous, then (4) is satisfied, too. (We have e.g. $|\varphi_z'(h, z) - \Phi_z'(h, z)| \leq K_3 h^{1+\alpha}$ if f' is Hölder with exponent α.) If $f \in C^{n+1}$ (and, as before, $p = 1$, $\varphi(h, z) = z + hf(z)$), then (5) and (6) are satisfied, $n = 0, 1, \dots$. In various special cases [**7**], $K_2 = K_2(f, n)$ depends only on $|f|_{p+n+1}$. (In what follows, an additional parameter ε will be intoduced. In some important special cases [**7**], also the dependence of K_2 (and consequently, of Ω_1, Ω_2, h, K etc.) on ε can be explicitely given. See also Corollary 2.8.)

Given a Banach space $\mathcal{Z}$, a function $\mu\colon \mathcal{Z} \to [0,1]$ is called a **cut-off-function** if it is (globally) Lipschitzian and $\mu(z) = 0$ whenever $|z| \geq 1$ and 1 whenever $|z| \leq \Delta$ for some $\Delta > 0$. The simplest way of defining cut-off-functions is possibly to take $\mu(z) = \lambda(|z|)$ where λ is a suitable piecewise linear real function. On the other hand, the existence of C^n, $n = 1, 2, \ldots$, cut-off-functions depends crucially on the finer structure of $\mathcal{Z}$. For details, examples and counterexamples, see the corresponding remarks in [**21**], [**32**], and the references therein. We recall here only the simple fact that finite-dimensional spaces and Hilbert spaces admit C^∞ cut-off-functions.

In the next two sections, some local qualitative properties about equilibria of ordinary differential equations will be investigated. For this reason, we assume, as usual, that $f \in C^1$, $f(z) = Cz + c(z)$ for all $z \in \mathcal{Z}$, $C \in L(\mathcal{Z}, \mathcal{Z})$, $c(0) = 0$, $c'(0) = 0$ and consider the ordinary differential equation

$$\dot{z} = Cz + c(z;\varepsilon) \tag{7}$$

where μ is a cut-off-function and

$$c(z;\varepsilon) = \mu(z/\varepsilon)c(z), \qquad z \in \mathcal{Z}, \ \ \varepsilon > 0.$$

The induced solution flow can be written as

$$\Phi(t, z; \varepsilon) = e^{Ct}z + r(t, z; \varepsilon), \qquad t \in \mathbf{R}, \ \ z \in \mathcal{Z}, \ \ \varepsilon > 0.$$

With (7), we consider also its h-disretized equation $Z = \varphi(h, z; \varepsilon)$ and the **modified h-discretized equation**

$$Z = e^{Ch}z + \psi(h, z; \varepsilon) \tag{8}$$

where

$$\psi(h, z; \varepsilon) = \mu(z)\big(\varphi(h, z; \varepsilon) - \Phi(h, z; \varepsilon)\big) + r(h, z; \varepsilon), \qquad h > 0, \ \ z \in \mathcal{Z}, \ \ \varepsilon > 0.$$

Remark 1.1. The basic existence results of linearization and invariant manifold theory are formulated [**21**], [**32**] in the Lipschitz category. Though we are completely aware of the fact that inequality (3) can hardly be required for all members of the family $\{f(z) = Cz + \mu(z/\varepsilon)c(z)\}_{\varepsilon > 0}$ without assuming $\mu, c \in C^{p+1}$, our treatment of one-step discretization methods tries to be faithful to this Lipschitz tradition.

Proposition 1.2. *The function* $r(t,\cdot;\varepsilon)\colon \mathcal{Z}\to\mathcal{Z}$ *is bounded and Lipschitzian. More precisely, there exists a bounded continuous function* $\Omega_1=\Omega_1(\cdot,f)\colon (0,\infty)\to\mathbf{R}^+$ *with* $\Omega_1(\varepsilon)\to 0$ *as* $\varepsilon\to 0$ *such that*

$$|r(t,\cdot;\varepsilon)|\le\Omega_1(\varepsilon)\varepsilon t \quad \text{whenever} \quad t\in[0,1], \quad \varepsilon>0,$$
$$\operatorname{Lip}(r(t,\cdot;\varepsilon))\le\Omega_1(\varepsilon)t \quad \text{whenever} \quad t\in[0,1], \quad \varepsilon>0.$$

Proof. The differentiability assumptions on c imply that, for a suitable bounded continuous function $\Omega=\Omega(\cdot,f)\colon(0,\infty)\to\mathbf{R}^+$ satisfying $\Omega(\varepsilon)\to 0$ as $\varepsilon\to 0$, there holds

$$|c(z;\varepsilon)-c(w;\varepsilon)|\le\Omega(\varepsilon)|z-w| \quad \text{whenever} \quad z,w\in\mathcal{Z}, \quad \varepsilon>0.$$

In particular, $|c(z;\varepsilon)|\le\Omega(\varepsilon)\varepsilon$ for all $z\in\mathcal{Z}$, $\varepsilon>0$. Since

$$\dot r(t,z;\varepsilon)=Cr(t,z;\varepsilon)+c(\Phi(t,z;\varepsilon);\varepsilon) \quad \text{and} \quad r(0,z;\varepsilon)=0,$$

an elementary application of Gronwall lemma yields that

$$|\dot r(t,z;\varepsilon)|\le|C||r(t,z;\varepsilon)|+\Omega(\varepsilon)\varepsilon,$$
$$|r(t,z;\varepsilon)|\le\Omega(\varepsilon)\varepsilon\big(e^{|C|t}-1\big)/|C|\le\Omega_1(\varepsilon)\varepsilon t$$

whenever $t\in[0,1]$, $z\in\mathcal{Z}$, $\varepsilon>0$. Also the second inequality follows from Gronwall lemma (when applied to the difference of equations

$$\dot r(t,z;\varepsilon)=Cr(t,z;\varepsilon)+c(\Phi(t,z;\varepsilon);\varepsilon),$$
$$\dot r(t,w;\varepsilon)=Cr(t,w;\varepsilon)+c(\Phi(t,w;\varepsilon);\varepsilon). \quad)$$

□

Proposition 1.3. *The function* $\psi(h,\cdot;\varepsilon)\colon\mathcal{Z}\to\mathcal{Z}$ *is bounded and Lipschitzian. More precisely, there exists a bounded continuous function* $\Omega_2=\Omega_2(\cdot,f)\colon (0,\infty)\to\mathbf{R}^+$ *with* $\Omega_2(\varepsilon)\to 0$ *as* $\varepsilon\to 0$ *and, for each* $\varepsilon>0$, *there exists an* $h(\varepsilon)=h(\varepsilon,f)>0$ *such that*

$$|\psi(h,\cdot;\varepsilon)|\le\Omega_2(\varepsilon)\varepsilon h \quad \text{whenever} \quad h\in(0,h(\varepsilon)], \quad \varepsilon>0,$$
$$\operatorname{Lip}(\psi(h,\cdot;\varepsilon))\le\Omega_2(\varepsilon)h \quad \text{whenever} \quad h\in(0,h(\varepsilon)], \quad \varepsilon>0.$$

Proof. In virtue of (3), (4) and of the previous Proposition, we have that

$$\begin{aligned}|\psi(h,\cdot;\varepsilon)|&\le|\varphi(h,\cdot;\varepsilon)-\Phi(h,\cdot;\varepsilon)|+|r(h,\cdot;\varepsilon)|\\&\le K_1(\varepsilon)h^{p+1}+\Omega_1(\varepsilon)\varepsilon h\end{aligned}$$

and

$$\begin{aligned}\operatorname{Lip}(\psi(h,\cdot;\varepsilon))\le{}&\operatorname{Lip}(\mu)|\varphi(h,\cdot;\varepsilon)-\Phi(h,\cdot;\varepsilon)|+\operatorname{Lip}(\varphi(h,\cdot;\varepsilon)-\Phi(h,\cdot;\varepsilon))\\&+\operatorname{Lip}(r(h,\cdot;\varepsilon))\le\operatorname{Lip}(\mu)K_1(\varepsilon)h^{p+1}+\omega(h,\varepsilon)h+\Omega_1(\varepsilon)h\end{aligned}$$

for all $h\in(0,\min\{h_0,1\})$, $\varepsilon>0$ and the desired estimates follow immediately. (Here, of course, with a slight abuse of notation, $K_1(\varepsilon)=K_1(f_\varepsilon)$ and $\omega(h,\varepsilon)=\omega(h,f_\varepsilon)$ where $f_\varepsilon(z)=Cz+c(z;\varepsilon)$, $z\in\mathcal{Z}$.) □

Proposition 1.4. *There exists an* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $h \in (0, h(\varepsilon)]$, *the mapping* $z \to e^{Ch}z + \psi(h, z; \varepsilon)$ *defines a homeomorphism of* $\mathcal{Z}$ *onto itself.*

Proof. This is a simple consequence of Proposition 1.3 and of the global Lipschitz inverse function theorem [**21**, Ex. C. 11]. □

The previous three propositions are elementary. They prepare applications to Grobman-Hartman Lemma and to generalized stable manifolds. Both applications are related to the local behaviour of ordinary differential equations about equilibria. Even if both the conjugacy in Grobman-Hartman Lemma and the generalized stable manifold (in the center-stable case) depend on the particular form of the cut-off function μ, the use of cut-off techniques seems to be unavoidable. Local results on

$$\dot{z} = Cz + c(z) \tag{9}$$

follow easily from global results on (7). In order to extend this "local from globalized" principle to discretization results concerning (2) and (9), we

(C) assume the existence of a continuous function $\delta = \delta(\cdot, f) \colon (0, h_0] \to \mathbf{R}^+$ satisfying $\delta(h) \to 0$ as $h \to 0$ such that for all $h \in (0, h_0]$ and $z \in \mathcal{Z}$, $\varphi(h, z)$ is determined solely by the restriction of f to the set $\{w \in \mathcal{Z} | |w - z| \leq \delta(h)\}$.

As a trivial consequence of (C), given $\varepsilon > 0$ arbitrarily, there holds $\varphi(h, z) = \varphi(h, z; \varepsilon)$ for all $|z| \leq \varepsilon\Delta/2$ and h sufficiently small. Since $\Phi(h, z) = \Phi(h, z; \varepsilon)$ whenever $|z| \leq \varepsilon\Delta/2$ and h sufficiently small, global conjugacy and/or invariant manifold results for (7) can be obviously interpreted as local results for (9). We remark also that $\varphi(h, z; \varepsilon) = e^{Ch}z + \psi(h, z; \varepsilon)$ for all $h > 0$, $|z| \leq \Delta$ and $\varepsilon > 0$. The reason for introducing the modified h-discretized equation (8) is that ψ is bounded while $\varphi - \Phi$ (as it is shown by the trivial example $\mathcal{Z} = \mathbf{R}$, $\varphi(h, z) = \varphi(h, z; \varepsilon) = z + hf(z)$) is not necessarily bounded **on the entire space** $\mathcal{Z}$. (See how Propositions 1.2. and 1.3. are used in proving Corollary 2.3.) The "local from globalized" principle can be partially reversed. It is not hard e.g. to show that convergence results like (17) (when stated for the local stable manifolds of (9) and (2)) imply, in suitable C^p manifold topologies, convergence results for the global stable manifolds of (9) and (2). We shall return to this point in our forthcoming paper on discretizations and normally hyperbolic invariant manifolds.

2. Discretizations and Grobman-Hartman Lemma

In an interesting paper, Beyn [**6**] has shown that, in the vicinity of hyperbolic equilibria, any trajectory of the original continuous-time dynamical system is approximated, within the order of the discretization method, by a trajectory of the discretized system and, similarly, as long as remaining near to the stationary point, any trajectory of the discretized system approximates some trajectory of the original continuous-time system. However, an easy analysis of the construction in [**6**] shows that the pairing defined by the initial points of the corresponding trajectories is neither continuous nor invariant. Thus, though establishing the preservation of the saddle-point structure under discretization (trajectory pairing(s) with sharp estimates plus convergence of stable/unstable manifolds of the discretized systems to the stable/unstable manifolds of the original dynamical system, itself a remarkable result rediscovered in [**17**]), the pairing constructed in [**6**] does not conform to any of the usual equivalence concepts of dynamical systems theory. In the present section, we construct a continuous and invariant pairing which allows us, in the vicinity of hyperbolic equilibria, to interprete time-discretization as an invariant coordinate transformation. We do this by putting the whole problem into the general framework of Grobman-Hartman Lemma. What we recall here is a technical version of the original result due to Hartman.

Theorem 2.1. [**21**, Thm. 5.14.] *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces, $U \in L(\mathcal{X}, \mathcal{X})$, $V \in L(\mathcal{Y}, \mathcal{Y})$, $T \in L(\mathcal{Z}, \mathcal{Z})$, $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, $T = \operatorname{diag}(U, V)$. Let $\xi, \eta \in C^0(\mathcal{Z}, \mathcal{Z})$ be Lipschitzian and* $\operatorname{Lip}(\xi)$, $\operatorname{Lip}(\eta) \leq \kappa$. *Assume that T is invertible and*

$$\kappa < \min\{1 - a, 1/|V^{-1}|\}$$

where $a = \max\{|U^{-1}|, |V|\}$ and $a < 1$. Then there exists a unique $g \in C^0(\mathcal{Z}, \mathcal{Z})$ such that

$$(T + \eta)(\mathrm{id} + g) = (\mathrm{id} + g)(T + \xi). \tag{10}$$

Moreover, $\mathrm{id} + g$ *is a homeomorphism and is thus a topological conjugacy from $T + \xi$ to $T + \eta$.*

The **proof** [**21**] begins with the observation that, in virtue of the Lipschitz inverse function theorem, $T + \xi$ is a homeomorphism of $\mathcal{Z}$ onto itself. Therefore, $\tilde{T}g = Tg(T + \xi)^{-1}$ defines a continuous linear operator on $C^0(\mathcal{Z}, \mathcal{Z})$. The core of the proof is to point out that $\mathrm{id} - \tilde{T} \in L(C^0(\mathcal{Z}, \mathcal{Z}),\ C^0(\mathcal{Z}, \mathcal{Z}))$ is invertible, $|(\mathrm{id} - \tilde{T})^{-1}| \leq 1/(1 - a)$ and that the function $C^0(\mathcal{Z}, \mathcal{Z}) \to C^0(Z, Z)$, $g \to \eta(id + g)(T + \xi)^{-1} - \xi(T + \xi)^{-1}$ is Lipschitzian with constant κ. Thus,

$$F_{\xi,\eta}(g) = (\mathrm{id} - \tilde{T})^{-1}\{\eta(\mathrm{id} + g)(T + \xi)^{-1} - \xi(T + \xi)^{-1}\}$$

defines a contraction on $C^0(\mathcal{Z}, \mathcal{Z})$ and the contraction constant is $\operatorname{Lip}(F_{\xi,\eta}) \leq \kappa/(1 - a) < 1$. But $g = F_{\xi,\eta}(g)$ is a reformulation of (10). The unique fixed point g satisfies $|g| \leq (1 - a - \kappa)^{-1}|\xi - \eta|$. The last trick is to point out that $\mathrm{id} + g$ is a homeomorphism of $\mathcal{Z}$ onto itself. (Note that $(\mathrm{id} + g)^{-1} - \mathrm{id} \in C^0(\mathcal{Z}, \mathcal{Z})$

and $|(\mathrm{id} + g)^{-1} - \mathrm{id}| = |g|$. We remark also that $g(0) = 0$ provided that $\xi(0) = \eta(0) = 0$.)

Now we return to equation (9) and assume that C admits a **pseudo-hyperbolic splitting** i.e. there exist closed subspaces $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{Z}$, $A \in L(\mathcal{X}, \mathcal{X})$, $B \in L(\mathcal{Y}, \mathcal{Y})$ such that $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, $C = \operatorname{diag}(A, B)$ and

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B)\} < \beta < \alpha < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}.$$

Proposition 2.2. *There is a constant $h_0 > 0$ such that, by passing to an equivalent norm, we may assume that $|e^{-Ah}| < 1 - h\alpha$, $|e^{Bh}| < 1 + h\beta$, $|e^{-Ah}||e^{Bh}| < 1 - h(\alpha - \beta)$ for all $h \in (0, h_0]$.*

Proof. We restrict ourselves to prove the assertion for A. Choose $\alpha_1 > 0$ so that $\alpha + 3\alpha_1 < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$. Since $\sigma(\mathrm{id} - rA) = 1 - r\sigma(A)$ and $|1 - r\lambda|^2 = 1 - 2r\operatorname{Re}\lambda + r^2|\lambda|^2$ for all $r \in \mathbf{R}$, $\lambda \in \sigma(A)$, there is a constant $k > 0$ such that $\sigma(\mathrm{id} - kA) \subset \{\lambda \in \mathbf{C} \mid |\lambda| \le 1 - k(\alpha + 2\alpha_1)\}$. In virtue of the spectral radius theorem, there is an equivalent norm on $\mathcal{X}$ such that $|\mathrm{id} - kA| < 1 - k(\alpha + \alpha_1)$. Using Taylor expansion, we have that $|e^{-Ah}| \le |\mathrm{id} - Ah| + Kh^2$ for some constant K and, consequently, $|e^{-Ah}| \le (1 - h/k)|\mathrm{id}| + (h/k)|\mathrm{id} - kA| + Kh^2 \le 1 - h/k + (h/k)(1 - k(\alpha + \alpha_1)) + Kh^2 = 1 - h(\alpha + \alpha_1) + Kh^2$ for all $h \in (0, h_0]$. The rest is clear. □

Corollary 2.3. *Assume that the pseudo-hyperbolic splitting $C = \operatorname{diag}(A, B)$ is hyperbolic i.e. $\beta < 0 < \alpha$ and that the one-step discretization method satisfies* (3) *and* (4). *Then there is an $\varepsilon_0 > 0$, and, for all $\varepsilon \in (0, \varepsilon_0]$, there are positive constants $h(\varepsilon)$, $K(\varepsilon)$ with the properties as follow. Given $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$, there exists a unique map $g = g(h, \varepsilon) \in C^0(\mathcal{Z}, \mathcal{Z})$ such that $\mathrm{id} + g$ is a conjugacy from $\Phi(h, z; \varepsilon)$, the time-h-map of* (7), *to $e^{Ch}z + \psi(h, z; \varepsilon)$, the modified h-discretization* (8). *Further, $|g| \le K(\varepsilon)h^p$.*

Proof. Setting $U = e^{Ah}$, $V = e^{Bh}$, $T = e^{Ch}$, $\xi = r(h, z; \varepsilon)$, $\eta = \psi(h, z; \varepsilon)$, one checks easily that the conditions of Theorem 2.1 are all satisfied. In fact, by Proposition 2.2, U is invertible and $|U^{-1}| \le 1 - h\alpha$, $|V| \le 1 + h\beta$. Since V is invertible and $|V^{-1}| \le e^{|B|h}$ for all $h > 0$, the crucial inequality $\kappa < \min\{1 - a, 1/|V^{-1}|\}$ is a corollary of Propositions 1.2 and 1.3. Finally, we have that

$$1 - a - \kappa \ge 1 - \max\{1 - h\alpha, 1 + h\beta\} - \max\{\Omega_1(\varepsilon), \Omega_2(\varepsilon)\}h \ge k(\varepsilon)h$$

for some positive constant $k(\varepsilon)$ and therefore,

$$\begin{aligned} |g| &\le (1 - a - \kappa)^{-1}|r - \psi| \le (1 - a - \kappa)^{-1}|\varphi - \Phi| \\ &\le (k(\varepsilon)h)^{-1}K_1(\varepsilon)h^{p+1} = K(\varepsilon)h^p. \end{aligned}$$ □

Repeating the proof of Corollary 2.3 with $\xi = 0$, we obtain the existence of a unique $G = G(h; \varepsilon) \in C^0(\mathcal{Z}, \mathcal{Z})$ such that $\mathrm{id} + G$ is a conjugacy from $e^{Ch}z$, the time-h-map of $\dot{z} = Cz$, the linearization of (7), to $e^{Ch}z + \psi(h, z; \varepsilon)$, the modified h-discretization (8).

It is well-known that Grobman-Hartman Lemma applies for flows [**21**, Thm. 5.25]. As for the ordinary differential equation (7), it states that, given ε sufficiently small, there is a unique $G = G(\varepsilon) \in C^0(\mathcal{Z}, \mathcal{Z})$ such that $\mathrm{id} + G$ is a homeomorphism of $\mathcal{Z}$ onto itself and that

$$\Phi(t, z + G(z); \varepsilon) = e^{Ct}z + G(e^{Ct}z) \quad \text{for all} \quad z \in \mathcal{Z}, \quad t \in \mathbf{R}. \tag{11}$$

Thus, $\mathrm{id} + G$ is a conjugacy from $\dot{z} = Cz$ to the nonlinear differential equation (7). Our remarks following Theorem 2.1 imply that $|G| \leq \Omega(\varepsilon)\varepsilon$ where $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Corollary 2.4. *Under the previous conditions, there holds*

$$|G(\varepsilon) - G(h; \varepsilon)| \leq K(\varepsilon)h^p \quad \text{for all} \quad h \in (0, h(\varepsilon)].$$

Proof. **Part A.** Substituting $h = t$, $T = e^{Ch}$, $\xi = 0$, $\eta = r(h, \cdot; \varepsilon)$, the conjugacy equation (10) goes over into (11). Hence, $G = G(\varepsilon)$ is the unique solution to equation

$$g = F_{0, r(h, \cdot; \varepsilon)}(g) \tag{12}$$

in $C^0(\mathcal{Z}, \mathcal{Z})$. In particular, the solution of (12) does not depend on h.

Part B. By letting $T = e^{Ch}$, $\xi = 0$, $\eta = \psi(h, \cdot; \varepsilon)$, we see that $G = G(h; \varepsilon)$ is the unique solution to equation

$$g = F_{0, \psi(h, \cdot; \varepsilon)}(g) \tag{13}$$

in $C^0(\mathcal{Z}, \mathcal{Z})$. The right-hand side of (13) defines a contraction on $C^0(\mathcal{Z}, \mathcal{Z})$ and the contraction constant is less than

$$\kappa/(1 - a) < \Omega_2(\varepsilon)h/(1 - \max\{1 - h\alpha, 1 + h\beta\})$$

which, in turn, is less than $1/2$ for ε sufficiently small. The same is true for $F_{0,r}(h, \cdot; \varepsilon)$. Using Proposition 2.2 again, observe that

$$\begin{aligned} |F_{0, r(h, \cdot; \varepsilon)} - F_{0, \psi(h, \cdot; \varepsilon)}| &\leq |(I - \tilde{T})^{-1}| \cdot |r - \psi| \leq (1 - a)^{-1} \cdot |\varphi - \Phi| \\ &\leq (h \min\{\alpha, -\beta\})^{-1} K_1(\varepsilon) h^{p+1} = K(\varepsilon)h^p. \end{aligned}$$

Part C. The desired inequality is a trivial coonsequence of the **parametrized contraction mapping principle** applied to (12) and (13): Let (M, d) be a complete metric space, $f_1, f_2 \colon M \to M$ contractions with constant $q < 1$ and

$d(f_1(m), f_2(m)) \leq \nu$ for all $m \in M$. Then $d(m_1, m_2) \leq (1-q)^{-1}\nu$ where m_i is the unique fixed point of f_i in M, $i = 1, 2$. □

For completeness, we give a **second proof for Corollary** 2.4 (though it is much shorter than the one already presented, this second proof does not rely on perturbation methods and so it is somewhat outside of the general set-up of the present paper): By the uniqueness property in Grobman-Hartman Lemma, there holds

$$(\mathrm{id} + g)(\mathrm{id} + G(\varepsilon)) = \mathrm{id} + G(h; \varepsilon).$$

Hence, in virtue of Corollary 2.3,

$$\begin{aligned} |G(\varepsilon) - G(h;\varepsilon)| &= |(\mathrm{id} + G(\varepsilon)) - (\mathrm{id} + G(h;\varepsilon))| \\ &= |(\mathrm{id} + G(\varepsilon)) - (\mathrm{id} + g)(\mathrm{id} + G(\varepsilon))| = |g(\mathrm{id} + G(\varepsilon))| \leq K(\varepsilon)h^p. \end{aligned}$$

In order to pass from (7) to (9) and to interprete the above corollaries as local results on (9), we assume that our discretization method satisfies (C) and recall that $\Phi(h,z) = \Phi(h,z;\varepsilon)$ and $\varphi(h,z) = e^{Ch} + \psi(h,z;\varepsilon)$ whenever $|z| \leq \varepsilon\Delta/2$ and h sufficiently small. Similarly, $|g(0)| \leq \varepsilon\Delta/2$ for h sufficiently small. Consider now, in a vicinity of the origin, the pairing $z \longleftrightarrow z + g(z)$ where g is taken from Corollary 2.3. It is easy to check that this (conjugacy) pairing has all the good properties of the (somewhat unnatural – unnatural, in the context of the classical qualitative theory –) pairing constructed by Beyn [**6**]. The pairing $z \longleftrightarrow z + g(z)$ is a local coordinate transformation from the original to the discretized equation. Corollary 2.4 concerns also the saddle-point structure about hyperbolic equilibria. Stating that the linearizing conjugacy is the limit of the linearizing conjugacies for the discretized equations, it has also a simple geometrical meaning and clarifies the link between linearization and discretization.

Remark 2.5. For certain parabolic partial equations, the results of Beyn has recently been generalized by Alouges and Debussche [**1**]. They pointed out that, locally, in a neighbourhood of nondegenerate equilibria, any positive semi trajectory of the time–discretized system is "shadowed" by a positive semi trajectory of the original parabolic partial equation and vice-versa. This was done by combining Beyn's original ideas [**6**] with invariant manifold theory for parabolic equations [**4**]. The result is conceptually clear [**1**]: the underlying perturbation theory for stable/unstable manifolds works equally well [**21**, Thm. 6.23] for noninvertible mappings. From numerical point of view, a comparison between individual (positive semi) trajectories may be completely satisfactory. Nevertheless, on behalf of classical qualitative theory, the local phase-portrait (as a whole) of the discretized system had to be compared to the local phase portrait (as a whole) of the original partial equation. This is certainly a hard task because, not even as a decoupling result, Grobman-Hartman Lemma does not [**3**] remain true for noninvertible mappings. Neither the positive results [**3**], [**12**], [**26**], [**30**] on Grobman-Hartman Lemma for noninvertible mappings seem to be directly applicable.

Corollary 2.4 yields, of course, some results on stable/unstable manifolds and discretization. We clarify this point in some details now. The classical stable manifolds of (7) and (8) are denoted by $\mathcal{M}_\varepsilon = \{(u_\varepsilon(y), y) \in \mathcal{X} \times \mathcal{Y} = \mathcal{Z} \mid y \in \mathcal{Y}\}$, and $\mathcal{M}_{h,\varepsilon} = \{(u_{h,\varepsilon}(y), y) \in \mathcal{X} \times \mathcal{Y} = \mathcal{Z} \mid y \in \mathcal{Y}\}$ respectively. It is well-known that the functions $u_\varepsilon, u_{h,\varepsilon} \colon \mathcal{Y} \to \mathcal{X}$ are bounded and Lipschitzian. Reconsidering the proof of the stable manifold theorem, it is easy to show that $\mathrm{Lip}\,(u_\varepsilon)$, $\mathrm{Lip}\,(u_{h,\varepsilon}) \le L(\varepsilon)$ for some constant $L(\varepsilon)$ (independent of h). The stable linear subspace for $\dot z = Cz$ is $\{0\} \times \mathcal{Y}$. Since $\mathcal{M}_\varepsilon = (\,\mathrm{id}\, + G(\varepsilon))(\{0\} \times \mathcal{Y})$ and $\mathcal{M}_{h,\varepsilon} = (\,\mathrm{id}\, + G(h,\varepsilon))(\{0\} \times \mathcal{Y})$, it follows immediately from Corollary 2.4 that, for all $\varepsilon \in (0, \varepsilon_0]$,

$$|u_\varepsilon - u_{h,\varepsilon}| \le (1 + L(\varepsilon))K(\varepsilon)h^p \quad \text{whenever} \quad h \in (0, h(\varepsilon)]. \tag{14}$$

Now we assume that $f, \mu, \varphi \in C^{p+1}$. A careful analysis of the proof [**21**], [**32**] of the C^{p+1} version of the stable manifold theorem shows that

$$|u_{h,\varepsilon}|_p \le m(\varepsilon) \quad \text{for some constant} \quad m(\varepsilon) \quad \text{independent of} \quad h. \tag{15}$$

Since $u_\varepsilon \in C^{p+1}$, it follows that

$$|u_\varepsilon - u_{h,\varepsilon}|_p \le M(\varepsilon) \quad \text{whenever} \quad h \in (0, h(\varepsilon)]. \tag{16}$$

As an immediate application of the following theorem [**31**], [**24**] (which goes back to Landau, 1913 and Hadamard, 1914), inequality (14) resp. (16) is the first resp. last inequality in a finite chain of interpolation inequalities.

Theorem 2.6. *Let $\mathcal{Y}$, $\mathcal{X}$ be Banach spaces and let $u \in C^p(\mathcal{Y}, \mathcal{X})$, $p \in \mathbf{N}$. Then*

$$|u^{(j)}|^p \le \mathit{const}(j,p) \cdot |u|^{p-j} \cdot |u^{(p)}|^j, \qquad j = 0, 1, \dots, p.$$

Proof. The first nontrivial case is $j = 1$, $p = 2$. We show that

$$|u'|^2 \le 4|u| \cdot |u''| \quad \text{for all} \quad u \in C^2. \tag{17}$$

In fact, for all $y, w \in \mathcal{Y}$, $w \ne 0$, there holds

$$\begin{aligned} |u'(y)w/|w|| &\le |w|^{-1}(|u(y+w) - u(y) - u'(y)w| + |u(y+w) - u(y)|) \\ &\le |w|^{-1}(|\int_0^1 (u'(y+\tau w) - u'(y))w d\tau| + 2|u|) \\ &\le |w|^{-1}(2^{-1}|u''| \cdot |w|^2 + 2|u|). \end{aligned}$$

Taking minimum for $|w|$ on the right-hand side, we obtain

$$|u'(y)w/|w|| \le 2|u|^{1/2}|u''|^{1/2}$$

and (17) follows immediately. The general case is settled by a double induction on j and p. In case of $\mathcal{X} = \mathcal{Y} = \mathbf{R}$, also the best constants $\mathrm{const}_0(j,p)$ are known [**31**], [**24**]. □

Corollary 2.7. *Assume, in addition, that $f, \mu, \varphi \in C^{p+1}$. Then, in the C^{p-1}-norm, $\mathcal{M}_{h,\varepsilon} \to \mathcal{M}_\varepsilon$ as $h \to 0$, for all ε sufficiently small. If also* (C) *is satisfied, then the local stable manifold of* (9) *is, in the C^{p-1} norm, the limit of the local stable manifolds of its discretizations* (2) *as the discretization parameter approaches zero.*

Proof. Set $u = u_\varepsilon - u_{h,\varepsilon}$. Starting form (14) and (16), a simple application of the previous theorem yields that

$$|u_\varepsilon^{(j)} - u_{h,\varepsilon}^{(j)}| \leq K(\varepsilon) h^{p-j}, \quad j = 0, 1, 2, \ldots, p.$$

In particular, $|u_\varepsilon - u_{h,\varepsilon}|_{p-1} \leq K(\varepsilon) h$. □

For the Euler method, assuming $f, \mu \in C^2$, the above considerations yield that $|u_\varepsilon - u_{h,\varepsilon}| \leq K(\varepsilon) h$ and $|u_\varepsilon - u_{h,\varepsilon}|_1 \leq K(\varepsilon)$. (The very same result follows also from Corollary 3.7 of the next section. In case of $f, \mu \in C^3$, Corollary 3.7 implies that $|u_\varepsilon - u_{h,\varepsilon}|_1 \leq K(\varepsilon) h$ and $|u_\varepsilon - u_{h,\varepsilon}|_2 \leq K(\varepsilon)$.) On the other hand — still assuming $f, \mu \in C^2$—there holds [**17**]

$$|u_\varepsilon - u_{h,\varepsilon}|_2 \leq K \quad \text{for some constant} \quad K = K(f). \tag{18}$$

Consequently, in virtue of (17), $|u_\varepsilon - u_{h,\varepsilon}|_1 \leq K h^{1/2}$. (This point is missed in [**17**] where, in the special case $\mathcal{Z} = \mathbf{R}^n$, (18) is followed by an Arzela-Ascoli argument leading to $|u_\varepsilon - u_{h,\varepsilon}|_1 \to 0$ as $h \to 0$). Actually, even a stronger result is true. (The ultimate reason is a peculiarity of the Euler method: $\varphi(h, z) = z + hf(z)$ is smooth in h.)

Corollary 2.8. *Assume that $f, \mu \in C^2$, $\varphi(h, z) = z + hf(z)$, the classical Euler method. Then*

$$|u_\varepsilon - u_{h,\varepsilon}|_1 \leq Kh \quad \textit{and} \quad |u_\varepsilon - u_{h,\varepsilon}|_2 \leq K$$

for some constant K independent of h and ε.

Proof. For $\varepsilon > 0$, define $f_\varepsilon(z) = Cz + \mu(z/\varepsilon) c(z)$, $z \in \mathcal{Z}$. It is elementary to check that $|z + hf_\varepsilon(z) - \Phi(h, z; \varepsilon)| \leq Kh^2$, $|\mathrm{id} + hf_\varepsilon'(z) - \Phi_z'(h, z; \varepsilon)| \leq Kh^2$, $|hf_\varepsilon''(z) - \Phi_{zz}''(h, z; \varepsilon)| \leq Kh$ for some constant $K = K(f)$. (This is better what, on the basis of (5) and (6), $p = 1$, $n = 0$, one might hope for. More precisely, the latter inequalities correspond to (5) and (6) — in case of $p = 1$, $n = 1$.) In a technically much simpler situation, the reasoning we use in proving Corollary 3.7 can be repeated. (A careful analysis of the role of inequalities (5) and (6) is needed.)□

The next section is devoted to various generalizations of Corollary 2.7.

3. Discretization and the Hierarchy of Invariant Manifolds of Equilibria

The aim of this section is to demonstrate that $M_{h,\varepsilon} \to M_\varepsilon$ as $h \to 0$ where M_ε and $M_{h,\varepsilon}$ are suitable invariant manifolds of (7) and (8), respectively. A simple result into this direction is Corollary 2.7 above. It concerns stable manifolds of hyperbolic equilibria, the prototype of all invariant manifolds. The unique equilibria of (7) and (8) are 0 and $g(0)$, respectively. The stable manifolds of (7) and (8) (as well as the local stable manifolds of (9) and (2) providing (C) is satisfied) belong actually to these equilibria.

To pass to more general invariant manifolds, we relax the hyperbolicity condition on the splitting $C = \mathrm{diag}\,(A, B)$. On the other hand, we assume that 0 is an equilibrium of (8) i.e. of (2). This can be ensured e.g. by assuming that

$$\text{if} \quad f(z) = 0, \quad \text{then} \quad \varphi(h, z) = z \quad \text{for all} \quad h \in (0, h_0]. \tag{19}$$

The case of the classical center/center-stable manifold was investigated by Beyn and J. Lorenz [**7**]. It is well-known that center manifolds need not be unique. Different center manifolds of the same equilibrium have, both from the (differential) topological [**33**] and from the dynamical [**9**] point of view, very similar properties. Fortunately, in a well-defined technical sense, the center manifold of (7) is unique (but depends on μ: multiplication by $\mu(z/\varepsilon)$ singles out a local center manifold and makes it global). Given M_ε, the center manifold of (7), Beyn and Lorenz proved [**7**, Thm. 3.10] the existence of an invariant manifold $M_{h,\varepsilon}$ of (8) such that, with respect to the C^n-norm, M_ε and $M_{h,\varepsilon}$ are $O(h^p)$-near providing $f, \mu, \varphi \in C^{p+n+2}$. They pointed out that, in contrast to the stable manifold case, the detailed structure of the dynamics on $M_{h,\varepsilon}$ could vary dramatically with h.

In order to keep the technical difficulties limited [**7**], Beyn and Lorenz did not make any attempt to minimize the smoothness assumptions on f. We continue their investigations. The smoothness assumptions will be weakened and/but sharper estimates proved. Besides, we consider general invariant manifolds of equilibria and not only the classical stable/unstable and center/center-stable manifolds. If also (C) is satisfied, the results we present for (7) and (8) can, as in the previous section, be reformulated in the context of local invariant manifolds.

The section is organised as follows. Results on discretization methods are presented alternately with abstract theorems on invariant manifolds. As existence and smoothness statements, all these abstract theorems (Theorem 3.1, Propositions 3.3 and 3.5) are well-known from standard invariant manifold theory [**21**], [**32**]. The novelty (and this is what Corollaries 3.2, 3.4, 3.7 and 3.8 are based on) is that the contraction estimates we derive are more effective/efficient than usual.

We begin with a technical version of Irwin's result [**21**] on the existence of generalized stable manifolds. Actually, what we need is not the existence result itself but the weighted norm $||\cdot||$ and the estimates from the proof below.

Theorem 3.1. *Consider a mapping of the form*

$$Q\colon \mathcal{X}\times\mathcal{Y}\to\mathcal{X}\times\mathcal{Y},\qquad (x,y)\to(X,Y)=(Dx+s(x,y),Ey+t(x,y))$$

where $D\in L(\mathcal{X},\mathcal{X})$, $E\in L(\mathcal{Y},\mathcal{Y})$ *and* $s\colon\mathcal{X}\times\mathcal{Y}\to\mathcal{X}$, $t\colon\mathcal{X}\times\mathcal{Y}\to\mathcal{Y}$ *are Lipschitzian. In particular, there exist constants* s_x, s_y, t_x, t_y *such that, for all* $x,\tilde{x}\in\mathcal{X}$, $y,\tilde{y}\in\mathcal{Y}$, *there hold*

$$\begin{aligned}|s(x,y)-s(\tilde{x},\tilde{y})|&\le s_x|x-\tilde{x}|+s_y|y-\tilde{y}|,\\ |t(x,y)-t(\tilde{x},\tilde{y})|&\le t_x|x-\tilde{x}|+t_y|y-\tilde{y}|.\end{aligned}$$

Further, assume that $s(0,0)=0$, $t(0,0)=0$. *Finally, assume that* D *is invertible and there is a positive constant* L *such that*

$$|D^{-1}|\{L(|E|+s_x+t_y)+t_xL^2+s_y\}\le L \tag{20}$$

and

$$b_1=|D^{-1}|(|E|+2t_xL+s_x+t_y)<1. \tag{21}$$

Then there is a unique Lipschitzian mapping $v\colon\mathcal{Y}\to\mathcal{X}$ *with* $\mathrm{Lip}\,(v)\le L$ *for which* $\mathrm{graph}\,(v)=\{(v(y),y)\|y\in\mathcal{Y}\}\subset\mathcal{X}\times\mathcal{Y}$ *is (positively) invariant and* $v(0)=0$. *In addition,* $(x,y)\in\ \mathrm{graph}\,(v)$ *if and only if*

$$|x_k|,|y_k|\le\ \mathrm{Const}\,(x_0,y_0)\cdot(|E|+t_xL+t_y)^k\quad\text{for all}\quad k\in\mathbf{N} \tag{22}$$

where (x_k,y_k) *denotes the* k*-th iterate of* $(x,y)=(x_0,y_0)$ *under* Q, $k=1,2,\dots$.

Proof. Equipped with the weighted norm

$$||\vartheta||=\sup\{|\vartheta(y)|/|y|\ \big|\ 0\ne y\in\mathcal{Y}\}$$

and with the usual algebraic operations, the set

$$\begin{aligned}\mathcal{W}=\{\vartheta\colon\mathcal{Y}\to\mathcal{X}\text{ is a continuous function }|\text{ there exists a constant}\\ K=K(\vartheta)\text{ such that }|\vartheta(y)|\le K|y|\text{ for all }y\in\mathcal{Y}\}\end{aligned}$$

is a Banach space and

$$\mathcal{W}_L=\{\vartheta\in\mathcal{W}\ \big|\ |\vartheta(y)-\vartheta(\tilde{y})|\le L|y-\tilde{y}|\quad\text{for}\quad\text{all}\quad y,\tilde{y}\in\mathcal{Y}\}$$

is closed subset of $\mathcal{W}$.

If $v\in\mathcal{W}_L$ satisfies the invariance equation

$$Dx+s(x,y)\big|_{x=v(y)}=v(Ey+t(x,y))\big|_{x=v(y)},$$

then $v = G_{s,t}(v)$ where $G_{s,t}\colon \mathcal{W}_L \to \mathcal{W}_L$ is defined by

$$(G_{s,t}(v))(y) = D^{-1}v(Ey + t(v(y),y)) - D^{-1}s(v(y),y), \qquad y \in \mathcal{Y}.$$

We claim that $G_{s,t}(\mathcal{W}_L) \subset \mathcal{W}_L$ and that $G_{s,t}$ **is a contraction** with constant $\mathrm{Lip}_{||\cdot||}(G_{s,t}) \le b_1 < 1$.

In fact, for all $v \in \mathcal{W}_L$, $y, \tilde{y} \in \mathcal{Y}$, we have that

$$\begin{aligned}
&|(G_{s,t}(v))(y) - (G_{s,t}(v))(\tilde{y})| \\
&\qquad \le |D^{-1}|\{L(|Ey - E\tilde{y}| + |t(v(y),y) - t(v(\tilde{y}),\tilde{y})|) + |s(v(y),y) - s(v(\tilde{y}),\tilde{y})|\} \\
&\qquad \le |D^{-1}|\{L(|E| + t_xL + t_y) + s_xL + s_y\}|y - \tilde{y}|
\end{aligned}$$

and consequently, in virtue of (20), $G_{s,t}(v) \in \mathcal{W}_L$. Similarly, for all $v, \tilde{v} \in \mathcal{W}_L$, $y \in \mathcal{Y}$, there holds

$$\begin{aligned}
|(G_{s,t}(v))(y) - (G_{s,t}(\tilde{v}))(y)| \le |D^{-1}|\{&|v(Ey + t(v(y),y)) - v(Ey + t(\tilde{v}(y),y))| \\
&+ |v(Ey + t(\tilde{v}(y),y) - \tilde{v}(Ey + t(\tilde{v}(y),y))| + |s(v(y),y) - s(\tilde{v}(y),y)|\}.
\end{aligned}$$

Thus, for all $v, \tilde{v} \in \mathcal{W}_L$, we have that

$$||G_{s,t}(v) - G_{s,t}(\tilde{v})|| \le |D^{-1}|\{|E| + 2t_xL + s_x + t_y\}||v - \tilde{v}||$$

which, by (21), is the desired contraction estimate.

It remains to prove the growth order characterization (22). If $(x,y) = (x_0,y_0) \in \mathrm{graph}(v)$, then $|y_1| = |Ey_0 + t(v(y_0),y_0)| \le (|E| + t_xL + t_y)|y_0|$ and further, by induction, $|y_k| \le (|E| + t_xL + t_y)^k|y_0|$ and also $|x_k| \le L|y_k| \le L(|E| + t_xL + t_y)^k|y_0|$ for all $k \in \mathbf{N}$. With $\mathrm{Const}\,(x_0,y_0) = \max\{1, L\}|y_0|$, inequality (22) follows immediately.

To prove that $x = v(y)$ is implied by inequality (22), a little more care is needed. For all $(x,y) = (x_0,y_0) \in \mathcal{X} \times \mathcal{Y}$, we claim that

$$|x_k - v(y_k)| \ge (1/|D^{-1}| - t_xL - s_x)^k|x_0 - v(y_0)|, \qquad k \in \mathbf{N}.$$

In fact,

$$\begin{aligned}
|x_1 - v(y_1)| &\ge |x_1 - Dv(y_0) - s(v(y_0),y_0)| - |v(y_1) - Dv(y_0) - s(v(y_0),y_0)| \\
&= |Dx_0 + s(x_0,y_0) - Dv(y_0) - s(v(y_0),y_0)| - |v(y_1) - v(Ey_0 + t(v(y_0),y_0))| \\
&\ge |D(x_0 - v(y_0))| - |s(x_0,y_0) - s(v(y_0),y_0)| - L|y_1 - Ey_0 - t(v(y_0),y_0)| \\
&\ge |D(x_0 - v(y_0))| - s_x|x_0 - v(y_0)| - L|Ey_0 + t(x_0,y_0) - Ey_0 - t(v(y_0),y_0)| \\
&\ge (1/|D^{-1}|)|x_0 - v(y_0)| - s_x|x_0 - v(y_0)| - Lt_x|x_0 - v(y_0)|
\end{aligned}$$

and the claim follows by induction. Using (22), we conclude that

$$\begin{aligned}(1/|D^{-1}| - t_x L - s_x)^k |x_0 - v(y_0)| &\le |x_k - v(y_k)| \\ &\le |x_k| + L|y_k| \le \operatorname{Const}(x_0, y_0) \cdot (|E| + t_x L + t_y)^k\end{aligned}$$

for all $k \in \mathbf{N}$. Observe that, as a trivial consequence of (21),

$$1/|D^{-1}| - t_x L - s_x > |E| + t_x L + t_y.$$

Thus, by letting $k \to \infty$, we arrive at $|x_0 - v(y_0)| = 0$. □

Now we return to the differential equation (7) and consider the modified h-discretized equation (8). Assume that (19) is satisfied and, as in Section 2, assume that the splitting $C = \operatorname{diag}(A, B)$ is pseudo-hyperbolic i.e.

$$\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B)\} < \beta < \alpha < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$$

for some constants α and $\beta < \alpha$. Setting $D = e^{Ah}$, $E = e^{Bh}$, $\mathcal{X} \times \mathcal{Y} = \mathcal{Z}$, $(x, y) = z$, $(s(x,y), t(x,y)) = \psi(h, z; \varepsilon)$, $L = 1$, one checks easily that, for all $\varepsilon \in (0, \varepsilon_0]$ and $h \in (0, h(\varepsilon)]$, the conditions of Theorem 3.1 are all satisfied. In fact, by Proposition 1.3, $\psi(h, \cdot; \varepsilon)$ is Lipschitzian and $\operatorname{Lip}(\psi(h, \cdot; \varepsilon)) \le \Omega_2(\varepsilon) h$ for all $h \in (0, h(\varepsilon)]$, $\varepsilon > 0$. Property $\psi(h, 0; \varepsilon) = 0$ is a direct consequence of (19). Estimates (20), (21) follow from Proposition 2.2. In particular, applying Proposition 2.2 with $\tilde{\alpha} \in (\alpha, \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\})$, $\tilde{\beta} \in (\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B)\}, \beta)$, we see there is no generality in assuming that $b_1 < 1 - (\alpha - \beta)h$ for all $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$. Thus, equation $v = G_{\psi(h, \cdot; \varepsilon)}(v)$ has a unique solution in $\mathcal{W}_L = \mathcal{W}_1$. This solution is denoted by $v_{h,\varepsilon}$. Graph $(v_{h,\varepsilon})$ is an invariant manifold for (8). In fact, by (22), graph $(v_{h,\varepsilon})$ is a positively invariant (Lipschitz) manifold for (8). By Proposition 1.4, the mapping $z \to e^{Ch} + \psi(h, z; \varepsilon)$ is onto and invertible. Therefore, also negative invariance makes sense and follows from (22).

By the same arguments, for all $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$, equation $v = G_{r(h, \cdot; \varepsilon)}(v)$ has a unique solution in $\mathcal{W}_L = \mathcal{W}_1$. This solution is denoted by $w_{h,\varepsilon}$. Graph $(w_{h,\varepsilon})$ is an invariant manifold for $\Phi(h, \cdot; \varepsilon)$, the time-$h$-map of the solution flow of (7).

In virtue of (22), $(x, y) = z = z_0 = (x_0, y_0) \in \operatorname{graph}(w_{h,\varepsilon})$ if and only if $|\Phi(kh, z_0; \varepsilon)| \le \operatorname{Const}(z_0) \cdot (|E| + t_x L + t_y)^k$ for all $k \in \mathbf{N}$. Observe that

$$\begin{aligned} & \{z_0 \in \mathcal{Z} \mid |\Phi(kh, z_0; \varepsilon)| \le \operatorname{Const}(z_0) \cdot (|E| + t_x L + t_y)^k \quad \text{for all} \quad k \in \mathbf{N}\} \\ = \; & \{z_0 \in \mathcal{Z} \mid |\Phi(t, z_0; \varepsilon)| \le \operatorname{Const}(z_0) \cdot (|E| + t_x L + t_y)^{t/h} \quad \text{for all} \quad t \ge 0\}. \end{aligned}$$

(Proof. Inclusion $\supset$ is trivial. Conversely, assume that, with some constants K_1 and Q, $|\Phi(kh, z_0; \varepsilon)| \le K_1 Q^k$ for all $k \in \mathbf{N}$. Given $t \ge 0$ arbitrarily, choose $k = k(t)$ so that $kh \le t < (k+1)h$ and consider the initial value problem $\dot{z} = Cz + c(z; \varepsilon)$,

$z(kh) = \Phi(kh, z_0; \varepsilon)$. Since $|c(z;\varepsilon)| \le M|z|$ for some constant M, an easy application of Gronwall lemma yields that $|z(t)| = |\Phi(t, z_0; \varepsilon)| \le e^{(|C|+M)(t-kh)} K_1 Q^k \le K_2 Q^{k-t/h} Q^{t/h} \le K_2 \max\{1/Q, 1\} Q^{t/h} = K_3 Q^{t/h}$.) The conclusion is that graph$(w_{h,\varepsilon})$ **does not depend on** h. We point out this by analyzing the growth order characterization

$$\mathrm{graph}\,(w_{h,\varepsilon}) = \{z_0 \in \mathcal{Z} \bigm| |\Phi(t, z_0; \varepsilon)| \le \mathrm{Const}\,(z_0) \cdot (|E| + t_x L + t_y)^{t/h} \text{ for all } t \ge 0\}.$$

Actually, in virtue of (20), (21) and of Proposition 1.2, we have shown that

$$\mathrm{graph}\,(w_{h,\varepsilon}) = \{z_0 \in \mathcal{Z} \bigm| |\Phi(t, z_0; \varepsilon)| \le \mathrm{Const}\,(z_0) \cdot (\eta + \Omega(\varepsilon)h)^{t/h} \quad \text{for all} \quad t \ge 0\}$$

for all positive numbers η satisfying $\delta(\eta + \Omega(\varepsilon)h) < 1$ where $\eta = |e^{Bh}|$, $\delta = |e^{-Ah}|$ and $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. The greatest and the smallest choice for η are limited by Proposition 2.2. For ε sufficiently small, the greatest choice is $1 + \alpha h$ and the smallest choice is $1 + \beta h$. Consequently, given $\gamma \in [\beta, \alpha]$ arbitrarily, we have that

$$\mathrm{graph}\,(w_{h,\varepsilon}) = \{z_0 \in \mathcal{Z} \bigm| |\Phi(t, z_0; \varepsilon)| \le \mathrm{Const}\,(z_0) \cdot e^{\gamma t} \quad \text{for all} \quad t \ge 0\}$$

for all $\varepsilon \in (0, \varepsilon_0]$ and $h \in (0, h(\varepsilon)]$ (and a slightly more careful analysis shows that $\mathrm{Const}\,(z_0) \le K|z_0|$ where K is independent of γ). It follows immediately that graph$(w_{h,\varepsilon})$ is (positively and negatively) invariant for (7) and does not depend on h.

Thus, we are justified in writing $w_{h,\varepsilon} = v_\varepsilon$ and in calling graph(v_ε) the **generalized stable** (or **pseudo-stable**) **manifold** of (7) related to the splitting $C = \mathrm{diag}\,(A, B)$.

Corollary 3.2. *Assume that the splitting* $C = \mathrm{diag}\,(A, B)$ *is pseudo-hyperbolic and that the one-step discretization method satisfies* (5)*, with* $n = 1$*, and* (19)*. Then there is an* $\varepsilon_0 > 0$*, and for all* $\varepsilon \in (0, \varepsilon_0]$*, there are positive constants* $h(\varepsilon)$*,* $K(\varepsilon)$ *with the properties as follow. Given* graph(v_ε)*, the generalized stable manifold of* (7) *related to the splitting* $C = \mathrm{diag}\,(A, B)$*, then, for all* $h \in (0, h(\varepsilon)]$*,* (8) *has an invariant manifold of the form* graph$(v_{h,\varepsilon})$ *where* $v_{h,\varepsilon}$ *is unique in the function class* $\mathcal{W}_L = \mathcal{W}_1$ *and satisfies* $||v_{h,\varepsilon} - v_\varepsilon|| \le K(\varepsilon)h^p$*.*

Proof. **Part A.** By the preceding considerations, for all $h \in (0, h(\varepsilon)]$, v_ε is the unique solution to equation

$$v = G_{r(h,\cdot;\varepsilon)}(v) \tag{23}$$

in $\mathcal{W}_L = \mathcal{W}_1$. In particular, the solution of (23) does not depend on h.

Part B. Consider $v_{h,\varepsilon}$, the unique solution of

$$v = G_{\psi(h,\cdot;\varepsilon)}(v) \tag{24}$$

in $\mathcal{W}_L = \mathcal{W}_1$. As we already observed, graph$(v_{h,\varepsilon})$ is an invariant manifold of (8). For all $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$, the right-hand side of (24) defines a contraction on $\mathcal{W}_L = \mathcal{W}_1$ and the contraction constant is not greater than $b_1 < 1 - (\alpha - \beta)h$. The same is true for $G_{r(h,\cdot;\varepsilon)}$. We claim that

$$||G_{r(h,\cdot;\varepsilon)} - G_{\psi(h,\cdot;\varepsilon)}|| \le K(\varepsilon)h^{p+1}.$$

For brevity, we write $z = (x,y)$, $r(h,z;\varepsilon) = (\tilde{s}(x,y), \tilde{t}(x,y)) \in \mathcal{X} \times \mathcal{Y}$, $\psi(h,z;\varepsilon) = (\tilde{\tilde{s}}(x,y), \tilde{\tilde{t}}(x,y)) \in \mathcal{X} \times \mathcal{Y}$. For $v \in \mathcal{W}_L = \mathcal{W}_1$ and $y \in \mathcal{Y}$, using (19) and (5) (with $n = 1$) again, we have that

$$\begin{aligned}
&\big|\big(G_{r(h,\cdot;\varepsilon)}(v)\big)(y) - \big(G_{\psi(h,\cdot;\varepsilon)}(v)\big)(y)\big| \\
&\quad \le \big|e^{-Ah}\big|\big\{|e^{Bh}y + \tilde{t}(v(y),y) - e^{Bh}y - \tilde{\tilde{t}}(v(y),y)| + |\tilde{s}(v(y),y) - \tilde{\tilde{s}}(v(y),y)|\big\} \\
&\quad \le 2|e^{-Ah}| \cdot |r(h,(v(y),y);\varepsilon) - \psi(h(v(y),y);\varepsilon)| \\
&\quad = 2|e^{-Ah}| \cdot \Big| \int_0^1 [\varphi_z'(h, \tau(v(y),y);\varepsilon) - \Phi_z'(h,\tau(v(y),y);\varepsilon)](v(y),y)d\tau\Big| \\
&\quad \le 2|e^{-Ah}|K_2(\varepsilon)h^{p+1}|(v(y),y)| \le K(\varepsilon)h^{p+1}|y|
\end{aligned}$$

and the claim follows immediately.

Part C. The desired inequality is a trivial consequence of the parametrized contraction mapping principle applied to (23) and (24). □

Let $\mathcal{V} = C^0(\mathcal{Y}, \mathcal{X})$ and, for $L > 0$, define

$$\mathcal{V}_L = \{\vartheta \in \mathcal{V} \mid \vartheta(0) = 0 \quad \text{and} \quad |\vartheta(y) - \vartheta(\tilde{y})| \le L|y - \tilde{y}| \quad \text{for all} \quad y, \tilde{y} \in \mathcal{Y}\}.$$

It is clear that $\mathcal{V}_L$ is a closed subset of $\mathcal{V}$. In what follows we analyse $G_{s,t}$ on $\mathcal{V}_L$ and, for the norm $|\cdot|(= |\cdot|_0)$, we prove a result analogous to Corollary 3.2.

Proposition 3.3. *Assume that the conditions of Theorem* 3.1 *are satisfied. The conclusion is the existence of a positively invariant Lipschitz manifold* graph(v). *Further, assume that* $s \colon \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ *is bounded and*

$$b_0 = |D^{-1}|(t_x L + s_x + 1) < 1. \tag{25}$$

Then $G_{s,t}(\mathcal{V}_L) \subset \mathcal{V}_L$ *and* $G_{s,t}$ *is a contraction on* $\mathcal{V}_L$ *with constant* $\operatorname{Lip}_{|\cdot|}(G_{s,t}) \le b_0 < 1$. *Further,* $v \in \mathcal{V}_L$.

Proof. For all $v \in \mathcal{V}_L$, $(G_{s,t}(v))(0) = 0$, $|G_{s,t}(v)| \le |D^{-1}|(|v| + |s|)$. Further, for all $v \in \mathcal{V}_L$, $y, \tilde{y} \in \mathcal{Y}$, we have that

$$\begin{aligned}
&\big|(G_{s,t}(v))(y) - (G_{s,t}(v))(\tilde{y})\big| \\
&\qquad \le |D^{-1}|\big\{L(|Ey - E\tilde{y}| + |t(v(y),y) - t(v(\tilde{y}),\tilde{y})|) + |s(v(y),y) - s(v(\tilde{y}),\tilde{y})|\big\} \\
&\qquad \le |D^{-1}|\{L(|E| + t_x L + t_y) + s_x L + s_y\}|y - \tilde{y}|
\end{aligned}$$

and consequently, in virtue of (20), $G_{s,t}(v) \in \mathcal{V}_L$. Similarly, for all $v, \tilde{v} \in \mathcal{V}_L$, $y \in \mathcal{Y}$, there holds

$$\begin{aligned}\big|(G_{s,t}(v))(y) - (G_{s,t}(\tilde{v}))(y)\big| &\leq |D^{-1}|\big\{|v(Ey + t(v(y), y)) - v(Ey + t(\tilde{v}(y), y))| \\ &\quad + |v(Ey + t(\tilde{v}(y), y)) - \tilde{v}(Ey + t(\tilde{v}(y), y))| + |s(v(y), y) - s(\tilde{v}(y), y)|\big\} \\ &\leq |D^{-1}|\{Lt_x|v(y) - \tilde{v}(y)| + |v - \tilde{v}| + s_x|v(y) - \tilde{v}(y)|\} \leq b_0|v - \tilde{v}|.\end{aligned}$$

Hence, by the contraction mapping principle, $G_{s,t}$ has a unique fixed point u in $\mathcal{V}_L$. From set-theoretical point of view, $\mathcal{V}_L \subset \mathcal{W}_L$. Therefore, $u = v$.

Corollary 3.4. *Assume that the splitting* $C = \operatorname{diag}(A, B)$ *is pseudo-hyperbolic,* $\alpha > 0$ *and that the one-step discretization method satisfies* (3), (4) *and* (19). *Then there is an* $\varepsilon_0 > 0$ *and, for all* $\varepsilon \in (0, \varepsilon_0]$, *there are positive constants* $h(\varepsilon)$, $K(\varepsilon)$ *with the properties as follow. Given* $\operatorname{graph}(v_\varepsilon)$, *the generalized stable manifold of* (7) *related to the splitting* $C = \operatorname{diag}(A, B)$, *then* $v_\varepsilon \in \mathcal{V}_L = \mathcal{V}_1$ *and, for all* $h \in (0, h(\varepsilon)]$, (8) *has an invariant manifold of the form* $\operatorname{graph}(v_{h,\varepsilon})$ *where* $v_{h,\varepsilon}$ *is unique in the function class* $\mathcal{V}_L = \mathcal{V}_1$ *and satisfies* $|v_{h,\varepsilon} - v_\varepsilon| \leq K(\varepsilon)h^p$.

Proof. Step by step, with $G_{s,t}$ defined on $(\mathcal{V}_L, |\cdot|)$ and not on $(\mathcal{W}_L, ||\cdot||)$, the proof of Corollary 3.2. can be repeated. The extra assumption $\alpha > 0$ yields $|D^{-1}| = |e^{-Ah}| < 1 - h\alpha < 1$ and this is needed to ensure (25). Boundedness of s (and of t) follows from Propositions 1.2 and 1.3. The last step in proving Part B is somewhat easier. The upper bound on $|r(h, (v(y), y); \varepsilon) - \psi(h, (v(y), y); \varepsilon)|$ follows directly from (3). ☐

For $L > 0$, set $c(0) = L$, $\mathcal{V}^0_{c(0)} = \mathcal{V}_L$ and fix a nonnegative integer j. Given a finite sequence of positive numbers $\{c(i)\}_0^j$, we define, inductively, for $i = 1, 2, \ldots, j$,

$$\begin{aligned}\mathcal{V}^i_{c(i)} = \big\{\vartheta \in \mathcal{V}^{i-1}_{c(i-1)} \;\big|\; &\vartheta \text{ is } i \text{ times continuously differentiable and} \\ &\big|\vartheta^{(i)}(y) - \vartheta^{(i)}(\tilde{y})\big| \leq c(i)|y - \tilde{y}| \;\text{ for all }\; y, \tilde{y} \in \mathcal{Y}\big\}.\end{aligned}$$

By standard differential calculus [**28**], $|\vartheta^{(i)}(y)| \leq c(i-1)$ for all $\vartheta \in \mathcal{V}^i_{c(i)}$ and $y \in \mathcal{Y}$. Further, with respect to the norm $|\cdot|_i$, $\mathcal{V}^i_{c(i)}$ is a closed subset of $C^i(\mathcal{Y}, \mathcal{X})$, $i = 1, 2, \ldots, j$. (A somewhat more advanced differentiable calculus yields also that, with respect to the norm $|\cdot| = |\cdot|_0$, $\mathcal{V}^i_{c(i)}$ is a closed subset of $C^0(\mathcal{Y}, \mathcal{X})$. (For two different proofs of this latter statement, see [**11**, Prop. A2] and [**25**, Lemma 2.2]. The argument in [**25**] is based on inequality (17).)) In what follows we analyse $G_{s,t}$ on $\mathcal{V}^i_{c(i)}$ and, for the norm $|\cdot|_i$, we prove a result analogous to Corollary 3.4.

Proposition 3.5. *Assume that the conditions of Theorem* 3.1 *are satisfied. The conclusion is the existence of a positively invariant Lipschitz manifold* $\operatorname{graph}(v)$. *Further, assume that* $s \in C^{j+1}(\mathcal{X} \times \mathcal{Y}, \mathcal{X})$, $t \in C^{j+1}(\mathcal{X} \times \mathcal{Y}, \mathcal{Y})$ *and*

$$b_i = |D^{-1}|\big((|E| + t_xL + t_y)^i + t_xL + s_x\big) < 1, \quad i = 0, 1, \ldots, j + 1. \tag{26}$$

Then, for a suitable choice of the finite sequence $\{c(i)\}_0^j$, $G_{s,t}(\mathcal{V}^i_{c(i)}) \subset \mathcal{V}^i_{c(i)}$ *and, with respect to a norm* $|||\cdot|||_i$ *equivalent to* $|\cdot|_i$, $G_{s,t}$ *is a contraction on* $\mathcal{V}^i_{c(i)}$ *with constant*

$$\mathrm{Lip}_{\,|||\cdot|||_i}(G_{s,t}) \leq \max\{(1+b_m)/2 \mid m = 0,1,\ldots,i\} < 1, \qquad i = 0,1,\ldots,j.$$

Finally, $v \in \mathcal{V}^j_{c(j)}$.

Proof. With $|||\cdot|||_0 = |\cdot|_0 = |\cdot|$, the $j=0$ case is already settled by Proposition 3.3.

Assume that $j \geq 1$. If $v \in \mathcal{V}^1_{c(1)}$ is arbitrarily chosen, then $G_{s,t}(v)$ is continuously differentiable and

$$(G_{s,t}(v))' = D^{-1}\check{v}'(E + t'_x v' + t'_y) - D^{-1}(s'_x v' + s'_y) \tag{27}$$

where $\check{v}$ denotes v with argument $Ey + t(v(y), y)$. Recall that $|v'| \leq L$. It is easy to check that $(G_{s,t}(v))'$ is Lipschitzian with constant

$$c(1)|D^{-1}|\big(|E| + t_x L + t_y)^2 + Lt_x + s_x\big) + K_1 = c(1)b_2 + K_1$$

where K_1 is a constant depending only on $D, E, s, t,\ c(0) = L$. We proceed by induction. If $v \in \mathcal{V}^i_{c(i)}$ is arbitrarily chosen, $i = 2,3,\ldots,j$, then $G_{s,t}(v)$ is i times continuously differentiable and

$$\big(G_{s,t}(v)\big)^{(i)} = D^{-1}\check{v}^{(i)}(E + t'_x v' + t'_y)^i + D^{-1}(\check{v}' t'_x - s'_x)v^{(i)} + R_i \tag{28}$$

where the remainder R_i is a polynomial function in the variables $v', v'', \ldots, v^{(i-1)}$, $s'_x, \ldots, s_x^{(i)}, s''_{xy}, \ldots, t_y^{(i)}$. It follows easily that $\big(G_{s,t}(v)\big)^{(i)}$ is Lipschitzian with constant

$$c(i)|D^{-1}|\big((|E| + t_x L + t_y)^{i+1} + Lt_x + s_x\big) + K_i = c(i)b_{i+1} + K_i$$

where K_i is a constant depending only on $D, E, s, t, c(0) = L, c(1), \ldots, c(i-1)$. More precisely, for $i = 1,2,\ldots,j$, K_i is a polynomial in the variables $c(0), c(1), \ldots,$ $c(i-1)$ and the coefficient of each term is a nonconstant polynomial of $|s'_x|, \ldots,$ $|s_x^{(i+1)}|, |s''_{xy}|, \ldots, |t_y^{(i+1)}|$.

Now we are in a position to specify $\{c(i)\}_0^j$. Recall (26) and that $c(0) = L$. Thus, the recursion $c(i) = (1 - b_{i+1})^{-1}K_i$ makes sense and, in virtue of the previous considerations, $G_{s,t}(\mathcal{V}^i_{c(i)}) \subset \mathcal{V}^i_{c(i)}$, $i = 1,2,\ldots,j$.

In order to prove the desired contraction estimate, observe that

$$|(G_{s,t}(v))^{(m)} - (G_{s,t}(\tilde{v})^{(m)}| \leq \sum_{k=0}^{m} K_m^k |v^{(k)} - \tilde{v}^{(k)}|$$

whenever $v, \tilde{v} \in \mathcal{V}^m_{c(m)}$, $m = 0, 1, 2, \ldots, j$. Here, of course K^k_m, $m = 0, 1, 2, \ldots, j$, $k = 0, 1, 2, \ldots, m$ are positive constants independent of v and $\tilde{v}$. By the proof of Proposition 3.3, K^0_0 can be chosen for b_0. A further analysis of (27), (28) shows that K^m_m can be chosen for b_m, $m = 1, 2, \ldots, j$. Finally, it is easily seen that, for $m = 1, 2, \ldots, j$ and $k = 0, 1, \ldots, m-1$, K^k_m can be taken for a polynomial in the variables $c(0)$, $c(1)$, …, $c(m-1)$ where the coefficient of each term is a nonconstant polynomial of $|s'_x|, \ldots, |s^{(m+1)}_x|, |s''_{xy}|, \ldots, |t^{(m+1)}_y|$.

Now we are in a position to define $|||\cdot|||_i$. For $v \in C^i(\mathcal{Y}, \mathcal{X})$ arbitrary, let

$$|||v|||_i = \sum_{m=0}^{i} d(m)|v^{(m)}|, \qquad i = 0, 1, 2, \ldots, j,$$

where $d(0) = 1$ and $\{d(m)\}^j_1$ is a finite sequence of positive constants specified later. Obviously, for any choice of $\{d(m)\}^j_1$, $|||\cdot|||_i$ and $|\cdot|_i$ are equivalent norms on $C^i(\mathcal{Y}, \mathcal{X})$, $i = 1, 2, \ldots, j$. By definition,

$$\begin{aligned}|||G_{s,t}(v) - G_{s,t}(\tilde{v})|||_i &= \sum_{m=0}^{i} d(m)|(G_{s,t}(v))^{(m)} - (G_{s,t}(\tilde{v}))^{(m)}|\\ &\le \sum_{m=0}^{i} d(m) \sum_{k=0}^{m} K^k_m |v^{(k)} - \tilde{v}^{(k)}| = \sum_{m=0}^{i}\sum_{k=m}^{i} d(k)K^m_k|v^{(m)} - \tilde{v}^{(m)}|\\ &\le \sum_{m=0}^{i}\Big(d(m)b_m + \sum_{k=m+1}^{i} d(k)K^m_k\Big)|v^{(m)} - \tilde{v}^{(m)}|\end{aligned}$$

and

$$|||v - \tilde{v}|||_i = \sum_{m=0}^{i} d(m)|v^{(m)} - \tilde{v}^{(m)}|$$

whenever $v, \tilde{v} \in \mathcal{V}^i_{c(i)}$, $i = 0, 1, 2, \ldots, j$ (and $\sum^i_{i+1} = 0$).

Next we compare the coefficients of $|v^{(m)} - \tilde{v}^{(m)}|$. Using (26), a suitable choice of $\{d(m)\}^j_1$ yields that

$$d(m)b_m + \sum_{k=m+1}^{i} d(k)K^m_k \le 2^{-1}(1 + b_m)d(m) \tag{29}$$

for all $i = 0, 1, \ldots, j$ and $m = 0, 1, \ldots, i$. Consequently,

$$|||G_{s,t}(v) - G_{s,t}(\tilde{v})|||_i \le \max\{2^{-1}(1 + b_m) \mid m = 0, 1, 2, \ldots, i\}|||v - \tilde{v}|||_i$$

for all $v, \tilde{v} \in \mathcal{V}^i_{c(i)}$, $i = 0, 1, 2, \ldots, j$.

Finally, returning to our invariant manifold graph(v), condition (26), the estimate derived above and the contraction mapping principle yields that $v \in \mathcal{V}^j_{c(j)}$.□

Remark 3.6. Actually, $v \in \mathcal{V}^j_{c(j)} \cap C^{j+1}(\mathcal{Y}, \mathcal{X})$. Existence and continuity of the remaining $(j+1)$th derivative is well-known and can be proved at least by four different methods — see the thirty-line survey [**2**, Remark 8] on sharp smoothness results in invariant manifold and foliation theory. See also [**20**] — the fifth method. A particularly simple proof can be modelled after the one of [**2**, Thm. 4].

Corollary 3.7. *Assume that the conditions of Corollary* 3.4 *are satisfied. Further, for some non-negative integer* n, *assume that* $c \in C^{p+n+1}(\mathcal{Z}, \mathcal{Z})$, $\mu \in C^{p+n+1}(\mathcal{Z}, \mathbf{R})$ *and* $(p+n+1)\beta < \alpha$. *In addition, assume that the one-step discretization method satisfies* $\varphi \in C^{p+n+1}$. *Then, using the notation of Corollary* 3.4, $v_{h,\varepsilon}, v_\varepsilon \in C^{p+n+1}(\mathcal{Y}, \mathcal{X})$ *and*

$$|v_{h,\varepsilon} - v_\varepsilon|_{n+m} \le K(\varepsilon) h^{p-m}, \qquad m = 0, 1, 2, \ldots, p. \tag{30}$$

Proof. **Part A.** Setting $D = e^{Ah}$, $E = e^{Bh}$, $\mathcal{X} \times \mathcal{Y} = \mathcal{Z}$, $(x, y) = z$, $(s(x,y), t(x,y)) = r(h, z; \varepsilon)$, $L = 1$, $j = p + n$, one checks easily that, for all $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$, the conditions of Proposition 3.5 are all satisfied. In fact, by Proposition 2.2, the spectral gap condition $(p+n+1)\beta < \alpha$ implies that $|D^{-1}| \cdot |E|^i = |e^{-Ah}| \cdot |e^{Bh}|^i < (1 - h\alpha)(1 + h\beta)^i < 1 - h(\alpha - i\beta) + h^2 K$ for some constant K, $i = 0, 1, 2, \ldots, p + n + 1$. Using Proposition 1.2, it follows immediately that (26) is satisfied and there is a constant $\kappa > 0$ such that $b_i < 1 - 2\kappa h$ for all $i = 0, 1, 2, \ldots, p + n + 1$. The conclusion is that $v_\varepsilon \in \mathcal{V}^{p+n}_{c(p+n)}$. In virtue of Remark 3.6, $v_\varepsilon \in C^{p+n+1}(\mathcal{Y}, \mathcal{X})$.

Part B. Actually, Proposition 3.5 yields that, for all $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h(\varepsilon)]$ and with respect to the norm $||| \cdot |||_i$, $G_{r(h,\cdot;\varepsilon)}$ is a contraction on $\mathcal{V}^i_{c(i)}$ with constant $1 - \kappa h$, $i = 0, 1, 2, \ldots, p + n$. The same is true for $G_{\psi(h,\cdot,\varepsilon)}$. In particular, $v_{h,\varepsilon} \in \mathcal{V}^{p+n}_{c(p+n)} \cap C^{p+n+1}(\mathcal{Y}, \mathcal{X})$.

For each $\varepsilon \in (0, \varepsilon_0]$, $i = 0, 1, \ldots, p + n - 1$, it is absolutely essential that $||| \cdot |||_i$ does not depend on h.

First we prove that $\{c(i)\}_0^{n+p-1}$ is independent of h. Recall that $c(0) = L = 1$ and, with K_i taken from the proof of Proposition 3.5, $c(i) = (1 - b_{i+1})^{-1} K_i \le (2\kappa h)^{-1} K_i$, $i = 1, 2, \ldots, p + n$. We proceed by introduction. Since $(s(x,y), t(x,y)) = r(h, z; \varepsilon)$ and $(s(x,y), t(x,y)) = \psi(h, z; \varepsilon) = \mu(z)(\varphi(h, z; \varepsilon) - \Phi(h, z; \varepsilon)) + r(h, z; \varepsilon)$, K_i is a polynomial in the variables $c(0), c(1), \ldots, c(i-1)$ and the coefficient of each term is a nonconstant polynomial of $|r'_z|, \ldots, |r_z^{(i+1)}|$ and $|r'_z|, \ldots, |r_z^{(i+1)}|$, $|\varphi'_z - \Phi'_z|, \ldots, |\varphi_z^{(i+1)} - \Phi_z^{(i+1)}|$, respectively, $i = 1, 2, \ldots, p + n$. It is clearly enough to prove that $|r_z^{(m)}|$, $|\varphi_z^{(m)} - \Phi_z^{(m)}|$ are of order h, $m = 1, 2, \ldots, p + n$. First, consider $|r_z^{(m)}|$. Since $r_z^{(m)}$ is continuously differentiable in t, the desired result follows from $r(0, z; \varepsilon) = 0$. Secondly, consider $|\varphi_z^{(m)} - \Phi_z^{(m)}|$. The desired result follows from (5) and/or (6). The crucial property is that $|\varphi_z^{(p+n)} - \Phi_z^{(p+n)}| \le K_2 h$. (The same argument shows that $c(p+n)$ is of order h^{-1}. Consequently, (30) is valid for $m = p+1$ as well.)

Analyzing (29), a similar argument shows that, when appropriately chosen, $\{d(m)\}_0^{p+n-1}$ is independent of h. In fact, recall that $d(0) = 1$. In virtue of (26), it is clearly enough to prove that K_k^m, the coefficient of $d(k)$ in (29), satisfies $K_k^m \le Ch$ for some constant C, $m = 0, 1, \ldots, p+n-1$, $k = m+1, m+2, \ldots, p+n-1$. Since $\{c(i)\}_0^{p+n-2}$ is independent of h, K_k^m is a nonconstant polynomial in the variables $|r_z'|, \ldots, |r_z^{(k+1)}|$, $|\varphi_z' - \Phi_z'|, \ldots, |\varphi_z^{(k+1)} - \Phi_z^{(k+1)}|$, $k = 1, 2, \ldots, p+n-1$, $m = 0, 1, \ldots, k-1$. As before, the desired result follows from (5) and/or (6). (The same argument shows that $d(p+n)$ is of order h^{-1}.) Thus, for $i = 0, 1, 2, \ldots, p+n-1$, we have shown that $|||\cdot|||_i$ can be chosen independently of h.

Now we claim that, for all $v \in \mathcal{V}_{c(n+m)}^{n+m}$,

$$|||G_{r(h,\cdot;\varepsilon)}(v) - G_{\psi(h,\cdot;\varepsilon)}(v)|||_{n+m} \le K(\varepsilon)h^{p+1-m}, \qquad m = 0, 1, \ldots, p-1.$$

In fact, argueing as above, it is easily seen that, for all $v \in \mathcal{V}_{c(i)}^i$, there holds

$$\begin{aligned}
|(G_{r(h,\cdot;\varepsilon)}(v))^{(i)} - (G_{\psi(h,\cdot;\varepsilon)}(v))^{(i)}| &\le K \sum_{j=0}^{i} |r_z^{(j)}(h,z;\varepsilon) - \psi_z^{(j)}(h,z;\varepsilon)| \\
&= K \sum_{j=0}^{i} |(\mu(z)(\varphi(h,z;\varepsilon) - \Phi(h,z;\varepsilon)))_z^{(j)}| \\
&\le K_0 \sum_{j=0}^{i} |\varphi_z^{(j)}(h,z;\varepsilon) - \Phi_z^{(j)}(h,z;\varepsilon)|
\end{aligned}$$

for some constants K, K_0 and $i = 0, 1, \ldots, p+n-1$. In virtue of (5), and/or (6) and of the definition of the norm $|||\cdot|||_i$, the claim follows immediately.

Part C. Now we are in a position to prove (30). We distinguish two cases according as $m = p$ or $m \ne p$. If $m = p$, (30) follows directly from $v_\varepsilon, v_{h,\varepsilon} \in \mathcal{V}_{c(p+n-1)}^{p+n-1} \cap C^{p+n}$ and from the fact that $c(p+n-1)$ does not depend on h. In the other case, a simple application of the parametrized contraction mapping principle yields that

$$|||v_{h,\varepsilon} - v_\varepsilon|||_{n+m} \le K(\varepsilon)h^{p-m}, \quad m = 0, 1, 2, \ldots, p-1.$$

By the equivalence of the norms $|||\cdot|||_{n+m}$ and $|\cdot|_{n+m}$ on $\mathcal{V}_{c(n+m)}^{n+m}$, $m = 0, 1, 2, \ldots, p-1$, — nothing depends on h —, we are done. □

Our last result deals with the link between center manifolds and discretizations.

Corollary 3.8. *As before, consider* (7) *and* (8) *but assume that* $\mathcal{Z} = \mathcal{X} \times \mathcal{Z}_0 \times \mathcal{Y}$ *where* $\mathcal{Z}_0$ *is also a Banach space,* $C = \operatorname{diag}(A, C_0, B)$, $C_0 \in L(\mathcal{Z}_0, \mathcal{Z}_0)$ *and there are real constants* $\delta < \gamma \le 0 \le \beta < \alpha$ *such that*

$$\begin{aligned} &\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B)\} < \delta < \gamma < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(C_0)\} \\ \le\ &\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(C_0)\} < \beta < \alpha < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}. \end{aligned}$$

The differentiability assumptions are, as in Corollary 3.7, $c, \mu, \varphi \in C^{p+n+1}$. *We assume also that the spectral gap conditions* $(p+n+1)\beta < \alpha$ *and* $\delta < (p+n+1)\gamma$ *are satisfied. Finally, assume that* φ *satisfies* (3) *and* (19). *The generalized center manifolds* [**21**], [**32**] *of* (7) *and* (8) *are given as* $\operatorname{graph}(w_\varepsilon)$ *and* $\operatorname{graph}(w_{h,\varepsilon})$, *respectively. Then* $w_\varepsilon, w_{h,\varepsilon} \in C^{p+n+1}(\mathcal{Z}_0, \mathcal{X} \times \mathcal{Y})$ *and*

$$|w_{h,\varepsilon} - w_\varepsilon|_{n+m} \le K(\varepsilon) h^{p-m}, \qquad m = 0, 1, 2, \ldots, p.$$

Proof. As in [**21**], [**32**], the generalized center manifold is defined as the intersection of the center-stable (the generalized stable manifold with respect to the splitting $A \longleftrightarrow \operatorname{diag}(C_0, B)$) and of the center-unstable (with the time reversed (recall Proposition 1.4), the generalized stable manifold with respect to the splitting $\operatorname{diag}(A, C_0) \longleftrightarrow B$) manifolds. Via a simple analysis of the (parametrized) implicite function theorem, the desired result follows from a twofold application of Corollary 3.7. □

Remark 3.9. It is natural to ask whether Corollaries 3.4 and 3.7 (the spectral gap condition $(p+n+1)\beta < \alpha$ has to be replaced by $\beta < (p+n+1)\alpha$) remain valid for $\alpha \le 0$. The answer is in all certainty affirmative but we did not check all the details. The main peculiarity of the $\alpha \le 0$ case is that, by the local uniqueness property of strongly stable manifolds, cut-off techniques are avoidable. Thus, instead of (7) and (8), one has to consider (9) and (2) directly and to keep control on the domains. This does not fit into the general framework of the present paper and, apart from the $\alpha = 0$ case, got little attention in the literature. Therefore, though the underlying abstract existence and smoothness results are well-known [**21**], [**32**], it would not be honest to state that 'mutatis mutandis, the $\alpha \le 0$ case could also be settled'. Complete proofs, with all technical details fully elaborated, would require an extra amount of computation.

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