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# THE CONGRUENCE VARIETY OF METAABELIAN GROUPS IS NOT SELF–DUAL

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Abstract. A lattice identity is given such that it holds but its dual fails in the normal subgroup lattices of metaabelian groups. Thus the congruence variety of metaabelian groups is not self-dual; this is the first example for a modular congruence variety which is not self-dual.

For a ring $R$ with unit element let $\mathcal{L}(R)$ denote the class of lattices embeddable in submodule lattices of $R$-modules. Then $\mathbf{H}\mathcal{L}(R)$, the variety generated by $\mathcal{L}(R)$, is a self-dual congruence variety by Hutchinson [**6**, Thm. 7, and 5]. On the other hand, non-modular congruence varieties need not be self-dual by Day and Freese [**2**]. The $\mathbf{H}\mathcal{L}(R)$ have been the only known congruence varieties for a long time, leading to the impression that the congruence variety of Abelian groups, alias $\mathbf{H}\mathcal{L}(\mathbf{Z})$, could be the largest modular congruence variety. This picture was refuted in two steps. First, an unpublished work of Kiss and Pálfy [**7**] showed that the congruence lattice of a certain metaabelian group cannot be embedded in the congruence lattice of any Abelian group. Developing these ideas further, Pálfy and Szabó [**8, 9**] have recently shown that the congruence variety of certain group varieties are not subvarieties of $\mathbf{H}\mathcal{L}(\mathbf{Z})$. This leads to the problem whether every modular congruence variety is self-dual, cf. Pálfy and Szabó [**9**, Problem 4.2] for a slightly different formulation. The aim of the present paper is to give a negative solution.

For a variety $V$ let $\mathbf{Con}(V)$ denote the congruence variety of $V$, i.e., the lattice variety generated by the congruence lattices of all algebras in $V$. Let $\mathbf{M}$ be the variety of metaabelian groups. $\mathbf{M}$ is defined by the identity $[x,y]z = z[x,y]$ where $[x,y] = x^{-1}y^{-1}xy$. By the elementary properties of the commutator (cf., e.g.,

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Gorenstein [**3**, Ch. 2.2]) it is easy to see that **M** satisfies the identities

$$
(1)\qquad \begin{aligned}
&[a,b]^{-1}=[b,a]=[a^{-1},b]=[a,b^{-1}]\\
&ba=ab[a,b]^{-1}\\
&[ab,c]=[a,c][b,c],\quad [a,bc]=[a,b][a,c]\\
&b^l a^k=a^k b^l [a,b]^{-kl}\quad (k,l\in \mathbf{Z}).
\end{aligned}
$$

Let **A** be the variety of Abelian groups and let $\mathbf{M}_4$ be the variety generated by the quaternion group. Then $\mathbf{M}_4$ is a subvariety of **M** , and it is defined by the identities $[x,y]z=z[x,y]$, $x^4=1$ and $[x,y]^2=1$. Pálfy and Szabó [**8, 9**] gave identities satisfied in $\mathbf{Con}(\mathbf{A})$ but not in $\mathbf{Con}(\mathbf{M}_4)$. However, the duals of their identities do the same, so we have to consider another identity. In the variables $\alpha_1,\alpha_2,\dots,\alpha_{13}$ let us consider the following lattice terms:

$$
\begin{aligned}
p&=(\alpha_{12}+\alpha_{13})\big(\alpha_4+\alpha_5+(\alpha_1+\alpha_6+\alpha_7)(\alpha_2+\alpha_8+\alpha_9)(\alpha_3+\alpha_{10}+\alpha_{11})\big)\\
q_1&=\alpha_1+\alpha_2+\alpha_3,\quad q_2=\alpha_6+\alpha_7+\alpha_{12}+\alpha_{13},\quad q_3=\alpha_1+\alpha_4+\alpha_5+\alpha_{10}\\
q_4&=\alpha_3+\alpha_8+\alpha_9,\quad q_5=\alpha_2+\alpha_4+\alpha_{10}+\alpha_{11}+\alpha_{12},\\
q_6&=\alpha_2+\alpha_{11}+\alpha_{12}+\alpha_{13},\quad q_7=\alpha_4+\alpha_5+\alpha_7+\alpha_8+\alpha_9,\\
q_8&=\alpha_1+\alpha_3+\alpha_5,\quad q_9=\alpha_6+\alpha_7+\alpha_8+\alpha_{10}+\alpha_{11}\\
q_{10}&=\alpha_3+\alpha_6+\alpha_9+\alpha_{12}+\alpha_{13},\quad q_{11}=\alpha_4+\alpha_5+\alpha_{10}+\alpha_{11}\\
q_{12}&=\alpha_1+\alpha_2+\alpha_{13},\quad q_{13}=\alpha_6+\alpha_7+\alpha_8+\alpha_9,\quad \text{and}\\
q&=q_1+(q_2q_3+q_4q_5)(q_6q_7+q_8q_9)(q_{10}q_{11}+q_{12}q_{13}).
\end{aligned}
$$

Let $\mu_{13}$ denote the identity

$$p\le q,$$

and let $\mu_{13}^d$ denote the dual of $\mu_{13}$ . Note that $\mu_{13}$ was found by modifying, in fact weakening, the dual of the identity in Pálfy and Szabó [**8**].

**Theorem.**

(A) $\mu_{13}$ *holds in* $\mathbf{Con}(\mathbf{M})$.
(B) $\mu_{13}^d$ *fails in* $\mathbf{Con}(\mathbf{M})$.

We will actually show that $\mu_{13}^d$ fails even in $\mathbf{Con}(\mathbf{M}_4)$. Therefore the modular congruence varieties $\mathbf{Con}(\mathbf{M})$ and $\mathbf{Con}(\mathbf{M}_4)$ are not self-dual.

*Proof.* (B) The rather long calculations required by this part of the proof were done by a personal computer; here we outline the algorithm only. (A Pascal program, Borland's Turbo Pascal 6.0, is available from the author upon request.) The Wille — Pixley algorithm [**10, 11**] offers a standard way to check if a lattice identity holds in the congruence variety of a variety with permuting congruences. Like in [**6**], we can construct a strong Mal'cev condition (MC) such that (MC) holds in $\mathbf{M}_4$ iff $\mu_{13}^d$ holds in $\mathbf{Con}(\mathbf{M}_4)$. This Mal'cev condition is a finite collection of $n$-ary term symbols $f_k$ and equations of the form

$$f_l(x_{1C}, x_{2C}, \dots , x_{nC}) = f_r(x_{1C}, x_{2C}, \dots , x_{nC}) \quad \text{or} \tag{2}$$

$$f_l(x_{1C}, x_{2C}, \dots , x_{nC}) = x_j \tag{3}$$

where $C$ is a partition on the set $\{1, 2, \dots , n\}$ and $iC$ denotes the smallest element of the $C$-block containing $i$. Suppose $\mu_{13}^d$ holds in $\mathbf{Con}(\mathbf{M}_4)$, then there exist group terms $f_k$ such that all the equations (2) and (3) of (MC) are valid identities in $\mathbf{M}_4$. By Pálfy and Szabó [**9**] or the identities (1) each $n$-ary group term $g(x_1, \dots , x_n)$ in $\mathbf{M}_4$ can be uniquely represented in the form

$$\prod_{i=1}^{n} x_i^{a_i} \prod_{i<j} [x_i, x_j]^{b_{ij}} \tag{4}$$

where $a_i \in \mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $b_{ij} \in \mathbf{Z}_2 = \{0, 1\}$. Here $\prod_{i=1}^{n} x_i^{a_i}$ and $\prod_{i<j}[x_i, x_j]^{b_{ij}}$ are called the Abelian part and the commutator part of $g$, respectively.

The variety of Abelian groups of exponent four is a subvariety of $\mathbf{M}_4$, whence (MC) holds in it. Since it is term equivalent to the variety of modules over $\mathbf{Z}_4$, we can use the algorithm described in [**6**] to determine the $a_i^{(k)}$, the exponents occurring in the Abelian part of $f_k$ according to (4). Luckily enough, these $a_i^{(k)}$ are uniquely determined by (MC).

Now let $C_1, \dots , C_w$ be the blocks of a partition $C$ such that the minimal representatives $c_i \in C_i$ satisfy $c_1 < c_2 < \dots < c_w$. For a term $g$ of the form (4) the term $g(x_{1C}, \dots , x_{nC})$ can be written in the (unique) form

$$\prod_{i=1}^{w} x_{c_i}^{d_i} \prod_{i<j} [x_{c_i}, x_{c_j}]^{t_{ij}}.$$

Here $d_i = \sum_{j\in C_i} a_j$. To determine the $t_{ij}$ for $i < j$ let us consider an $u \in C_i$ and a $v \in C_j$. If $u < v$ then $[x_u, x_v]^{b_{uv}}$ turns into $[x_{c_i}, x_{c_j}]^{b_{uv}}$. If $u > v$ then $[x_v, x_u]^{b_{vu}}$ turns into $[x_{c_j}, x_{c_i}]^{b_{vu}} = [x_{c_i}, x_{c_j}]^{-b_{vu}}$ and exchanging the places of $x_{c_j}^{a_v}$ and $x_{c_i}^{a_u}$ in the Abelian part enters $[x_{c_i}, x_{c_j}]^{-a_u a_v}$ as well. Combining all these effects we obtain that

$$t_{ij} = \sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu} + a_u a_v). \tag{5}$$

Therefore, if the $a_i$ and $b_{ij}$ for $f_k$ are denoted by $a_i^{(k)}$ and $b_{ij}^{(k)}$, (2) implies

$$\begin{aligned} &\sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv}^{(l)} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu}^{(l)} + a_u^{(l)} a_v^{(l)}) \\ &\qquad = \sum_{\substack{u<v \\ u\in C_i,\ v\in C_j}} b_{uv}^{(r)} - \sum_{\substack{u>v \\ u\in C_i,\ v\in C_j}} (b_{vu}^{(r)} + a_u^{(r)} a_v^{(r)}) \end{aligned} \tag{6}$$

for all meaningful $i < j$. The equations (6) and the analogous equations derived from (3) constitute a system of linear equations over the two-element field with the $b_{uv}^{(k)}$ being the unknowns. Using some reductions, including the one offered by [**1**, Prop. 2] or its special case for groups [**9**, Lemma 1.1], the system eventually considered consists of 130 equations for 108 unknowns. Since this system proved to be unsolvable, $\mu_{13}^d$ fails in $\mathbf{Con}(\mathbf{M})$.

(A) Assume that $\alpha_1, \alpha_2, \dots, \alpha_{13}$ are congruences of a metaabelian group $G \in \mathbf{M}$ and $y_1$ is an element of $[1]p$, the $p(\alpha_1, \alpha_2, \dots, \alpha_{13})$-block of the group unit 1. From the permutability of group congruences and $(1, y_1) \in p$ we infer that there exists an element $y_2 \in G$ such that $(1, y_2) \in \alpha_{12}$ and $(y_2, y_1) \in \alpha_{13}$. Parsing the lattice term $p$ further we obtain elements $y_3, y_4, \dots, y_{13} \in G$ such that

$$\begin{array}{llll} (1, y_4) \in \alpha_4, & (y_4, y_3) \in \alpha_5, & (y_3, y_5) \in \alpha_1, & (y_5, y_7) \in \alpha_6, \\ (y_7, y_1) \in \alpha_7, & (y_3, y_6) \in \alpha_2, & (y_6, y_9) \in \alpha_8, & (y_9, y_1) \in \alpha_9, \\ (y_3, y_8) \in \alpha_3, & (y_8, y_{10}) \in \alpha_{10}, & (y_{10}, y_1) \in \alpha_{11}. & \end{array}$$

Consider the group elements

$$\begin{aligned} f_1 &= y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][y_3, y_6]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_2 &= y_3^{-1} y_5^{-1} y_6 y_8 [y_1, y_3][y_1, y_5]^{-1}[y_2, y_5][y_2, y_8]^{-1}[y_3, y_5]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_3 &= y_3^{-1} y_6 y_8 [y_2, y_3][y_2, y_8]^{-1}[y_3, y_9][y_6, y_9]^{-1}, \\ f_4 &= y_1 y_5^{-1} y_8 [y_2, y_5][y_2, y_8]^{-1}. \end{aligned}$$

We claim that

$$\begin{array}{lllll} & (1, f_2) \in q_1, & (f_2, f_1) \in q_{11}, & (f_2, f_1) \in q_{10}, & \\ & (f_1, y_1) \in q_{12}, & (f_1, y_1) \in q_{13}, & (f_2, f_3) \in q_3, & \\ (7) & (f_2, f_3) \in q_2, & (f_3, y_1) \in q_4, & (f_3, y_1) \in q_5, & (f_2, f_4) \in q_6, \\ & (f_2, f_4) \in q_7, & (f_4, y_1) \in q_8, & (f_4, y_1) \in q_9. & \end{array}$$

Each of the relations of (7) follows easily from (1) and the definitions. E.g., to verify $(f_2, f_1) \in q_{11}$ we can compute as follows. Since 1, $y_4$ and $y_3$ are pairwise congruent modulo $q_{11}$ and so are $y_1$ and $y_8$ we obtain

$$\begin{aligned} f_2 \; q_{11} \; & 1^{-1} y_5^{-1} y_6 y_1 [y_1, 1][y_1, y_5]^{-1}[y_2, y_5][y_2, y_1]^{-1}[1, y_5]^{-1}[1, y_9][y_6, y_9]^{-1} = \\ & y_5^{-1} y_6 y_1 [y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\ & y_5^{-1} y_1 y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\ & y_1 y_5^{-1}[y_1, y_5^{-1}]^{-1} y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} = \\ & y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1} \quad \text{and} \\ f_1 \; q_{11} \; & y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][1, y_6]^{-1}[1, y_9][y_6, y_9]^{-1} = \\ & y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1}, \end{aligned}$$

showing $(f_2, f_1) \in q_{11}$. From (7) it follows that $(1, y_1) \in q$. Therefore the $p$-class of 1 is included in the $q$-class of 1. By the canonical bijection between group congruences and normal subgroups we conclude that $\mu_{13}$ holds in $\mathbf{Con}(\mathbf{M})$.

**Problem.** Note that, in spite of some particular positive results of Haiman [**4**], it is still an open question if the variety generated by all linear lattices is self-dual. Thus it would be interesting to know if $\mu_{13}$ holds in every linear lattice, but we do not know even if it holds in the normal subgroup lattice of any group.

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