Acta Math. Univ. Comenianae
Vol. LXIII, 1(1994), pp. 141–153

ON THE CHARACTERIZATION OF PRIMAL PARTIAL ALGEBRAS BY STRONG REGULAR HYPERIDENTITIES

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1. Introduction

An identity $w \approx w'$ is called a hyperidentity in a total algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ iff $w = w'$ holds identically in $\underline{A}$ for every choice of term functions of $\underline{A}$ to represent the operation symbols of the corresponding arities appearing in w and w'. Hyperidentities of the algebra $\underline{A}$ correspond to the identities of the clone of term functions $T(\underline{A})$ of $\underline{A}$. A clone is a superposition closed set of finitary operations on a fixed set A containing all projections. Maximal clones of total operations are very important for primality (completeness) and are fully known ([**Ros; 65**], [**Ros; 70**]). In [**De-Pö; 88**] primality criteria were given by using hyperidentities satisfied in the maximal clones. Clones of partial operations also play an important role in the theory of partial algebras and in computer science (cf. e.g. [**Bur; 86**]).

Recently maximal partial clones were completely described by combinatorial properties ([**Had-Ros**]).

In this paper we introduce the concept of a strong regular hyperidentity for partial algebras. Then this concept will be used to get primality criteria for partial algebras. We give some strong regular hyperidentities satisfied by maximal partial clones. Further, two-element primal partial algebras are characterized by strong regular hyperidentities.

2. Preliminaries

Let A be a finite non-empty set. For every positive integer n, an n-ary partial operation on A is a map $f\colon D_f \to A$ where $D_f \subseteq A^n$. Denote by $P_A^{(n)}$ the set of all n-ary partial operations on A and put $P_A := \bigcup\limits_{n\geq 1} P_A^{(n)}$. Let $O_A \subset P_A$ be the set of all total operations defined on A.

Received January 1, 1993; revised November 30, 1993.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 08A55.

For $n, m \geq 1$, $f \in P_A^{(n)}$ and $g_1, \ldots, g_n \in P_A^{(m)}$, we define the superposition of f and $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n) \in P_A^{(m)}$, by setting

$$D_{f(g_1,\ldots,g_n)} := \{(a_1, \ldots, a_m) \in A^m : (a_1, \ldots, a_m) \in \bigcap_{i=1}^{n} D_{g_i} \\ \wedge (g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)) \in D_f\};$$

and

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in D_{f(g_1,\ldots,g_n)}$.

Let $J_A := \{e_i^n : 1 \leq i \leq n < \omega\}$ be the set of all total projections.

Definition 2.1. A partial clone C on A is a superposition closed subset of P_A containing J_A. If a partial clone C contains an n-ary operation f with $D_f \neq A^n$ then it is called proper partial. A clone C is called total if $C \subseteq O_A$.

Equivalently, partial clones can be regarded as subalgebras of the algebra $(P_A; *, \xi, \tau, \Delta, e_1^2)$ of the type $(2, 1, 1, 1, 0)$ where the operations $*, \xi, \tau, \Delta$ are defined in the following manner:
For $n, m \geq 1$, $f \in P_A^{(n)}$, $g \in P_A^{(m)}$, $t = m + n - 1$, $D_h = \{(a_1, \ldots, a_t) \in A^t : (a_1, \ldots, a_m) \in D_g \wedge (g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_t) \in D_f\}$ $h = f * g$ is defined by

$$h(a_1, \ldots, a_t) := f(g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_t)$$

for all $(a_1, \ldots, a_t) \in D_h$.
$\xi(f) \in P_A^{(n)}$, $\tau(f) \in P_A^{(n)}$, $\Delta(f) \in P_A^{(n-1)}$ are defined by

$$\begin{aligned} D_{\xi(f)} =& \{(a_1, \ldots, a_n) : (a_2, \ldots, a_n, a_1) \in D_f\}, \\ & \xi(f)(a_1, \ldots, a_n) = f(a_2, \ldots, a_n, a_1), \\ D_{\tau(f)} =& \{(a_1, \ldots, a_n) : (a_2, a_1, \ldots, a_n) \in D_f\}, \\ & \tau(f)(a_1, \ldots, a_n) = f(a_2, a_1, \ldots, a_n), \\ D_{\Delta(f)} =& \{(a_1, \ldots, a_{n-1}) : (a_1, a_1, a_2, \ldots, a_{n-1}) \in D_f\}, \\ & \Delta(f)(a_1, \ldots, a_{n-1}) = f(a_1, a_1, a_2, \ldots, a_{n-1}) \text{ if } n > 1, \end{aligned}$$

and $\Delta(f)(x) = \xi(f)(x) = \tau(f)(x) = f(x)$ for $n = 1$. e_1^2 is the binary projection on the first component.

The set of all partial clones on A, ordered by inclusion, forms an algebraic lattice $\mathcal{L}_A$ in which arbitrary infimum is the set-theoretical intersection. For $F \subseteq P_A$ the partial clone $\langle F \rangle$ generated by F is the least partial clone containing F.

Definition 2.2. A set $(f_i^A)_{i\in I}$ of partial operations is complete (or the partial algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ is primal) if $\langle (f_i^A)_{i\in I}\rangle = P_A$.

Let $\underline{A} = (A; (f_i^A)_{i\in I})$ be a partial algebra of a given type τ, i.e. a pair consisting of the set A and an indexed set of partial operations defined on A. The set $(f_i^A)_{i\in I}$ of partial operations corresponds to a set $(f_i)_{i\in I}$ of operation symbols of the type τ. To every partial algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ we can assign the partial clone generated by $(f_i^A)_{i\in I}$ denoted by $T(\underline{A})$. $T(\underline{A})$ is called clone of the algebra $\underline{A}$.

One of the striking differences between the total and the partial case is the fact that $T(\underline{A})$ does not agree with the set of all partial functions induced by terms constructed from the operation symbols $(f_i)_{i\in I}$ and variables.

Let $W_\tau(X)$ be the set of all terms constructed from variables $X = \{x_0, \dots\}$ and operation symbols of the type τ.

Every n-ary term w induces an n-ary term function w^A of the partial algebra $\underline{A}$ such that $w^A(x_1, \dots, x_n)$ is defined exactly by the following rules:

(i) If $w = x_i$ then w^A is everywhere defined and $w^A(x_1, \dots, x_n) = e_i^n(x_1, \dots, x_n) = x_i$ for all $x_1, \dots, x_n \in A$, $n \geq i$,

(ii) If $w = f(w_1, \dots, w_m)$ where f is an m-ary operation symbol and $w_1^A, \dots, w_m^A$ are the n-ary term functions induced by $w_1, \dots, w_m$ and $w_i^A(x_1, \dots, x_n)$ are defined and $w_i^A(x_1, \dots, x_n) = b_i$ $(1 \leq i \leq m)$ and $f^A(b_1, \dots, b_m)$ is defined then $w^A(x_1, \dots, x_n) = f^A(b_1, \dots, b_m)$.

Remark that in general in the partial case the set of all functions induced by terms is a proper subset of the clone of the partial algebra.

Some partial clones can be defined by relations.

Definition 2.3. Let h and n be positive integers, let ρ be an h-ary relation on A (i.e. $\rho \subseteq A^h$) and let f be an n-ary partial operation. We say that f preserves ρ if from $(a_{11}, \dots, a_{1h}) \in \rho, \dots, (a_{n1}, \dots, a_{nh}) \in \rho$ and $(a_{11}, \dots, a_{n1}) \in D_f, \dots, (a_{1h}, \dots, a_{nh}) \in D_f$ it follows $(f(a_{11}, \dots, a_{nh}), \dots, f(a_{1h}, \dots, a_{nh})) \in \rho$.

Let POL ρ denote the set of all $f \in P_A$ preserving ρ. (Clearly POL ρ is a partial clone).

Example. Let $0 \in A$ and let

$$\mathrm{POL}\,\{0\} := \bigcup_{n\geq 1} \{\, f \in P_A^{(n)} : (0, \dots, 0) \in D_f \Rightarrow f(0, \dots, 0) = 0 \,\},$$

then POL $\{0\}$ is a proper partial clone on A. (Remark that in the total case we write Pol ρ.)

Definition 2.4. A maximal partial clone is a coatom (dual atom) of the lattice $\mathcal{L}_A$ of all partial clones on A.

The knowledge of all maximal partial clones is basic for finding a general completeness criterion. For $|A| = 2$ all maximal partial clones are determined by

Freivald ([**Frei; 66**]) and for $|A| = 3$ by Romov ([**Rom; 80**]) and Lau ([**Lau; 77**]). For an arbitrary $|A| \geq 2$ Haddad and Rosenberg ([**Had-Ros; 87**], [**Had-Ros**]) determined all maximal partial clones.

Let $f, g \in P_A^{(n)}$. Then g is called a subfunction of f, symbolically $g \leq f$ if $D_g \subseteq D_f$ and if $f|_{D_g} = g$. A partial clone $C \subseteq P_A$ is strong if it is closed under taking subfunctions. Clearly, the set POL ρ is a strong partial clone for a h-ary relation ρ. Let M be a maximal partial clone on A. Then there are the following cases:

(1) If M is not strong then $M = O_A \cup \{o^n \mid n \in \mathbb{N}^*\}$, where o^n is the nowhere defined n-ary operation (the n-ary partial function with empty domain, $\mathbb{N}^*$ is the set of all positive integers).
(2) If M is strong then $M = \mathrm{POL}\,\rho$ where ρ is a relation from one of the classes described in [**Had-Ros; 87**] or in [**Had-Ros**].

For $A = \{0,1\}$ there are exactly the following maximal partial clones.

Theorem 2.5 ([**Frei; 66**]). *$\mathcal{L}_{\{0,1\}} =: \mathcal{L}_2$ is co-atomic and has exactly 8 co-atoms:*

(1) $M = O_A \cup \{o^n \mid n \in \mathbb{N}^*\}$
(2) $\mathrm{POL}\,\{0\}$
(3) $\mathrm{POL}\,\{1\}$
(4) $\mathrm{POL}\,\{(0,1)\}$
(5) $\mathrm{POL}\,\{(0,0), (0,1), (1,1)\}$
(6) $\mathrm{POL}\,\{(0,1), (1,0)\}$
(7) $\mathrm{POL}\,R_1$ *with* $R_1 = \{(x,x,y,y) : x,y \in \{0,1\}\} \cup \{(x,y,y,x) : x,y \in \{0,1\}\}$,
(8) $\mathrm{POL}\,R_2$ *with* $R_2 = R_1 \cup \{(x,y,x,y) : x,y \in \{0,1\}\}$.

A subset F of $P_2 := P_{\{0,1\}}$ is complete iff F is not contained in one of the maximal partial clones listed in (1)–(8).

3. Strong Regular Hyperidentities of Partial Algebras

Let $\underline{A} = (A; (f_i^A)_{i \in I})$ be a partial algebra of a given type τ and let w, w' be terms of this type. Assume that in w and in w' the same variables occur.

Definition 3.1. $w \approx w'$ is called a strong regular identity of $\underline{A}$ if the partial functions w^A and w'^A induced by w and w' agree (and if w and w' contain the same variables). That means: $w \approx w'$ is a strong identity of $\underline{A}$ if the right hand side is defined whenever the left hand side is defined and if both sides agree ($\underline{A} \models_s w \approx w'$).

Definiton 3.2. The strong regular identity $w \approx w'$ of $\underline{A}$ is called a strong regular hyperidentity of the partial algebra $\underline{A}$ if for every substitution of n-ary functions from $T(\underline{A})$ for n-ary operation symbols of w, w' the result is a strong regular identity of $\underline{A}$: $\underline{A} \underset{s\ hyp}{\models} w \approx w'$ or $T(\underline{A}) \underset{s\ hyp}{\models} w \approx w'$.

4. One-point Extension

At first we will investigate hyperidentities built up from unary operation symbols only (unary hyperidentities). In the total case such hyperidentities are studied in [**De-Pö; 88**]. To use these results we need a method to come from the partial to the total case.

Let $f \in P_A^{(n)}$ and $\infty \notin A$. Then we define a total function (the one-point-extension of f) $f^+ \colon B^n \to B$ with $B = A \cup \{\infty\}$ by

$$f^+(a_1, \dots, a_n) = \begin{cases} f(a_1, \dots, a_n), & \text{if } (a_1, \dots, a_n) \in D_f \\ \infty, & \text{otherwise.} \end{cases}$$

f^+ is a function defined on B with the property $f^+(\infty, \dots, \infty) = \infty$, i.e. $f^+ \in \mathrm{POL}\,\{\infty\} \subset O_B$.

Every strong (regular) hyperidentity of a partial algebra $\underline{A}$ which contains only unary operation symbols corresponds to an identity of the monoid of all unary operations of the clone of the partial algebra $\underline{A}$; i.e. of $\underline{T}^{(1)}(A) = (T^{(1)}(\underline{A}); \circ; id_A)$ where $\circ$ is the composition of unary functions and id_A is the identity function defined on A (clearly, identities and hyperidentities built up from unary operation symbols only are regular).

Remark that the mapping $+ \colon P_A \longrightarrow O_B$ defined by $f \mapsto f^+$ is no clone embedding since for $a \neq \infty$ we have $e_1^2(a, \infty) = \infty \neq e_1^{2,B}(a, \infty) = a$. Let $\underline{A} = (A; (f_i^A)_{i \in I})$ be a partial algebra and let $T^{(1)}(\underline{A})$ be the set of all unary operations of the clone of the algebra $\underline{A}$. Consider $(T^{(1)}(\underline{A}))^+ = \{f^+ \mid f \in T^{(1)}(\underline{A})\}$; i.e. the set of all one-point-extensions of unary operations of the clone of the algebra $\underline{A}$.

Proposition 4.1. $(T^{(1)}(\underline{A}))^+$ *is closed with respect to the composition of total functions.*

Proof. Let $f_1^+ \in (T^{(1)}(\underline{A}))^+$ and $f_2^+ \in (T^{(1)}(\underline{A}))^+$, i.e. $f_1 \in T^{(1)}(\underline{A})$ and $f_2 \in T^{(1)}(\underline{A})$. Since $T(\underline{A})$ is a clone we get $f_1 \circ f_2 \in T^{(1)}(\underline{A})$ and $(f_1 \circ f_2)^+ \in (T^{(1)}(\underline{A}))^+$. $(f_1 \circ f_2)^+$ is defined by

$$(f_1 \circ f_2)^+(a) = \begin{cases} (f_1 \circ f_2)(a), & \text{if } a \in D_{f_1 \circ f_2} \\ \infty, & \text{otherwise} \end{cases} = \begin{cases} f_1(f_2(a)), & \text{if } a \in D_{f_1 \circ f_2} \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly, $a \in D_{f_1 \circ f_2}$ iff $a \in D_{f_2}$ and $f_2(a) \in D_{f_1}$. $a \in D_{f_2}$ means $f_2(a) = f_2^+(a)$.

Consequently,

$$\begin{aligned} (f_1 \circ f_2)^+(a) &= \begin{cases} f_1(f_2^+(a)), & \text{if } a \in D_{f_2} \text{ and } f_2(a) \in D_{f_1} \\ \infty, & \text{otherwise} \end{cases} \\ &= \begin{cases} f_1(f_2^+(a)), & \text{if } f_2^+(a) \in D_{f_1} \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Further, if $f_2^+(a) \in D_{f_1}$ then $f_1(f_2^+(a)) = f_1^+(f_2^+(a))$ and $(f_1 \circ f_2)^+(a) = (f_1^+ \circ f_2^+)(a) = \infty$ if $f_2^+(a) \notin D_{f_1}$. It follows that $(f_1 \circ f_2)^+ = f_1^+ \circ f_2^+ \in (T^{(1)}(\underline{A}))^+$. □

Proposition 4.2. $\underline{(T^{(1)}(A))}^{(+)} = ((T^{(1)}(\underline{A}))^+; \circ, id_A)$ *is isomorphic to* $\underline{T^{(1)}(A)} = (T^{(1)}(\underline{A}); \circ, id_A)$.

Proof. Because of $(f \circ g)^+ = f^+ \circ g^+$ we have only to show that the mapping $+ \colon T^{(1)}(\underline{A}) \to (T^{(1)}(\underline{A}))^+$ defined by $f \mapsto f^+$ is one-to-one. By the definition of $(T^{(1)}(\underline{A}))^+$ the mapping $+$ is surjective.

Let $f^+ = g^+$, i.e. $f^+(a) = g^+(a)$ for all $a \in B = A \cup \{\infty\}$. If $a \in D_g \cap D_f$ then $f(a) = f^+(a) = g^+(a) = g(a)$. $a \in D_g$ and $a \notin D_f$ is impossible since otherwise $g(a) = g^+(a) \neq \infty$, but $f^+(a) = \infty$. Similarly $a \in D_f$, $a \notin D_g$ is impossible. If $a \notin D_f$ and $g \notin D_g$ then $f^+(a) = g^+(a) = \infty$ and g and f are not defined. This means that $f = g$. □

This isomorphism shows that to every unary hyperidentity satisfied in $\underline{A}$ there corresponds a unary hyperidentity satisfied in $\underline{A^+}$.

5. Unary Hyperidentities Satisfied in Maximal Partial Clones

We mention some definitions and results of [**De-Pö; 88**] concerning unary hyperidentities of total algebras.

Definition 5.1. Let $A = \{1, \dots, n\}$, $n > 1$, and let f be a total unary function defined on A. f^m is defined as the m-fold composition of f with $f^o := id$. Put $\kappa(n) = lcm\,\{1, \dots, n\}$. We set $\operatorname{Im} f := \{f(a) : a \in A\}$ (image of f). Let $\lambda(f)$ denote the least non-negative integer m such that $\operatorname{Im} f^m = \operatorname{Im} f^{m+1}$ and put $o(f) = \operatorname{ordn}(f|\operatorname{Im} f^{\lambda(f)})$ where $f|\operatorname{Im} f^{\lambda(f)}$ is a permutation and $\operatorname{ordn}(g)$ is the order of a permutation.

Proposition 5.2. *Let* $|A| = n$ *and* $f \in O_A^{(1)}$. *Then*

(i) $o(f)|\kappa(n)$,
(ii) $0 \leq \lambda(f) \leq |Im\, f|$ *and* $\lambda(f) \leq n - 1$,
(iii) $\lambda(f) = 0 \Leftrightarrow f$ *is a permutation,*
(iv) $\lambda(f) = n - 1 \Leftrightarrow$ *there exists an element* $d \in A$ *such that*

$$A = \{d, f(d), f^2(d), \dots, f^{n-1}(d), f^n(d) = f^{n-1}(d)\},$$

(v) *If* $m' \geq m$ *and* $p|p'$ *then* $f^m = f^{m+p}$ *implies* $f^{m'} = f^{m'+p'}$,
(vi) $f^m = f^{m'} \Leftrightarrow m, m' \geq \lambda(f)$ *and* $m \equiv m' \bmod o(f)$,
(vii) $f^{\lambda(f)} = f^{\lambda(f)+o(f)} = f^{\lambda(f)+\kappa(n)}$.

Further in [**De-Pö; 88**] the following was proved.

Lemma 5.3.

(i) $O_A \underset{hyp}{\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$.

$O_A \underset{hyp}{\not\models} \varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$.

$O_A \underset{hyp}{\not\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x)$, *if* $\kappa(n) \neq \kappa(n-1)$, *i.e. if* n *is a prime power.*

(ii) $\operatorname{Pol} B \underset{hyp}{\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x) \quad (\emptyset \neq B \subset A)$,

(iii) $\operatorname{Pol} \rho_s \underset{hyp}{\models} \varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$ *with* $\rho_s = \{(a, s(a)) : a \in A$ *and* s *is a permutation on* $A\}$.

Remark that if M is a total maximal clone containing not all unary functions then

$$M \underset{hyp}{\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x) \quad \text{or} \quad M \underset{hyp}{\models} \varphi^{n-2}(x) \approx^{n-2+\kappa(n)} (x).$$

Using Lemma 5.3 and Proposition 5.2 we get:

Theorem 5.4.

(i) $P_A \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n)}(x)$, $P_A \underset{s\ hyp}{\not\models} \varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$, *(if* $\kappa(n) \neq \kappa(n-1)$, *i.e. if* n *is a prime power),* $P_A \underset{s\ hyp}{\not\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$,

(ii) *If* f *is a proper partial function, i.e.* $\emptyset \neq D_f \subset A$, *then* $f^n(x) = f^{n+\kappa(n-1)}(x)$,

(iii) *For the non-strong maximal partial clone we get* $O_A \cup \{o^n \mid n \in \mathbb{N}^*\} \underset{s\ hyp}{\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$,

(iv) *Let* M *be a strong maximal partial clone and let* M_t *be the subclone of all total operations of* M. *If* $M_t \underset{s\ hyp}{\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n-1)}(x)$ *then* $M \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$.

Proof.

(i) $(P_A^{(1)})^+$ is included in $\operatorname{Pol}\{\infty\} \subset O_B$. Therefore, by Lemma 5.3(ii) we have $(P_A^{(1)})^+ \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n)}(x)$ and Proposition 4.2 shows $P_A^{(1)} \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n)}(x)$ and therefore $P_A \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n)}(x)$. $\varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$ is not satisfied for certain permutations, therefore $P_A \underset{s\ hyp}{\not\models} \varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$. By Proposition 5.2(iv) $\lambda(f^+) = n$ iff there exists an element $d \in B$ such that $B = \{d, f^+(d), \dots, f^{n+1}(d) = (f^+)^n(d)\}$. Then $(f^+)^n(x) \approx (f^+)^{n+\kappa(n)}(x)$ and $f^n(x) = f^{n+\kappa(n)}(x)$ and $P_A \underset{s\ hyp}{\not\models} \varphi^{n-1}(x) \approx \varphi^{n-1+\kappa(n)}(x)$.

(ii) If $D_f \subset A$ then there is an element $i \in \{1, \dots, n\}$ with $f^+(i) = \infty$, therefore $o(f^+) \leq \kappa(n-1)$ and $(f^+)^n(x) = (f^+)^{n+\kappa(n-1)}(x)$ and $f^n(x) = f^{n+\kappa(n-1)}(x)$.

(iii) follows from Lemma 5.3 (i).

(iv) Let M_p be the set of all proper partial operations then $M_p \cup \{o_n \mid n \in \mathbb{N}^*\} \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$. Altogether we have $M \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n-1)}(x)$.

Remark. If for the set of all total functions of a partial clone we have $M_t \underset{s\ hyp}{\models} \varphi^{n-2}(x) \approx \varphi^{n-2+\kappa(n)}(x)$, then altogether we have $M \underset{s\ hyp}{\models} \varphi^n(x) \approx \varphi^{n+\kappa(n)}(x)$ and M cannot be " separated " from P_A by unary hyperidentities with one operation symbol.

6. A Characterization of Two-Element Primal Partial Algebras by Strong Regular Hyperidentities

For the maximal partial subclones (1)–(5) of Theorem 2.5 of P_2 we get the following strong regular hyperidentities (using Theorem 5.4)

$$
\begin{aligned}
M = O_2 \cup \{o^n : n \in \mathbb{N}^*\} &\underset{s\ hyp}{\models} \varphi(x) \approx \varphi^3(x)\\
\mathrm{POL}\,\{0\} &\underset{s\ hyp}{\models} \varphi^2(x) \approx \varphi^3(x)\\
\mathrm{POL}\,\{1\} &\underset{s\ hyp}{\models} \varphi^2(x) \approx \varphi^3(x)\\
\mathrm{POL}\,\{(01)\} &\underset{s\ hyp}{\models} \varphi^2(x) \approx \varphi^3(x)\\
\mathrm{POL}\,\{(00),(01),(11)\} &\underset{s\ hyp}{\models} \varphi^2(x) \approx \varphi^3(x).
\end{aligned}
$$

By Theorem 5.4(i) $P_2 \underset{s\ hyp}{\not\models} \varphi^2(x) \approx \varphi^3(x)$ and $P_2 \underset{s\ hyp}{\not\models} \varphi(x) \approx \varphi^3(x)$, i.e. M, POL $\{0\}$, POL $\{1\}$, POL$\{(01)\}$, POL$\{(00),(01),(11)\}$ can be "separated" from P_2 using these hyperidentities. The maximal clones POL R_1 and POL R_2 contain all unary partial functions defined on $\{0,1\}$. Consequently, these clones cannot be separated from P_2 by unary hyperidentities. But we show

Lemma 6.1. *Let ε be the following regular hyperidentity: $\varepsilon : H_1(x,y) \approx H_2(x,y)$ with*

$$\begin{aligned}
H_1 &= F(F(S_1, F(S_1, S_1)), F(S_1, S_1)),\\
H_2 &= F(F(S_2, S_2), F(F(S_2, S_2), S_2)),\\
S_1 &= F(F(T_1, T_1), F(F(T_1, T_1), T_1)),\\
S_2 &= F(F(T_2, T_2), F(T_2, T_2)),\\
T_1 &= F(F(x, y), F(F(y, x), F(y, y))),\\
T_2 &= F(F(x, y), F(F(F(y, x), F(y, y)), F(x, y)))
\end{aligned}$$

and F is a binary operation symbol.

Then $\mathrm{POL}\,\{(0,1),(1,0)\} \underset{s\ hyp}{\models} \varepsilon$, $\mathrm{POL}\,R_1 \underset{s\ hyp}{\models} \varepsilon$, $\mathrm{POL}\,R_2 \underset{s\ hyp}{\models} \varepsilon$, *but* $P_2 \underset{s\ hyp}{\not\models} \varepsilon$.

Proof. We prove the following facts:

fact1: Every binary partial function on $\{0,1\}$ which is not everywhere defined satisfies ε,

fact2: Every binary total function from $\mathrm{POL}\,\{(0,1),(1,0)\}$, $\mathrm{POL}\,R_1$, and $\mathrm{POL}\,R_2$ satisfies ε,

fact3: There is a binary function from P_2 which does not satisfy ε.

Proof of fact 1:

Consider two cases: case 1: $x = a$, $y = b$, $a, b \in \{0,1\}$, $a \neq b$ and case 2: $x = y = a \in \{0,1\}$.

Case 1: Let f be a binary partial function on $\{0,1\}$ which is not everywhere defined. If f is not defined on one of the pairs (a,b), (b,a), (b,b) then T_1 and T_2 are not defined and H_1 and H_2 are not defined. If f is defined on (a,b), (b,a), (b,b), then f is not defined on (a,a). For T_1 and T_2 we have:

$$\begin{aligned}
T_1 &= f(f(a,b), f(f(b,a), f(b,b))) \quad \text{and}\\
T_2 &= f(f(a,b), f(f(f(b,a), f(b,b)), f(a,b))).
\end{aligned}$$

Assume that $f(a,b) = a$. If $f(f(b,a), f(b,b)) = a$ then T_1 and T_2 are not defined and therefore H_1 and H_2 are not defined. If $f(f(b,a), f(b,b)) = b$, then $T_1 = a$ and $T_2 = f(a, f(b,a))$. If $f(b,a) = a$ then T_2 is not defined. If $f(b,a) = b$ then $T_2 = a$. If $T_1 = a$ then S_1 and H_1 are not defined. If $T_2 = a$ or T_2 is not defined then S_2 and H_2 are not defined.

Now we assume that $f(a,b) = b$. If $f(f(b,a), f(b,b)) = b$ then $T_1 = f(b,b)$ and $T_2 = f(b, f(b,b))$. If $f(b,b) = b$ then $T_1 = T_2 = S_1 = S_1 = b$ and thus $H_1 = H_2 = b$. If $f(b,b) = a$ then $T_1 = a$, $T_2 = f(b,a)$. Then S_1 and H_1 are not defined. If $f(b,a) = a$ then S_2 and H_2 are not defined and if $T_2 = f(b,a) = b$ then

$S_2 = f(f(b,b), f(b,b)) = f(a,a)$ is not defined and therefore H_2 is not defined.
Let us assume that $f(f(b,a), f(b,b)) = a$. It follows that $T_1 = f(b,a)$ and $T_2 = f(b, f(a,b)) = f(b,b)$. Now we discuss all possibilities for $f(b,a)$ and $f(b,b)$. $f(b,a) = f(b,b) = a$ is impossible since otherwise $f(f(b,a), f(b,b)) = f(a,a)$ would not be defined. $f(b,a) = a$, $f(b,b) = b$ is also impossible since otherwise $f(f(b,a), f(b,b)) = f(a,b) = a$ in contradiction to the presumption $f(a,b) = b$. $f(b,a) = b, f(b,b) = a$ leads to a contradiction because of $f(f(b,a), f(b,b)) = f(b,a) = a = b$. The last case is that $f(b,a) = b, f(b,b) = b$. Then we have

$$T_1 = f(b, f(b,b)) = b,$$
$$T_2 = f(b, f(f(b,b), b)) = f(b, f(b,b)) = b.$$

Further we get $S_1 = S_2 = b$ and $H_1 = H_2 = b$. If $f(f(b,a), f(b,b))$ is not defined then T_1, T_2, S_1, S_2, H_1, H_2 are not defined.

Case 2: If $x = y = a$ then

$$T_1 = f(f(a,a), f(f(a,a), f(a,a))),$$
$$T_2 = f(f(a,a), f(f(f(a,a), f(a,a)), f(a,a))).$$

If f is not defined on (a,a) then T_1, T_2, S_1, S_2 and H_1, H_2 are not defined. Assume $f(a,a)$ is defined.
If $f(a,a) = a$ then $T_1 = f(a,a) = a$, $T_2 = f(a, f(f(a,a), a)) = a$, $S_1 = S_2 = a$ and $H_1 = H_2 = a$.
If $f(a,a) = b$ then $T_1 = f(b, f(b,b))$ and $T_2 = f(b, f(f(b,b), b))$.
Assume that $f(b,b)$ is not defined. Then T_1, S_1, H_1 and T_2, S_2, H_2 are not defined.
If $f(b,b) = b$ then $T_1 = b$, $T_2 = b$, $S_1 = S_2 = b$ and $H_1 = H_2 = b$. If $f(b,b) = a$ then $T_1 = f(b, f(f(a,a), f(a,a))) = f(b, f(b,b)) = f(b,a)$ and $T_2 = f(f(a,a), f(f(f(a,a), f(a,a)), f(a,a))) = f(b, f(f(b,b), b)) = f(b, f(a,b))$.
If $f(b,a)$ is defined then $f(a,b)$ is not defined and therefore T_2, S_2 and H_2 are not defined.
Assume that $f(b,a) = b$. Then $T_1 = b$, $S_1 = f(f(b,b), f(f(b,b), b)) = f(a, f(a,b))$ and S_1 is not defined. Consequently, H_1 is not defined.
Now assume that $f(b,a) = a$. Then $T_1 = a$ and $S_1 = f(f(a,a), f(f(a,a), a)) = f(b, f(b,a)) = f(b,a) = a$ and $H_1 = f(f(a, f(a,a)), f(a,a)) = f(f(a,b), b))$ is not defined.
If $f(b,a)$ is not defined then T_1, S_1, H_1 are not defined. If then $f(a,b)$ is not defined then T_2, S_2, H_2 are not defined. If $f(a,b) = a$ then $T_2 = f(b,a)$ is not defined and thus H_2 is not defined. If $f(a,b) = b$ then $T_2 = f(b,b) = a$. $S_2 = f(f(a,a), f(a,a)) = f(b,b) = a$ and $H_2 = f(f(a,a), f(f(a,a), a)) = f(b, f(b,a))$ is not defined.

Proof of fact 2: $e_1^2, e_2^2, Ne_1^2, Ne_2^2$ are all total binary functions of POL $\{(0,1),(1,0)\}$ (N denotes the negation), i.e. $O_2 \cap \mathrm{POL}\,\{(0,1),(1,0)\} = \{e_1^2, e_2^2, Ne_1^2, Ne_2^2\}$. e_1^2,

e_2^2, Ne_1^2, Ne_2^2, c_0^2, c_1^2 belong to POL R_1 and also to POL R_2. These functions are all binary functions not depending essentially on both variables.

Let f be a binary function defined on $\{0,1\}$. If $f \in$ POL R_1 then from $(x_1, y_1, y_1, x_1) \in R_1$, $(x_2, x_2, y_2, y_2) \in R_1$ for $x_1 \neq y_1$, $x_2 \neq y_2$ we obtain $(f(x_1, x_2)$, $f(y_1, x_2)$, $f(y_1, y_2)$, $f(x_1, y_2)) \in R_1$, i.e. $f(x_1, x_2) = f(y_1, x_2)$ and $f(y_1, y_2) = f(x_1, y_2)$ or $f(x_1, x_2) = f(x_1, y_2)$ and $f(y_1, x_2) = f(y_1, y_2)$. It follows $f \in \{e_1^2, e_2^2, c_0^2, c_1^2, Ne_1^2, Ne_2^2\}$. Consequently $O_2 \cap \mathrm{POL}^{(2)} R_1 = \{e_1^2, e_2^2, c_0^2, c_1^2, Ne_1^2, Ne_2^2\}$. It is easy to check that $x + y$, $N(x + y) \in$ POL R_2.

If f is a binary function and $f \in$ POL R_2 then from $(x_1, x_1, y_1, y_1) \in R_2$, $(x_2, y_2, y_2, x_2) \in R_2$, $(x_1, y_1, x_1, y_1) \in R_2$, $(x_2, y_2, y_2, x_2) \in R_2$ we get $(f(x_1, x_2)$, $f(x_1, y_2)$, $f(y_1, y_2)$, $f(y_1, x_2)) \in R_2$ and $f(x_1, x_2)$, $f(y_1, y_2)$, $f(x_1, y_2)$, $f(y_1, x_2)) \in R_2$, i.e.

$$\begin{array}{lcll} f(x_1,x_2)=f(x_1,y_2) & \text{and} & f(y_1,y_2)=f(y_1,x_2) & \text{or}\\ f(x_1,x_2)=f(y_1,x_2) & \text{and} & f(x_1,y_2)=f(y_1,y_2) & \text{or}\\ f(x_1,x_2)=f(y_1,y_2) & \text{and} & f(x_1,y_2)=f(y_1,x_2). & \end{array}$$

Similarly,

$$\begin{array}{lcll} f(x_1,x_2)=f(y_1,y_2) & \text{and} & f(x_1,y_2)=f(y_1,x_2) & \text{or}\\ f(x_1,x_2)=f(y_1,y_2) & \text{and} & f(y_1,y_2)=f(x_1,y_2) & \text{or}\\ f(x_1,x_2)=f(x_1,y_2) & \text{and} & f(y_1,y_2)=f(y_1,x_2). & \end{array}$$

It follows $f \in \{e_1^2, e_2^2, c_0^2, c_1^2, Ne_1^2, Ne_2^2, x + y, N(x + y)\}$ and therefore $O_2 \cap \mathrm{POL}^{(2)} R_2 = \{e_1^2, e_2^2, c_0^2, c_1^2, Ne_1^2, Ne_2^2, x+y, N(x+y)\}$. It is easy to check that all these functions fulfil ε.

Using fact 1 and fact 2 we have

$$\mathrm{POL}\,\{(01),(10)\} \underset{s\ hyp}{\models} \varepsilon, \qquad \mathrm{POL}\ R_1 \underset{s\ hyp}{\models} \varepsilon, \qquad \mathrm{POL}\ R_2 \underset{s\ hyp}{\models} \varepsilon.$$

Now we come to fact 3.

Proof of fact 3: Consider the total function $g(x,y) = Nx \wedge Ny$. For $x = y = 1$ we have

$$\begin{aligned} T_1 &= f(f(1,1), f(f(1,1), f(1,1))) = f(0, f(0,0)) = f(0,1) = 0,\\ S_1 &= f(f(0,0), f(f(0,0),0)) = f(1, f(1,0)) = f(1,0) = 0,\\ H_1 &= f(f(0, f(0,0)), f(0,0)) = f(f(0,1),1) = 0,\\ T_2 &= f(f(1,1), f(f(f(1,1), f(1,1)), f(1,1))) = f(0, f(f(0,0),0)) = f(0, f(1,0))\\ &= f(0,0) = 1, \end{aligned}$$

$$S_2 = f(f(1,1), f(1,1)) = f(0,0) = 1 \quad \text{and}$$
$$H_2 = f(f(1,1), f(f(1,1),1)) = f(0, f(0,1)) = f(0,0) = 1,$$

i.e. $H_1 \neq H_2$. This shows $P_2 \underset{hyp}{\not\models} \varepsilon$. □

Now we can prove the following primality criterion:

Theorem 6.2. *A two-element partial algebra $\underline{A} = (\{0,1\}; (f_i^{\{0,1\}})_{i\in I})$ is primal if and only if $\underline{A}$ satisfies none of the following strong regular hyperidentities:*

(i) $\varphi(x) \approx \varphi^3(x)$,
(ii) $\varphi^2(x) \approx \varphi^3(x)$,
(iii) $\varepsilon : H_1(x,y) \approx H_2(x,y)$ *with* H_1, H_2 *defined as in Lemma* 6.1.

Proof. $\underline{A} = (\{0,1\}; (f_i^{\{0,1\}})_{i\in I})$ is primal iff $(f_i^{\{0,1\}})_{i\in I})$ is complete, i.e. $\langle (f_i^{\{0,1\}})_{i\in I}\rangle = P_2$. A subset of P_2 is complete iff it is not contained in any maximal subclone of P_2.

Every clone contained in $M = O_2 \cup \{O^n : n \in \mathbb{N}^*\}$ satisfies $\varphi(x) \approx \varphi^3(x)$. Every clone contained in $\mathrm{POL}\,\{0\}$, $\mathrm{POL}\,\{1\}$, $\mathrm{POL}\,\{(01)\}$, $\mathrm{POL}\,\{(00),(01),(11)\}$ satisfies $\varphi^2(x) \approx \varphi^3(x)$ and every clone contained in $\mathrm{POL}\,\{(00),(01),(11)\}$, $\mathrm{POL}\,R_1$, $\mathrm{POL}\,R_2$ satisfies ε, i.e. if $\underline{A}$ is not primal then at least one of the three hyperidentities is satisfied. If $\underline{A}$ is primal then $(f_i^{\{0,1\}})_{i\in I}$ is not contained in one of the maximal subclones of P_2 and thus none of the hyperidentities is satisfied. □

Remark that it is easier to check a hyperidentity than to find it. The hyperidentity ε was given by Welke ([**Wel; 91**]) using a computer program.

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