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AVOIDANCE IN TRIPLE SYSTEMS

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1. Introduction

A **triple system** $\mathrm{TS}(v,\lambda)$ is a pair $(V,\boldsymbol{B})$ where V is a v-set of **elements**, and $\boldsymbol{B}$ is a collection of 3-subsets of V called **triples** or **lines** such that every 2-subset of V is contained in exactly λ triples. The number v is called the **order** of the triple system and λ its **index**. A triple system of index 1, $\mathrm{TS}(v,1)$, is also called a **Steiner triple system**, $\mathrm{STS}(v)$, and a triple system of index 2, $\mathrm{TS}(v,2)$ is sometimes called a **twofold triple system**, $\mathrm{TTS}(v)$.

If in the above definition of a triple system, "in exactly λ" is replaced by "in at most λ", we have a **partial** triple system $\mathrm{PTS}(v,\lambda)$. We will use the term **configuration** to describe a partial triple system with a small or fixed number of lines. We will often denote a configuration just by $\boldsymbol{C}$ rather than by $(v,\boldsymbol{C})$ when there is no danger of confusion or when it is irrelevant what the actual element-set is. In what follows we will always assume $v \geq 3$.

It is possible for a triple system with $\lambda > 1$ to contain the same triple $\{x,y,z\}$ more than once; such a triple is then termed **repeated**. A $\mathrm{TS}(v,\lambda)$ is **simple** if it contains no repeated triples.

If $\boldsymbol{C}$ is a configuration, we will say that a $\mathrm{TS}(v,\lambda)$ (or a $\mathrm{PTS}(v,\lambda)$) $(V,\boldsymbol{B})$ **contains** $\boldsymbol{C}$ if there exists a $\mathrm{PTS}(U,\boldsymbol{C}')$ with $U \subseteq V$, $\boldsymbol{C}' \subseteq \boldsymbol{B}$ and $\boldsymbol{C} \simeq \boldsymbol{C}'$. Otherwise, $(V,\boldsymbol{B})$ will be said to **avoid** $\boldsymbol{C}$ or **to be without** $\boldsymbol{C}$.

It has been shown by Hanani (cf. [**H**]) that a $\mathrm{TS}(v,\lambda)$ exists if and only if $v \in B(\lambda)$ where $B(\lambda)$ is the set of **admissible values** of v for given λ, i.e. the set of values of v satisfying the obvious arithmetic necessary conditions. Explicitly,

$$B(\lambda) = \begin{cases} \{\, v : v \equiv 1 \text{ or } 3 \pmod 6 & \text{if } \lambda \equiv 1 \text{ or } 5 \pmod 6 \,\}, \\ \{\, v : v \equiv 0 \text{ or } 1 \pmod 3 & \text{if } \lambda \equiv 2 \text{ or } 4 \pmod 6 \,\}, \\ \{\, v : v \equiv 1 \pmod 2 & \text{if } \lambda \equiv 3 \pmod 6 \,\}, \\ \{\, v : v \geq 3 & \text{if } \lambda \equiv 0 \pmod 6 \,\}. \end{cases}$$

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Let now $\boldsymbol{C}$ be a configuration. The **avoidance set** $\Omega(\boldsymbol{C},\lambda)$ for $\boldsymbol{C}$ and given λ is the set

$$\Omega(\boldsymbol{C},\lambda) = \{\, v : v \in B(\lambda) \text{ and } \exists\ \mathrm{TS}(v,\lambda) \text{ without } \boldsymbol{C}\}.$$

Whenever convenient, we may consider $U(\boldsymbol{C},\lambda) = B(\lambda) \setminus \Omega(\boldsymbol{C},\lambda)$, i.e.

$$U(\boldsymbol{C},\lambda) = \{v : v \in B(\lambda) \text{ and } \forall\ \mathrm{TS}(v,\lambda) \text{ contain } \boldsymbol{C}\}.$$

Similarly, if Σ is a set of configurations, the **simultaneous avoidance set** $\Omega(\Sigma,\lambda)$ for Σ and given λ is the set

$$\Omega(\Sigma,\lambda) = \{v : v \in B(\lambda) \text{ and } \exists\ \mathrm{TS}(v,\lambda) \text{ without } \boldsymbol{C} \text{ for all } \boldsymbol{C} \in \Sigma\}.$$

Trivially, $\Omega(\Sigma,\lambda) \subseteq \bigcap_{\boldsymbol{C}\in\Sigma} \Omega(\Sigma,\lambda)$.

Although not in this setting, in at least one case the problem of determining the avoidance set has been attempted in the literature: several papers are devoted to the spectrum problem for anti-Pasch Steiner triple systems [**B**], [**GMP**], [**SW**], a problem which is still not completely settled. In this paper, we determine the avoidance sets for all configurations with up to three lines, leaving just a couple of undecided individual cases in the case of two of the three-line configurations.

2. Avoiding Two-Line Configurations

The following observations are immediate.

Lemma 2.1. *If $\Sigma' \subseteq \Sigma$ are two sets of configurations then $\Omega(\Sigma,\lambda) \subseteq \Omega(\Sigma',\lambda)$ for any λ.*

Lemma 2.2. *If C, C' are two configurations and C contains C' then $\Omega(C',\lambda) \subseteq \Omega(C,\lambda)$.*

In what follows we will often use the following result on the existence of triple systems without repeated triples which was first proved by Dehon [**D**].

Lemma 2.3. *A simple TS(v,λ) exists if and only if $v \in B(\lambda)$ and $\lambda \le v-2$.*

There are exactly 4 nonisomorphic two-line configurations as shown in Fig. 1.

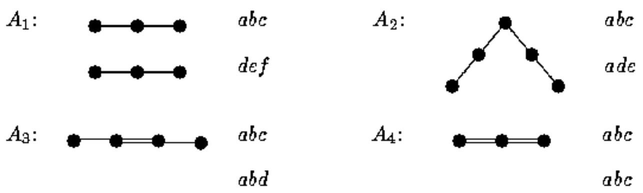

Figure 1

It is easily seen (and well known) that if $(V, \boldsymbol{B})$ is a $\mathrm{TS}(v, \lambda)$ with $v > 7$, there must exist two triples $B, B' \in \boldsymbol{B}$ such that $B \cap B' = \emptyset$. The following theorem is then almost immediate.

Theorem 2.4. *Let A_1, A_2, A_3, A_4 be the four two-line configurations as in Fig. 1. Then*

$$\begin{aligned}
\Omega(A_1, \lambda) &= B(\lambda) \cap \{3,4,5,6,7\},\\
\Omega(A_2, \lambda) &= B(\lambda) \cap \{3,4\},\\
\Omega(A_3, \lambda) &= B(1),\\
\Omega(A_4, \lambda) &= B(\lambda) \cap \{v : v \geq \lambda + 2\}.
\end{aligned}$$

Proof. The assertion of the theorem concerning A_4 is a restatement of Lemma 2.3. The rest is obvious. □

The following result on simultaneous avoidance sets is easy to establish; nevertheless, it will be useful for us in the next section, and so we state it explicitly.

Theorem 2.5.

$$\begin{aligned}
\Omega(\{A_1, A_2\}, \lambda) &= \Omega(A_2, \lambda) = B(\lambda) \cap \{3,4\},\\
\Omega(\{A_1, A_3\}, \lambda) &= \{3,7\},\\
\Omega(\{A_1, A_4\}, \lambda) &= \begin{cases} \{3,7\} & \text{if } \lambda = 1,\\ \{4,6\} & \text{if } \lambda = 2,\\ \{5\} & \text{if } \lambda = 3,\\ \emptyset & \text{if } \lambda \geq 4,\end{cases}\\
\Omega(\{A_2, A_4\}, \lambda) &= \Omega(\{A_1, A_2, A_4\}, \lambda) = \begin{cases} \{3\} & \text{if } \lambda = 1,\\ \{4\} & \text{if } \lambda = 2,\\ \emptyset & \text{if } \lambda \geq 3,\end{cases}\\
\Omega(\{A_2, A_3\}, \lambda) &= \Omega(\{A_1, A_2, A_3\}, \lambda) = \{3\},\\
\Omega(\{A_3, A_4\}, \lambda) &= \begin{cases} B(1) & \text{if } \lambda = 1,\\ \emptyset & \text{if } \lambda \geq 2,\end{cases}\\
\Omega(\{A_1, A_3, A_4\}, \lambda) &= \begin{cases} \{3,7\} & \text{if } \lambda = 1,\\ \emptyset & \text{if } \lambda \geq 2,\end{cases}\\
\Omega(\{A_2, A_3, A_4\}, \lambda) &= \Omega(\{A_1, A_2, A_3, A_4\}, \lambda) = \begin{cases} \{3\} & \text{if } \lambda = 1,\\ \emptyset & \text{if } \lambda \geq 2.\end{cases}
\end{aligned}$$

Proof. An exercise using Lemmas 2.1 and 2.2. □

3. Avoiding Three-Line Configurations

There are exactly 16 nonisomorphic three-line configurations in triple systems shown in Fig. 2. The first five of these contain no repeated pairs.

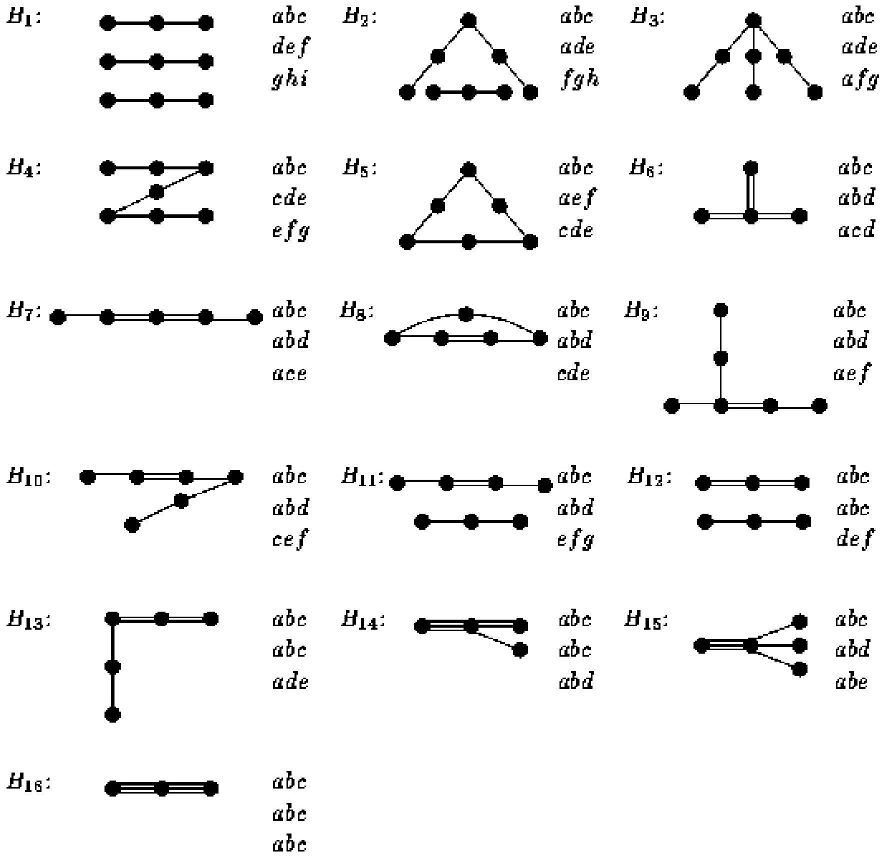

Figure 2.

We now turn to determining the avoidance sets for each B_i, $i = 1, 2, \ldots, 16$. The avoidance set for B_i is finite if $i \in \{1, 2, 3, 4, 5\}$ but is infinite for the remaining configurations.

Denote $N(i, j) = \{n : i \leq n \leq j, n \text{ integer}\}$. The following is a simple but useful lemma.

Lemma 3.1. *Let C be a* PTS(v,λ) *without* A_4 *(i.e., containing no repeated triples). Then* $v \in \Omega(C,\lambda)$ *implies* $v \in \Omega(C, m\lambda)$ *for any integer* $m \geq 1$.

Proof. Take a TS(v,λ) avoiding C, and repeat each triple m times. □

Lemma 3.2. *Any* TS(v,λ) *with* $v \geq 12$ *contains* B_1.

Proof. Let $(V, \boldsymbol{B})$ be a TS(v,λ) with $v \geq 12$. Then $\boldsymbol{B}$ must contain two disjoint triples, say, $\{a,b,c\}$ and $\{d,e,f\}$. Let $Y = V \setminus \{a,b,c,d,e,f\}$, and assume that $\boldsymbol{B}$ avoids B_1; then if $\{x,y,z\} \in \boldsymbol{B}$ and $\{x,y\} \subseteq Y$, we must have $z \in \{a,b,c,d,e,f\}$. A simple counting argument shows that this is impossible if $|Y| \geq 8$, i.e. if $v \geq 14$. Thus it remains only to consider $v \in \{12, 13\}$.

Let first $v = 13$, and let $Y = \{1,2,3,4,5,6,7\}$. Then for $u \in \{a,b,c,d,e,f\}$, the set $F_u = \{\{v,w\} : \{u,v,w\} \in \boldsymbol{B}, \{v,w\} \subset Y\}$ is a λ-factor of λK_7 on Y, and $\boldsymbol{F} = \{F_a, F_b, F_c, F_d, F_e, F_f\}$ is a λ-factorization of λK_7 on Y. It is easily seen that then there exist three edges which are pairwise disjoint and belong to three distinct λ-factors of $\boldsymbol{F}$, say, w.l.o.g., $\{m,n\} \in F_a$, $\{p,q\} \in F_b$, $\{r,s\} \in F_c$. But then $\{a,m,n\}$, $\{b,p,q\}$, $\{c,r,s\}$ are three disjoint triples forming B_1.

Let now $v = 12$. We present the proof for $\lambda = 2$; the proof for an arbitrary $\lambda = 2\mu$ is similar, with only the obvious modifications. Let now $Y = \{1,2,3,4,5,6\}$, and let $\{1,2,u\}$, $\{1,2,v\}$ be the two triples containing $\{1,2\}$. Distinguish the following three cases.

Case 1. $u \in \{a,b,c\}$, $v \in \{d,e,f\}$. Then there is at least one triple $\{x,y,w\}$ with $\{x,y\} \subset \{3,4,5,6\}$, $w \in \{a,b,c,d,e,f\} \setminus \{u,v\}$, say, $w = b$. But then $\{1,2,u\}$, $\{x,y,b\}$, $\{d,e,f\}$ are three parallel triples.

Case 2. $\{u,v\} \subset \{a,b,c\}$, $u \neq v$. Then $\boldsymbol{B}$ contains no triple of the form $\{x,y,w\}$ with $\{x,y\} \subset \{3,4,5,6\}$, $w \in \{a,b,c\}$ since otherwise we would have $\{x,y,w\}$, $\{d,e,f\}$ and $\{1,2,t\}$, $t \in \{u,v\}$, $t \neq w$, as three parallel triples. Thus all triples $\{x,y,w\}$ with $\{x,y\} \subset \{3,4,5,6\}$ must have $w \in \{d,e,f\}$. If $\{x,y,w\}$, $\{r,s,z\} \in \boldsymbol{B}$ with $\{x,y,r,s\} = \{3,4,5,6\}$ and $w \neq z$, $\{w,z\} \subset \{d,e,f\}$ then we have three parallel triples $\{x,y,w\}$, $\{r,s,z\}$ and $\{1,2,u\}$. It follows that the triples containing $\{x,y\} \subset \{3,4,5,6\}$ are, w.l.o.g., $\{3,4,d\}$, $\{5,6,d\}$, $\{3,5,e\}$, $\{4,6,e\}$, $\{3,6,f\}$, $\{4,5,f\}$, each taken twice. But then all 8 triples containing the pairs $\{1,x\}$, $x \in \{3,4,5,6\}$ must have as its third element $w \in \{a,b,c\}$ which is impossible.

Case 3. $u = v \in \{a,b,c\}$. In fact, if neither of the above cases is to occur, then all triples $\{x,y,w\}$ with $\{x,y\} \subset \{1,2,3,4,5,6\}$, $w \in \{a,b,c,d,e,f\}$ are repeated. W.l.o.g., let the triples containing 1 be $\{1,2,a\}$, $\{1,3,b\}$, $\{1,4,c\}$, $\{1,5,d\}$, and let $\{1,6,e\}$, each repeated twice. But then there can be no triple in $\boldsymbol{B}$ containing $\{1,f\}$. □

Theorem 3.3.

(i) $\Omega(B_1, 1) = \{3, 7\}$;
(ii) $\{3, 7\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 7, 9\}$ *if* $\lambda \equiv 1, 5 \pmod 6$, $\lambda > 1$;
(iii) $\Omega(B_1, 2) = \{3, 4, 6, 7, 9\}$;
(iv) $\{3, 4, 6, 7, 9\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 4, 6, 7, 9, 10\}$ *if* $\lambda \equiv 2, 4 \pmod 6$, $\lambda > 2$;
(v) $\{3, 5, 7\} \subseteq \Omega(B_1, 3) \subseteq \{3, 5, 7, 11\}$;
(vi) $\{3, 5, 7\} \subseteq \Omega(B_1, \lambda) \subseteq \{3, 5, 7, 9, 11\}$ *if* $\lambda \equiv 3 \pmod 6$, $\lambda > 3$;
(vii) $N(3, 9) \subseteq \Omega(B_1, \lambda) \subseteq N(3, 11)$ *if* $\lambda \equiv 0 \pmod 6$.

Proof. Trivially, any $\mathrm{TS}(v, \lambda)$ with $v \le 8$ avoids B_1, thus in view of Lemma 3.2 it suffices to consider $v \in \{9, 10, 11\}$. Also Lemma 3.2 proves (ii) and (vi). While the unique STS(9) is resolvable and thus contains four disjoint parallel classes (i.e. $B_1 s$), there exists a TTS(9) without B_1. In fact, exactly five out of 36 nonisomorphic TTS(9) , namely Nos. 13, 22, 34, 35, and 36 in the listing of [**MR**] avoid B_1. This together with Lemma 3.1 implies that $9 \in \Omega(B_1, \lambda)$ for all $\lambda \equiv 0 \pmod 2$. This proves (i), (iv) and (vii). An inspection of the 960 nonisomorphic TTS(10) (cf. [**GGMR**], [**CCHR**]) reveals that each of them contains B_1. This proves (iii). None of the 22 521 nonisomorphic TS(9, 3) avoids B_1 [**ML**] which proves (v). □

Let us remark that some of the undecided cases in Theorem 3.3 could be settled if one knew what is the largest possible index λ in an indecomposable $\mathrm{TS}(9, \lambda)$ with repeated triples.

Theorem 3.4. $\Omega(B_2, \lambda) = (B(\lambda) \cap N(3, 9)) \setminus \{8\}$.

Proof. It is easy to see that any $\mathrm{TS}(v, \lambda)$ with $v \ge 10$ contains B_2. At the same time, clearly any $\mathrm{TS}(v, \lambda)$ with $v \le 7$ avoids B_2, thus it is suffices to consider $v \in \{8, 9\}$. The unique STS(9) avoids B_2, and so $9 \in \Omega(B_2, \lambda)$ for all $\lambda \ge 1$ by Lemma 3.1. Finally, consider a $\mathrm{TS}(8, \lambda)$. It must contain two disjoint triples, say, abc, def. Then any triple through the two remaining elements must have as its third element one of a, b, c, d, e, f, yielding B_2. □

Theorem 3.5. $\Omega(B_3, \lambda) = B(\lambda) \cap N(3, 6)$.

Proof. Trivially, any $\mathrm{TS}(v, \lambda)$ with $v \le 6$ avoids B_3. To show that any $\mathrm{TS}(v, \lambda)$ with $v \ge 7$ must contain B_3, consider the **neighbourhood** of an element x. This is a multigraph whose vertices are elements other than x, and whose edges (with corresponding multiplicities) are pairs yz such that xyz is a triple. A copy of B_3 corresponds to a 3-edge matching in a neighbourhood. This observation immediately implies that any $\mathrm{TS}(v, \lambda)$ with $v \ge 8$ contains B_3. For a $\mathrm{TS}(7, \lambda)$ not to contain B_3, the neighbourhood of **every** element would have to consist (if we suppress the multiplicities) of two disjoint triangles; it is well known that this is impossible (cf. [**CR**]). □

Theorem 3.6. $\Omega(B_4, \lambda) = B(\lambda) \cap N(3,7)$.

Proof. Clearly any $\mathrm{TS}(v, \lambda)$ with $v \leq 6$ avoids B_4. The unique STS(7) avoids B_4, and so $7 \in \Omega(B_4, \lambda)$ for all $\lambda \geq 1$ by Lemma 3.1. Consider now an arbitrary $\mathrm{TS}(8, \lambda)$. It must contain two disjoint lines, say $X = \{a, b, c\}$, $Y = \{d, e, f\}$, and let p, q be the remaining two elements. The element p occurs in λ triples with q, and so it must occur in $5\lambda/2$ triples of type pzz' with $z, z' \in X \cup Y$. Similarly, q must occur in $5\lambda/2$ other triples of type qzz'. If one of these triples were of type pxy or qxy, $x \in X$, $y \in Y$, it would yield together with X and Y a copy of B_4. Assume therefore that there are no triples of type pxy or qxy, thus there are altogether 5λ triples of types pxx, pyy, qxx, qyy. Consider now all triples containing pairs xy, $x \in X$, $y \in Y$. Clearly, there are $9\lambda/2$ such triples, all of type either xxy or xyy. Thus there would by $9\lambda/2 + 5\lambda = 19\lambda/2$ pairs of type xx or yy but there are only $6\lambda - 6$ pairs of type xx or yy available, a contradiction. Thus any $\mathrm{TS}(8, \lambda)$ contains B_4. A proof that any $\mathrm{TS}(v, \lambda)$ with $v \geq 9$ contains B_4 presents no difficulties, and is left to the reader. □

Theorem 3.7. $\Omega(B_5, \lambda) = B(\lambda) \cap N(3,5)$.

Proof. Any $\mathrm{TS}(v, \lambda)$ with $v \leq 5$ avoids B_5. On the other hand, any $\mathrm{TS}(v, \lambda)$ with $v \geq 6$ must contain B_5. Let $(V, \boldsymbol{B})$ be a $\mathrm{TS}(v, \lambda)$. By Theorem 2.4, $\boldsymbol{B}$ must contain a copy of A_2, i.e. two lines, say 123 and 145. Let $X = V \setminus \{1, 2, 3, 4, 5\}$. Assume $\boldsymbol{B}$ does not contain B_5; then $\boldsymbol{B}$ does not contain lines $24x$, $25x$, $34x$, $35x$ with $x \in X$. Let now $x \in X$ be arbitrary. Let the multiplicity of lines $12x$, $13x$, $14x$, $15x$, $23x$, and $45x$ in $\boldsymbol{B}$ be a, b, c, d, e, and f, respectively. Assuming one of a, b, c, d, e, f equal to 0 leads quickly to a contradiction, thus all of a, b, c, d, e, f are > 0. Thus in turn implies that $\boldsymbol{B}$ does not contain lines 234, 235, 245, 345. The pairs 24 and 25 can now only occur in lines 124 and 125, respectively, both of which must have multiplicity λ giving a contradiction. □

Before proceeding further, we note that **any** $\mathrm{STS}(v)$ avoids every configuration containing repeated pairs.

Theorem 3.8.

(i) *If* $\lambda \not\equiv 0 \pmod 6$ *then*

$$\Omega(B_6, \lambda) = \begin{cases} B(1) & \textit{if } \lambda \equiv 1, 5 \pmod 6, \\ B(2) \setminus \{4\} & \textit{if } \lambda \equiv 2, 4 \pmod 6, \\ B(3) \setminus \{5\} & \textit{if } \lambda \equiv 3 \pmod 6. \end{cases}$$

(ii) *If* $\lambda \equiv 0 \pmod 6$ *then*

$$B(6) \setminus \{4, 5, 8, 14, 20\} \subseteq \Omega(B_6, \lambda) \subseteq B(6) \setminus \{4, 5\}.$$

Proof. Clearly, every TS$(4,\lambda)$ or TS$(5,\lambda)$ must contain B_6. In view of Lemma 3.1, in order to prove (i) it suffices to show that (i) holds for $\lambda = 1,2,3$. In view of the remark preceding Theorem 3.8, $\Omega(B_6,1) = B(1)$. In order to prove $\Omega(B_6,2) = B(2) \setminus \{4\}$, it obviously suffices to show $v \in \Omega(B_6,2)$ for all $v \equiv 0,4 \pmod 6$, $v \geq 6$. We proceed by establishing a sequence of claims.

I. $v \in \Omega(B_6,2) \implies 2v+4 \in \Omega(B_6,2)$.

Let $(X,\boldsymbol{B})$ be a TTS(v) avoiding B_6, with $X = \{x_1,\dots,x_v\}$. Take $V = X \cup Z_{v+4}$, and let $\boldsymbol{C}$ be the following set of triples:

$$\begin{aligned}\boldsymbol{C} = \{\,&\{x_j, i, i+j\} : i \in Z_{v+4}, j = 1,2,\dots \lfloor (v+4)/2 \rfloor\,\}\\ \cup\,\{\,&\{x_{j+\lfloor (v-2)/2 \rfloor}, i, i+j\} : i \in Z_{v+4},\ j = 4,5,\dots,\lfloor (v+3)/2 \rfloor\,\}\\ \cup\,\{\,&\{i,i+1,i+3\} : i \in Z_{v+4}\,\}.\end{aligned}$$

Then it is easily verified that $(V, \boldsymbol{B} \cup \boldsymbol{C})$ is a TTS$(2v+4)$. Suppose $(V, \boldsymbol{B} \cup \boldsymbol{C})$ contains a copy of B_6, say, abc, abd, acd. Clearly, at most one of a, b, c, d is in X since $(X,\boldsymbol{B})$ avoids B_6. If $a \in X$ then $|b-c| = |b-d| = |c-d| = (v+4)/3$ which is impossible since $v \equiv 0,1 \pmod 3$. If one of b, c $d \in X$, say, $b \in X$ then a, c, $d \in Z_{v+4}$, $|a-c| = |a-d|$ but then acd cannot be a triple of $\boldsymbol{C}$ since if $u,v,w \in Z_{v+4}$ and uvw is a triple then $\{|u-v|, |u-w|, |v-w|\} = \{1,2,3\}$.

II. $\{6,10,12\} \subset \Omega(B_6,2)$.

The case $v=6$ is settled by inspection of the unique TTS(6). Since $3 \in \Omega(B_6,2)$, by applying I. we get $10 \in \Omega(B_6,2)$. The following is a TTS(12) avoiding B_6: $V = Z_4 \times Z_3$, base triples are $0_0 1_0 2_1$, $0_0 1_0 3_1$, $0_0 2_0 1_2$, $0_0 2_0 3_2$, $0_0 0_1 0_2$, $0_0 0_1 0_2$ (mod $4,3$); the verification that this TTS(12) avoids B_6 is straightforward.

III. $\Omega(B_6,2) = B(2) \setminus \{4\}$.

Assume now that $v \equiv 0,4 \pmod 6$, $v \geq 16$, and that $u \in \Omega(B_6,2)$ for all $u < v$, $u \in B(2) \setminus \{4\}$. Then $w = (v-4)/2 \equiv 0,1 \pmod 3$ and $w < v$. Applying I. to w then yields $v \in \Omega(B_6,2)$.

Next we establish $\Omega(B_6,3) = B(3) \setminus \{5\}$, again through a sequence of claims.

IV. Consider the well-known TS$(v,3)$, with $V = Z_v$, and base triples $0\ i\ 2i$, $i = 1,2,\dots,(v-1)/2$ (cf. [**SS**]). In order to avoid B_6 say, abc, abd, acd (or bcd), it is necessary that $a,b \notin \{5b-4a, 5a-4b, (5a-b)/4, (5b-a)/4, (7a-5b)/2, (7b-5a)/2\}$; this is so if and only if $5 \nmid v$ and $7 \nmid v$.

V. $v \in \Omega(B_6,3) \implies 2v+1 \in \Omega(B_6,3)$.

Let $(V,\boldsymbol{B})$ be a TS$(v,3)$ on elements $\{1,2,\dots,v\}$ avoiding B_6. Let $\boldsymbol{F} = \{F_1, F_2, \dots, F_v\}$ be a 3-factorization of $3K_{v+1}$ on a set $X = \{x_1,\dots,x_{v+1}\}$ disjoint from V obtained by triplicating each edge of a 1-factorization of K_{v+1}. Let $\boldsymbol{C} = \{\,\{i, x_i, y_i\} : i \in \{1,2,\dots,v\}, \{x_i,y_i\} \in F_i\,\}$. Then $(V \cup X, \boldsymbol{B} \cup \boldsymbol{C})$ is a TS$(2v+1,3)$ avoiding B_6.

VI. $v \in \Omega(B_6, 3) \implies 2v + 7 \in \Omega(B_6, 3)$.

Let $(V, \boldsymbol{B})$ be a $\mathrm{TS}(v, 3)$ on elements $\{1, 2, \dots, v\}$ avoiding B_6, and let $X = \{x_1, \dots, x_{v+7}\}$ be a set disjoint from V. Let $\boldsymbol{D} = \{\,\{x_i, x_{i+1}, x_{i+3}\} : i \in Z_{v+7}\,\}$, and let G be the graph obtained from K_{v+7} on X by deleting all edges contained in triples of $\boldsymbol{D}$. Let $\boldsymbol{F} = \{F_1, \dots, F_v\}$ be a 3-factorization of $3G$ obtained by triplicating each edge of a 1-factorization of G; the latter exists by [**SL**]. Let $\boldsymbol{E} = \{\,\{i, x_i, y_i\} : i \in \{\,1, 2, \dots, v\,\}, \{x_i, y_i\} \in F_i\,\}$. Then $(V \cup X, \boldsymbol{B} \cup 3\boldsymbol{D} \cup \boldsymbol{E})$ is a $\mathrm{TS}(2v + 7, 3)$ avoiding B_6.

VII. $\Omega(B_6, 3) = B(3) \setminus \{5\}$.

Obviously, it suffices to consider $v \equiv 5 \pmod 6$. Clearly, each $\mathrm{TS}(5, \lambda)$ contains B_6. Since none of 11, 17, 23, 29 is divisible by either 5 or 7, $\{\,11, 17, 23, 29\,\} \subset \Omega(B_6, 3)$. Assume now $v \geq 35$ and $u \in \Omega(B_6, 3)$ for all u: $5 < u < v$, $u \equiv 5 \pmod 6$. Then either $(v - 1)/2$ or $(v - 7)/2$ is $\equiv 5 \pmod 6$, and so $v \in \Omega(B_6, 3)$ by V. or VI., as the case may be.

This completes the proof of part (i) of Theorem 3.8.

In order to prove (ii), it clearly suffices to consider the case $\lambda = 6$. We show first that if $v \equiv 0 \pmod 3$, $v \geq 12$, and there exists a $\mathrm{TS}(v, 6)$ avoiding B_6 then there exists a $\mathrm{TS}(2v + 2, 6)$ avoiding B_6. Indeed, let $(X, \boldsymbol{B})$ be a $\mathrm{TS}(v, 6)$ avoiding B_6 where $X = \{x_1, \dots, x_v\}$, and let $V = X \cup Z_{v+2}$. Denote by $\langle p^\alpha, q^\beta, r^\gamma, \dots\rangle$ the multigraph on Z_{v+2} in which two vertices i, j are joined by exactly α edges if $|i - j| = p$, by exactly β edges if $|i - j| = q$ etc. Let $F_0 = \langle 1^2, 5\rangle$, $F_1 = \langle 2^2, 5\rangle$, $F_2 = \langle 3^2, 5\rangle$, $F_3 = \langle 1^3\rangle$, $F_4 = \langle 2^3\rangle$, $F_5 = \langle 3^3\rangle$, $F_6 = F_7 = \langle 4^3\rangle$, $F_8 = \langle 5^3\rangle$, $F_9 = F_{10} = \langle 6^3\rangle, \dots, F_v = \langle \lceil (v+2)/2\rceil^3\rangle$. Each F_i is a regular graph of degree 6. Let $\boldsymbol{C} = \{\,\{i, i+1, i+3\} : i \in Z_{v+2}\,\}$, and for $x_k \in X$, let $\boldsymbol{D}_k = \{\,\{x_k, i, j\} : \{i, j\} \in F_k\,\}$. Then $(V, \boldsymbol{B} \cup \boldsymbol{C} \cup \bigcup \boldsymbol{D}_k)$ is a $\mathrm{TS}(v, 6)$ avoiding B_6. Indeed, suppose that this $\mathrm{TS}(v, 6)$ contains a copy of B_6, say, abc, abd, acd. Then X can contain at most one of a, b, c, d. If X contains none of a, b, c, d then $\{|a-b|, |a-c|, |b-c|\} = \{|a-b|, |a-d|, |b-d|\} = \{|a-c|, |a-d|, |c-d|\} = \{\,1, 2, 3\,\}$ which is clearly impossible since the triples of $\boldsymbol{C}$ are the only triples entirely on Z_{v+2}. If X contains exactly one of a, b, c, d, say d, then there are two possibilities. Either one of the triples of our B_6, say abc, is in $\boldsymbol{C}$ in which case $\{|a-b|, |a-c|, |b-c|\} = \{\,1, 2, 3\,\}$ and then the pairs ab, ac, bc have mutually distinct third elements in X; or none of the triples of B_6 is in $\boldsymbol{C}$ which implies that ab, ac, $bc \in \boldsymbol{D}_d$, and at the same time, if $\{|a-b|, |a-c|, |b-c|\} = \{\,p, q, r\,\}$ then either $p + q + r \equiv 0 \pmod{v+2}$ or the sum of two of p, q, r equals the third; by our definition of the F_i's, this is impossible if $v \geq 26$.

Since $v \in \Omega(B_6, 6)$ for all $v \equiv 0 \pmod 3$, $v \geq 12$, by part (i) and Lemma 3.1, it follows that $v \in \Omega(B_6, 6)$ for all $v \equiv 2 \pmod 6$, $v \geq 26$. This completes the proof of Theorem 3.8. $\square$

A 3-GDD **of type** $g_1^{u_1} g_2^{u_2} \dots g_n^{u_n}$ is a group divisible design (cf. [**H**] or [**SS**]) which has u_i groups of cardinality g_i $i = 1, 2, \dots, n$ and whose all blocks have cardinality 3.

Theorem 3.9.

$$\Omega(B_7, \lambda) = \Omega(B_8, \lambda) = \begin{cases} B(1) & \text{if } \lambda \equiv 1 \pmod 2, \\ B(2) \setminus \{6\} & \text{if } \lambda \equiv 0 \pmod 2. \end{cases}$$

Proof. For $v \equiv 1, 3 \pmod 6$, a λ-multiple of any STS(v) avoids B_7 (B_8, respectively) by Lemma 3.1. For $v \equiv 0, 4 \pmod 6$, it obviously suffices to show $v \in \Omega(B_i, 2)$, $i \in \{7, 8\}$.

So let $v \equiv 0, 4 \pmod{12}$, and consider a 3-GDD of type $4^{v/4}$ which always exists [**H**]. Form a TTS(v) by replacing each group with a TTS(4) and "doubling" each block (i.e. replacing each block with two identical triples). The resulting TTS(v) avoids B_7: if abc, abd, ace were the 3 triples of B_7 then abc, abd would have to belong to a TTS(4); but then acd is also a triple, thus ace cannot be a triple. Similarly, this TTS(v) avoids B_8.

Let now $v \equiv 6, 10 \pmod{12}$. Obviously, no TS$(6, \lambda)$ can avoid B_7 or B_8. We note next that $10 \in \Omega(B_7, 2)$; the corresponding TTS(10) avoiding B_7 is obtained from a 3-GDD of type 3^3 in which the 3 groups are extended by a (common) point, and the resulting 4-sets are replaced with a TTS(4). Similarly, $22 \in \Omega(B_7)$: take a 3-GDD of type 3^7, and proceed as above extending this time the 7 groups. For $v \equiv 6, 10 \pmod{12}$, $v \geq 30$, consider a 3-GDD of type $4^{(v-10)/4}10^1$ which exists whenever $v \geq 30$ [**RH**]. Form a TTS(v) by replacing each group of size 4 by a TTS(4), the group of size 10 by the above TTS(10), and doubling each block. Again, this TTS(v) clearly avoids B_7. The proof that it avoids B_8 is identical.

Consider now the case $v \equiv 5 \pmod 6$, $\lambda \equiv 3 \pmod 6$. We treat here in detail the case $\lambda = 3$, with the cases of larger such λ being similar. By [**CMS**], in any such TS$(v, 3)$ there exists a pair, say ab, not appearing in a doubly or triply repeated triple. (i) Let abc, abd be two of the three triples containing ab. There are further two triples containing the pair ac, and two triples containing the pair bc. The element d can be the third element of at most 3 of these 4 triples, and so one of these triples must be ace or bce which together with abc, abd form B_7. (ii) Let abc, abd, abe be the triples containing the pair ab. If cde is a triple then abc, abd, cde form B_8. Otherwise there are 3 triples containing cd, and 3 triples containing ce. At most two of these 6 triples can contain a as its third element, and at most two can contain b. Thus we must have a triple cdf or cef where f is another element, and, in any case, we have a B_8.

The considerations in the only remaining case $v \equiv 2 \pmod 6$, $\lambda \equiv 0 \pmod 6$ showing that B_7 or B_8 cannot be avoided are similar, and are left to the reader. □

Theorem 3.10.

$$\Omega(B_9,\lambda)=\begin{cases} B(1) & \textit{if } \lambda\equiv 1,5 \pmod 6,\\ B(1)\cup\{4\} & \textit{if } \lambda\equiv 2,4 \pmod 6,\\ B(1)\cup\{5\} & \textit{if } \lambda\equiv 3 \pmod 6,\\ B(1)\cup\{4,5\} & \textit{if } \lambda\equiv 0 \pmod 6.\end{cases}$$

Proof. Any STS(v) avoids B_9, thus by Lemma 3.1, the statement of the theorem holds for any $v\in B(1)$. Also, any TS(v,λ) trivially avoids B_9 if $v\in\{4,5\}$. Let now $v\notin B(1)$, i.e. $v\equiv 0 \pmod 2$ or $v\equiv 5 \pmod 6$. Then any TS(v,λ) contains two triples (say) abc, abd. If $\boldsymbol{B}_a$ is the set of triples containing a, then the number of triples of $\boldsymbol{B}_a$ containing b is λ, those containing c but not b is at most $\lambda-1$, and those containing d but not b or c is at most $\lambda-1$. It follows that our TS(v,λ) must contain a triple aef whenever $3\lambda-2<\lambda(v-1)/2$, i.e. $v>1+(6\lambda-4)/\lambda$, i.e. $v\ge 7$. When $v=6$, λ must be even, say, $\lambda=2\mu$. If the elements are $\{a,b,c,d,e,f\}$, then the number of blocks of $\boldsymbol{B}_a$ containing b is 2μ. If c,d are two elements which occur most times with b in the triples of $\boldsymbol{B}_a$, then the number of triples of $\boldsymbol{B}_a$ containing c or d but not b is less than or equal to 3μ. Since the number of triples of $\boldsymbol{B}_a$ is 5μ, it follows that there must exist a triple aef. The triples abc, abd, aef form a copy of B_9. □

Theorem 3.11.

$$\Omega(B_{10},\lambda)=\Omega(B_{11},\lambda)=\begin{cases} B(1) & \textit{if } \lambda\equiv 1,5 \pmod 6,\\ B(1)\cup\{4,6\} & \textit{if } \lambda\equiv 2,4 \pmod 6,\\ B(1)\cup\{5\} & \textit{if } \lambda\equiv 3 \pmod 6,\\ B(1)\cup\{4,5,6\} & \textit{if } \lambda\equiv 0 \pmod 6.\end{cases}$$

Proof. Any STS(v) avoids B_{10} and B_{11} which by Lemma 3.1 implies the statement for any $v\in B(1)$. Any TS(v,λ) trivially avoids B_{10} if $v\in\{4,5\}$, and B_{11} if $v\in\{4,5,6\}$. An inspection of the unique TTS(6) reveals that it avoids B_{10}.

Let now $v\notin B(1)$, $v\ge 8$. For any such v, any TS(v,λ) contains A_3 by Theorem 2.4. Let the two lines of this A_3 be, say, abc, abd. Let X be the set of remaining $v-4$ elements, and consider the pairs cx, dx where $x\in X$. There are $2(v-4)\lambda$ such pairs. If any triple containing such a pair has as its third element $y\in X$ then this triple together with abc, abd form a B_{10}. Assume therefore that this does not happen, i.e. all pairs cx, dx occur in triples where the third element is one of a, b, c, d. But there are at most λ triples cdx with $x\in X$, and therefore at most 4λ further such triples acx, adx, bcx, bdx which requires $5\lambda\ge 2(v-4)\lambda$ which is impossible if $v\ge 8$. Thus B_{10} cannot be avoided.

Let us show now that B_{11} cannot be avoided, either, if $v\notin B(1)$ and $v\ge 8$. Consider the case of $v=8$. Let the elements of TS$(8,\lambda)$ be $\{a,b\}\cup X\cup Y$ where

$X = \{1,2,3\}$, $Y = \{4,5,6\}$, and let 123 and 456 be two disjoint lines which exist by Theorem 2.4. Then there is a total of λ triples abz, $z \in X \cup Y$. Assume that our TS$(8,\lambda)$ avoids B_{11}. Then there are no triples of type axx, ayy, bxx or byy, and consequently there are $5\lambda/2$ triples axy and $5\lambda/2$ triples bxy with $x \in X$, $y \in Y$. Thus there are 2λ triples of type xxy or xyy, and consequently, $4\lambda/3$ triples $\{1,2,3\}$, $\{4,5,6\}$. Further $\lambda/2$ of the triples abz must be of type abx and the other $\lambda/2$ of type aby. Choose one of each type, say, $ab1$ and $ab4$, then the only triples of type xxy or xyy that can occur are 124, 134, 234, 145, 146, 156; otherwise, we would have a B_{11}. At most λ of the 2λ triples of type xxy and xyy can contain the pair 14, so at least λ of these must be the triples 234 and 156. But since there are $4\lambda/3$ triples 123 and 456, there are at most $2\lambda/3$ triples 234 and 156 (since $\{2,3\} \subset X$, $\{5,6\} \subset Y$) which is a contradiction.

Now assume $v > 8$. The proof is similar. Let $X = \{x_1, x_2, x_3\}$ be any line. Then there exists a disjoint line $Y = \{y_1, y_2, y_3\}$. Let $W = V \setminus (X \cup Y)$ and the multiplicities of triples of type xxx, xxy, xyy, yyy, and xyw be a, b, c, d, and e, respectively. As above there can be no triples of type xxw or yyw. Counting pairs of elements of type xx, xy, and yy then leads to $3(a+b+c+d)+e = 15\lambda$. Further counting of elements x and y gives that the total number of triples of type xww and yww is $3\lambda(v-6)-e$. Continuing, the total number of triples of type www, i.e. disjoint from both X and Y is $(\lambda(v-6)(v-13)+2e)/6$ and is positive for $v > 13$. In this case to avoid B_{11} we must have $b = c = 0$ and thus $a = d = \lambda$ which could only occur if $v \in B(1)$ by choosing the TS(v,λ) to be λ copies of an STS(v). We are just left with the cases $v \in \{10,11,12\}$. In these cases too if the number of triples of type www is positive then the above argument applies. Otherwise they can be disposed of in a manner similar to the case $v = 8$, and are left as exercises for the reader. This completes the proof. □

Theorem 3.12. $\Omega(B_{12},\lambda) = (B(\lambda) \cap \{v : v \geq \lambda+2\}) \cup N(3,7)$.

Proof. Every TS(v,λ) with $v \in \{3,4,5\}$ trivially avoids B_{12}. Both the unique TTS(6) and the unique STS(7) avoid B_{12}, thus $\{6,7\} \in \Omega(B_{12},\lambda)$ for any λ. It suffices now to observe that B_{12} contains A_4, and the rest follows from Lemma 2.2 and Theorem 2.4. □

Theorem 3.13. $\Omega(B_{13},\lambda) = (B(\lambda) \cap \{v : v \geq \lambda+2\}) \cup \{3,4\}$.

Proof. The proof is almost identical to the proof of Theorem 3.12. □

Theorem 3.14. $\Omega(B_{14},\lambda) = (B(\lambda) \cap \{v : v \geq \lambda+2\}) \cup \{3\}$.

Proof. The proof is identical to that of Theorem 3.13. □

Theorem 3.15.

$$\Omega(B_{15},\lambda) = \begin{cases} B(1) & \textit{if } \lambda \equiv 1 \pmod{2}, \\ B(2) & \textit{if } \lambda \equiv 0 \pmod{2}. \end{cases}$$

Proof. If $v \in B(1) \cup B(2)$, take an appropriate multiple of an STS(v) or of a TTS(v); these surely avoid B_{15}. Let now $(V, \boldsymbol{B})$ be a TS(v, λ) with $v \equiv 5 \pmod 6$; then $\lambda \equiv 0 \pmod 3$. For any $\{a, b\} \subset V$, let $\boldsymbol{N}_{a,b} = \{x : \{x, a, b\} \in \boldsymbol{B}\}$. If $|\boldsymbol{N}_{a,b}| > 2$ for some $\{a, b\} \subset \boldsymbol{B}$, there is nothing to prove. Otherwise, consider the complete graph with vertex-set V, and assign colours red and blue to its edges as follows. Start with an arbitrary edge ab. If $|\boldsymbol{N}_{a,b}| = 1$, assign the edge ab both red and blue colours. If $|\boldsymbol{N}_{a,b}| = 2$, say, $\boldsymbol{N}_{a,b} = \{x, y\}$ then the triple abx occurs in $\boldsymbol{B}$ i times, and the triple aby occurs in $\boldsymbol{B}$ $\lambda - i$ times where, without loss of generality, $i \leq \lambda - i$. Colour all three edges ab, ax, bx of the triangle abx red, and all three edges of the triangle aby blue. The edge ab has been coloured both red and blue but the edges ax, bx, ay, by have so far only one of the colours. If, say, $\boldsymbol{N}_{b,x} = \{a, u\}$, $\boldsymbol{N}_{b,y} = \{a, v\}$, colour the edges bx, bu, xu of the triangle bxu blue, and the edges by, bv, yv of the triangle byv red. Continue assigning the colours red and blue until there is no edge left with just one colour assigned. Two cases may occur:

Case 1. There are no uncoloured edges. Each pair of elements ab occurs in a triple abc which is repeated i times, and in a triple abd which is repeated $\lambda - i$ times, as indicated by the red and blue triangles, respectively. Then the set of red triangles, each taken as a triple just once, forms a TS$(v, 1)$, a contradiction, as $v \equiv 5 \pmod 6$.

Case 2. There remain uncoloured edges. If cd is an uncoloured edge and $\boldsymbol{N}_{c,d} = \{p, q\}$, say, then the triple cdp occurs in $\boldsymbol{B}$ j times and the triple cdq occurs in $\boldsymbol{B}$ $\lambda - j$ times. Colour the edges of the triangle cdp red, and those of the triangle cdq blue, and continue assigning colours to the edges in the above manner while possible, i.e. until there is no edge with only one colour assigned. After this stage is completed, if there still remain uncoloured edges, continue the process until no uncoloured edges remain. The set of red triangles, each taken as a triple just once, is a TS$(v, 1)$, again a contradiction. Thus any TS(v, λ) with $v \equiv 5 \pmod 6$ must contain B_{15}.

Let us observe that for $\lambda = 3$ this follows form [**CMS**] where it is shown that in any TS$(v, 3)$ with $v \equiv 5 \pmod 6$ there must exist a pair of elements not appearing in a doubly or triply repeated triple. The same follows by [**CM**] for any TS$(v, 6)$ with $v \equiv 2 \pmod 6$. The argument for TS(v, λ) with $\lambda \equiv 0 \pmod 6$, $\lambda > 6$, is similar to the one given above, and is therefore omitted. $\square$

Theorem 3.16. $\Omega(B_{16}, \lambda) = B(\lambda) \cap \{v : \lambda \leq 2v - 4\}$.

Proof. For $\lambda \leq v - 2$ the statement follows from Lemma 2.3. For $v - 2 < \lambda \leq 2v - 4$, the TS$(v, \lambda)$ whose set of triples consists of the union of the set of triples of a simple TS$(v, \lambda - v + 2)$ (which exists by Lemma 2.3) and the set of all triples on the same v-set clearly avoids B_{16}. On the other hand, every TS(v, λ) with $\lambda \geq 2v - 4$ must contain a triple repeated at least three times. $\square$

4. Configurations with Four or More Lines

The number of nonisomorphic configurations increases very rapidly with the number of lines. For example, even when restricted to configurations without a repeated pair, the number for four lines is 16 (see [**GRR**]). It follows from [**GRR**] but is also easily seen directly that the avoidance set $\Omega(C_i, 1)$ for all but two of the 16 configurations C_i, $i = 1, 2, \ldots, 16$, namely C_{14} and C_{16} (the Pasch configuration), is finite. While it is conjectured that $\Omega(C_{16}, 1) = B(1) \setminus \{7, 13\}$ (cf. [**B**], [**GMP**], [**SW**]) this remains far from proved. On the other hand, $\Omega(C_{14}, 1) = \{2^n - 1 : n > 1\}$, and the only STS$(2^n - 1)$ avoiding C_{14} is the projective space $PG(n-1, 2)$. But to determine the avoidance sets $\Omega(C_i, \lambda)$ for all $i \in N(1, 16)$ and all λ appears to be a formidable task (the same holds, for that matter, for the simultaneous avoidance sets, even in the case of three lines).

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