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NON–ARCHIMEDEAN SEQUENTIAL SPACES AND THE FINEST LOCALLY CONVEX TOPOLOGY WITH THE SAME COMPACTOID SETS

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ABSTRACT. For a non-Archimedean locally convex space (E, τ), the finest locally convex topology having the same as τ convergent sequences and the finest locally convex topology having the same as τ compactoid sets are studied.

INTRODUCTION

For a locally convex space E over the field of either the real numbers or the complex numbers, Webb investigated in [**13**] the finest locally convex topology on E having the same convergent sequences as the original topology. Also, he studied the finest locally convex topology which has the same precompact sets.

In this paper we look at analogous problems for non-Archimedean spaces. For a non-Archimedean locally convex space (E, τ), we study the sequential locally convex topology τ^s which is the finest locally convex topology with the same as τ convergent sequences. Passing from τ to τ^s, we get that the category of non-Archimedean sequential locally convex spaces and continuous linear maps is a full coreflective subcategory of the category of all locally convex spaces. If τ is the weak topology of c_0, then τ^s coincides with the norm topology of c_0 which of course is not true in the classical case. For a zero dimensional topological space X and a non-Archimedean locally convex space E, we look at the problem of when is the space $C(X, E)$, of all continuous E-valued functions on X, with the topology of either the pointwise convergence or the compact convergence, a sequential space. In case E is metrizable, it is shown that $C(X, E)$ is sequential iff it is bornological and this happens iff X is $\mathbb{N}$-replete, where $\mathbb{N}$ is the set of natural numbers.

For a non-Archimedean locally convex topology τ on E, we study the locally convex topology τ^c which coincides with the finest locally convex topology with the same as τ compactoid sets. The compactoid sets in non-Archimedean spaces are much more important than the precompact sets. As in the case of τ^s, we get that the category of all non-Archimedean locally convex spaces (E, τ) and continuous linear maps, for which $\tau = \tau^c$, is a full coreflective subcategory of the

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category of all locally convex spaces. If τ is the weak topology of ℓ^∞ and if the field is non-spherically complete, it is shown that τ^s coincides with the finest locally convex topology which agrees with τ on norm bounded sets and with the finest polar topology having the same as τ compactoid sets.

1. Preliminaries

All vector spaces considered in this paper will be over a complete non-Archimedean valued field $\mathbb{K}$ whose valuation is non-trivial.

For a subset S of a vector space E over $\mathbb{K}$, we will denote by $\operatorname{co}(S)$ the absolutely convex hull of S. The edged hull A^e, of an absolutely convex subset A of E, is defined by: $A^e = A$ if the valuation of $\mathbb{K}$ is discrete, and $A^e = \cap\{\lambda A : |\lambda| > 1\}$ if the valuation of $\mathbb{K}$ is dense (see [**9**]).

A subset B, of a locally convex space E over $\mathbb{K}$, is called compactoid if, for each neighborhood V of zero, there exists a finite subset S of E such that $B \subset \operatorname{co}(S)+V$. For a non-Archimedean seminorm p on E, we will denote by E_p the quotient space $E/_{\ker p}$ equipped with the norm $||[x]_p|| = p(x)$, where $\ker p = \{x : p(x) = 0\}$. By $\hat{E}_p$ we will denote the completion of E_p.

A seminorm p on E is called polar if $p = \sup\{|f| : f \in E^*,\ |f| \leq p\}$, where E^* is the algebraic dual space of E. The locally convex space E is called polar if its topology is generated by a family of polar seminorms (see [**9**]).
Equivalently, E is a polar space if it has a base at zero consisting of polar sets, i.e. sets V with $V = V^{00}$, where V^{00} is the bipolar of V. For other notions refering to non-Archimedean locally convex spaces and for related results we refer to [**9**].

2. Sequential Spaces

Definition 2.1. A subset V, of a locally convex space E, is called a sequential neighborhood of zero if every null sequence in E lies eventually in V. The space E is called sequential if every convex sequential neighborhood of zero is a neighborhood of zero.

We have the following easily established

Lemma 2.2. *Let V be an absolutely convex absorbing subset of a locally convex space E. Then, V is a sequential neighborhood of zero iff its Minkowski functional p_V is sequentially continuous.*

Proposition 2.3. *For a locally convex space E, the following are equivalent:*

(1) *E is a sequential space.*

(2) *Every sequentially continuous seminorm on E is continuous.*

(3) *For every locally convex space F, every sequentially continuous linear map from E to F is continuous.*

(4) *For every Banach space F, every sequentially continuous linear map from E to F is continuous.*

Proof. The equivalence of (1) and (2) follows from Lemma 2.2

(2) $\Rightarrow$ (3) Let $f\colon E \to F$ be linear and sequentially continuous. If V is a convex neighborhood of zero in F, then $f^{-1}(V)$ is a convex sequential neighborhood of zero in E and hence $f^{-1}(V)$ is a neighborhood of zero.

(4) $\Rightarrow$ (2) Let p be a sequentially continuous seminorm on E and consider the Banach space $\hat{E}_p$. The canonical mapping $\pi_p\colon E \to \hat{E}_p$ is sequentially continuous and hence continuous, which implies that p is continuous. $\square$

Let now (E, τ) be a locally convex space. The family of all convex sequential τ-neighborhoods of zero is a base at zero for a locally convex topology τ^s. The family of polar (with respect to the pair $\langle E, E^* \rangle$) sequential τ-neighborhoods of zero is a base at zero for a polar topology τ^s_π. We have the following

Proposition 2.4. 1) *τ^s coincides with the coarsest sequential topology finer than τ.*

2) *τ is sequential iff $\tau = \tau^s$.*

3) *τ^s is the finest locally convex topology on E having the same convergent sequences as τ.*

4) *If τ_1 is a locally convex topology on E such that every τ-null sequence is also τ_1-null, then τ_1 is coarser then τ^s.*

5) *The topologies τ and τ^s have the same bounded sets.*

6) *If F is a locally convex space and $f\colon E \to F$ a linear mapping, then f is τ^s-continuous iff it is sequentially τ-continuous.*

7) *τ^s is generated by the family of all non-Archimedean seminorms on E which are sequentially τ-continuous.*

8) *τ^s_π is the finest of all polar topologies τ_1 on E such that every τ-convergent sequence is also τ_1-convergent. If τ is polar, then $\tau \le \tau^s_\pi$ and the topologies τ and τ^s_π have the same convergent sequences.*

9) *$\tau^s_\pi \le \tau^s$.*

10) *τ^s_π is generated by the family of all sequentially τ-continuous polar seminorms on E.*

11) *τ^s_π is the largest of all polar topologies which are coarser than τ^s.*

Proposition 2.5. *Let (E, τ) and (F, τ_1) be locally convex spaces. If a linear map*

$$f\colon (E, \tau) \to (F, \tau_1)$$

is continuous, then f is also (τ^s, τ_1^s)-continuous and $(\tau^s_\pi, (\tau_1)^s_\pi)$-continuous.

Using the preceding Proposition we get that the category of non-Archimedean sequential locally convex spaces and continuous linear maps is a full coreflective subcategory of the category of all locally convex spaces.

Corollary 2.6. *If $(E,\tau) = \prod_{\alpha\in I}(E_\alpha, \tau_\alpha)$, then*

$$\tau^s \geq \prod_{\alpha\in I}(\tau_\alpha)^s \quad \text{and} \quad \tau^s_\pi \geq \prod_{\alpha\in I}(\tau_\alpha)^s_\pi \,.$$

Proposition 2.7. *Let $(E,\tau) = \prod_{k=1}^n (E_k, \tau_k)$. Then*

$$\tau^s = \prod_{k=1}^n (\tau_k)^s \quad \text{and} \quad \tau^s_\pi = \prod_{k=1}^n (\tau_k)^s_\pi \,.$$

Proof. Let $\tau_0 = \prod_{k=1}^n (\tau_k)^s$. By the preceding Corollary, we have $\tau_0 \leq \tau^s$. On the other hand, let V be a convex sequential τ-neighborhood of zero in E. If $j_k \colon E_k \to E$ is the canonical injection, then $V_k = j_k^{-1}(V)$ is a sequential τ_k-neighborhood of zero in E_k. It follows that $W = \prod_{k=1}^n V_k$ is a τ_0-neighborhood of zero with $W \subset V$, which proves that $\tau^s \leq \tau_0$.
The proof for the case of τ^s_π is analogous. □

Recall that a locally convex space E is called polarly bornological (see [**9**]) if every subset of E, which is polar with respect to the pair $\langle E, E^*\rangle$ and which absorbs bounded sets is a neighborhood of zero. Equivalently, E is polarly bornological if every polar seminorm on E, which is bounded on bounded sets, is continuous.

Proposition 2.8. *Let (E,τ) be a locally convex space.*
1) *If (E,τ) is bornological, then $\tau = \tau^s$.*
2) *If (E,τ) is polarly bornological, then $\tau^s_\pi \leq \tau$.*

Proof. 1) It follows from the fact that τ and τ^s have the same bounded sets.

2) Let p be a sequentially continuous polar seminorm. Then, p is bounded on bounded sets. In fact, let B be a bounded set with $\sup_{x\in B} p(x) = \infty$. Let $|\lambda| > 1$ and choose a sequence (x_n) in B with $p(x_n) > |\lambda|^n$. Now, $\lambda^{-n}x_n \to 0$ but $p(\lambda^{-n}x_n) \geq 1$ for all n, a contradiction. Since E is polarly bornological, p is continuous and the result follows from Proposition 2.4. □

Proposition 2.9. *If E is finite dimensional, then it is sequential.*

Proof. If E is Hausdorff, then E is topologically isomorphic to $\mathbb{K}^n$, where $n = \dim(E)$, and so E is sequential. If E is not Hausdorff, let $F = \overline{\{0\}}$. Since E/F is Hausdorff and finite-dimensional, its topology is given by some norm $\|\cdot\|$. If $\pi \colon E \to E/F$ is the quotient map, then it is easy to see that the topology of E is given by the seminorm $p(x) = \|\pi(x)\|$. Thus E is seminormable and hence sequential. □

We have also the following easily established

Proposition 2.10. *Let $\{E_\alpha : \alpha \in I\}$ be a family of locally convex spaces, E a vector space and, for each $\alpha \in I$, $f_\alpha \colon E_\alpha \to E$ a linear mapping. If each E_α is sequential and if E is equipped with the finest locally convex topology for which each f_α is continuous, then E is sequential.*

Corollary 2.11. *Quotient spaces and direct sums of sequential locally convex spaces are sequential.*

The result about direct sums of sequential spaces also follows using the general theory of coreflective subcategories.

For a locally convex space (E, τ), we will denote by E^s the space of all sequentially τ-continuous linear functionals on E. Clearly $E^s = (E, \tau^s)'$.

Definition 2.12. A subset B of E^* is called sequentially τ-equicontinuous if $x_n \xrightarrow{\tau} 0$ in E implies that $f(x_n) \to 0$ uniformly for $f \in B$, i.e. $\lim_{n\to\infty} \sup_{f\in B} |f(x_n)| = 0$. Clearly every sequentially τ-equicontinuous subset of E^* is contained in E^s and E^s is the union of all such subsets of E^*.

Lemma 2.13. *If $B \subset E^*$ is sequentially τ-equicontinuous, then its bipolar $B^{\square\square}$, with respect to the pair $\langle E^*, E\rangle$, is also sequentially τ-equicontinuous.*

Proof. It follows from the fact that for each $x \in E$ we have

$$\sup_{f\in B} |f(x)| = \sup_{f\in B^{\square\square}} |f(x)| .$$

$\square$

In the following Proposition, we will denote by $b(E^s, E)$ the strong topology on E^s.

Proposition 2.14. *If τ is polar, then every sequentially τ-equicontinuous subset H of E^s is $b(E^s, E)$-bounded.*

Proof. Assume that H is not strongly bounded and let A be a $\sigma(E, E^s)$-bounded subset of E such that

$$\sup_{x\in A,\ f\in H} |f(x)| = \infty .$$

Since $E' \subset E^s$, the set A is $\sigma(E, E')$-bounded and hence it is τ-bounded since τ is polar.

Let $|\lambda| > 1$ and choose a sequence (x_n) in A and a sequence (f_n) in H such that $|f_n(x_n)| \geq |\lambda|^n$ for all n. Since A is bounded, we have that $\lambda^{-n} x_n \xrightarrow{\tau} 0$. Moreover, $|f_n(\lambda^{-n} x_n)| \geq 1$ which contradict the fact that H is sequentially τ-equicontinuous. $\square$

Since, for each $f \in E^s$, the seminorm $p_f(x) = |f(x)|$ is polar and sequentially continuous, it is clear that $E^s = (E, \tau_\pi^s)'$.

Proposition 2.15. *τ_π^s coincides with the topology of uniform convergence on the sequentially τ-equicontinuous subsets of E^s.*

Proof. It is easy to see that a subset of E^s is τ_π^s-equicontinuous iff it is sequentially τ-equicontinuous. Now the result follows form the fact that τ_π^s is a polar topology. □

Notation. For a locally convex space (E, τ), we will denote by E^b the space of all bounded linear functionals on E, i.e. the space of all $f \in E^*$ which are bounded on τ-bounded sets. By τ^n we will denote the topology on E^b of uniform convergence on the τ-null sequences in E.

Proposition 2.16. *For an absolutely convex subset H of E^s, the following assertions are equivalent:*

(1) *H is sequentially τ-equicontinuous.*

(2) *H is τ^n-compactoid.*

Proof. (1) $\Rightarrow$ (2). Let p be the Minkowski functional of the polar H^0 of H in E. We have that

$$p(x) = \sup_{f \in H} |f(x)| \quad (x \in E).$$

Let $A = \{x_n : n \in \mathbb{N}\}$, where (x_n) is a τ-null sequence in E. Let $0 < |\mu| < 1$ and let $\varepsilon > 0$ with $4\varepsilon \le |\mu|$. There exists an index n_0 such that $p(x_n) < \varepsilon$ if $n > n_0$. We choose a basis $\{z_1, \ldots, z_k\}$ for $F = [x_1, \ldots, x_{n_0}]$ which is $\frac{1}{2}$-orthogonal with respect to the seminorm p. We may assume that $p(z_1) \ge p(z_2) \ge \cdots \ge p(z_k)$.
Let $m \le k$ be such that $p(z_i) > 0$ if $i \le m$ and $p(z_i) = 0$ if $i > m$. We may assume that $|\mu| \le p(z_i) < 1$ for $i \le m$. There are $g_1, \ldots, g_m$ in G', where $G = [z_1, \ldots, z_m]$, with $g_i(z_j) = 0$ if $i \ne j$ and $g_i(z_i) = 1$. For $x = \sum_{i=1}^m \lambda_i z_i \in G$, we have

$$p(x) \ge \frac{1}{2} \sup_{1 \le i \le m} |\lambda_i| p(z_i) \ge \sup_{1 \le i \le m} \frac{|\mu \lambda_i|}{2}.$$

Thus,

$$|g_i(x)| = |\lambda_i| \le \frac{2}{|\mu|} p(x).$$

Since p is a polar seminorm, there exists a continuous extension $\overline{g}_i$ of g_i to all of E such that

$$|\overline{g}_i(y)| \le \frac{4}{|\mu|} p(y)$$

for all $y \in E$. Note that $\overline{g}_i \in (E, \tau_\pi^s)' = E^s$.
Let now $f \in H$ and set $h = \sum_{i=1}^m f(z_i) \overline{g}_i$. Since $f \in H$, we have $|f| \le p$ and so $|f(z_i)| \le p(z_i) \le 1$, which implies that $h \in \operatorname{co}(\overline{g}_1, \ldots, \overline{g}_m)$. Moreover, $h = f$ on G. For $m < i \le k$, we have

$$|\overline{g}_i(z_i)| \le \frac{4}{|\mu|} p(z_i) = 0,$$

which implies that $h = f$ on F. Finally, for $n > n_0$, we have

$$|f(x_n)| \le p(x_n) < 1, \quad |\overline{g}_i(x_n)| \le \frac{4}{|\mu|} p(x_n) \le \frac{4\varepsilon}{|\mu|} \le 1 \,.$$

Therefore, $|(f-h)(x_n)| \le 1$ for all n and so $f - h \in A^0$, where A^0 is the polar of A in E^b. Thus

$$H \subset \mathrm{co}\,(\overline{g}_1, \dots, \overline{g}_m) + A^0,$$

which proves that H is τ^n-compactoid.

(2) $\Rightarrow$ (1). Let (x_n) be a τ-null sequence in E and set $A = \{x_n : n \in \mathbb{N}\}$. Since the polar A^0 of A in E^b is a τ^n-neighborhood of zero, given $\mu \neq 0$ in $\mathbb{K}$ there are $g_1, \dots, g_m$ in the linear hull $[H] \subset E^s$ of H such that

$$H \subset \mathrm{co}\,(g_1, \dots, g_m) + \mu A^0 \,.$$

Let n_0 be such that $|g_k(x_n)| \le |\mu|$, for $k = 1, \dots, m$, if $n \ge n_0$. Now

$$\sup_{f \in H} |f(x_n)| \le |\mu|$$

for all $n \ge n_0$. In fact, let $f \in H$. There exist $g \in \mathrm{co}\,(g_1, \dots, g_m)$ and $h \in A^0$ such that $f = g + \mu h$, which implies that, for $n \ge n_0$, we have $|f(x_n)| \le |\mu|$ since $|g(x_n)| \le |\mu|$. This proves that H is sequentially τ-equicontinuous. $\square$

3. The Topology τ^c

In this section we will study the finest locally convex topology on E having the same compactoid sets as a given locally convex topology.

Proposition 3.1. *Let τ_1, τ_2 be locally convex topologies on E such that every τ_1-compactoid is also τ_2-compactoid. Then, every τ_1-bounded set is also τ_2-bounded.*

Proof. Let A be a subset of E which is τ_1-bounded but not τ_2-bounded. We may assume that A is absolutely convex. Since A is not τ_2-bounded, given $\mu \in \mathbb{K}$, with $|\mu| > 1$, there exist a convex τ_2-neighborhood V of zero and a sequence (x_n) in A with $x_n \notin \mu^{2n} V$. The sequence (y_n), $y_n = \mu^{-n} x_n$, is τ_1-null and hence τ_1-compactoid, which implies that (y_n) is τ_2-compactoid. Therefore, (y_n) is τ_2-bounded and so $\mu^{-n} y_n \overset{\tau_2}{\to} 0$, which is a contradiction since $\mu^{-n} y_n \notin V$.

Let now τ be a locally convex topology on E and let $\mathbb{B}_\tau$ be the family of all convex absorbing subsets V of E with the following property: For each τ-compactoid subset A of E there exists a finite subset S of E such that $A \subset \mathrm{co}\,(S) + V$. Clearly every convex τ-neighborhood of zero is in $\mathbb{B}_\tau$ and, for each $V \in \mathbb{B}_\tau$ and each $\mu \notin 0$, we have $\mu V \in \mathbb{B}_\tau$. If V_1, V_2 are in $\mathbb{B}_\tau$, then $V = V_1 \cap V_2$

is also in $\mathbb{B}_\tau$. In fact, let A be an absolutely convex τ-compactoid and let $|\lambda| > 1$. There exists a finite subset $S = \{x_1, \dots, x_n\}$ of E such that

$$A \subset \mathrm{co}\,(S) + \lambda^{-2} V_1\,.$$

By [**3**, Lemma 1.2], there exists a finite subset $S_1 = \{y_1, \dots, y_n\}$ of λA such that

$$A \subset \mathrm{co}\,(S_1) + \lambda^{-1} V_1\,.$$

Since the set $B = [A + \mathrm{co}\,(S_1)] \cap (\lambda^{-1} V_1)$ is a τ-compactoid, using again [**3**, Lemma 1.2], we can find a finite subset S_2 of $\lambda B \subset V_1$ such that

$$B \subset \mathrm{co}\,(S_2) + V_2.$$

Now

$$A \subset \mathrm{co}\,(S_1 \cup S_2) + V_1 \cap V_2\,.$$

In fact, let $x \in A$. There exists $z_1 \in \mathrm{co}\,(S_1)$ such that $x - z_1 \in \lambda^{-1} V_1$. Since $x - z_1 \in B$, there exists $z_2 \in \mathrm{co}\,(S_2) \subset V_1$ such that $x - z_1 - z_2 \in V_2$. Since $x - z_1 \in B \subset \lambda^{-1} V_1 \subset V_1$, we have that $x - z_1 - z_2 \in V_1 \cap V_2$ and $z_1 + z_2 \in \mathrm{co}\,(S_1 \cup S_2)$, which completes the proof of our claim. This proves that $V_1 \cap V_2 \in \mathbb{B}_\tau$. It follows from the above that $\mathbb{B}_\tau$ is a base at zero for a locally convex topology τ^c finer than τ. □

Proposition 3.2. (1) *τ^c is the finest locally convex topology on E having the same compactoid sets as τ.*

(2) *If τ_1 is a locally convex topology on E such that every τ-compactoid is also τ_1-compactoid, then $\tau_1 \le \tau^c$.*

(3) *The topologies τ and τ^c have the same bounded sets.*

(4) *If (E, τ) is bornological, then $\tau = \tau^c$.*

(5) *If F is a locally convex space and $f \colon E \to F$ a linear mapping, then f is τ^c-continuous iff it maps τ-compactoid sets into compactoid sets in F.*

(6) *$\tau^c = \tau$ iff, for any locally convex space F, any linear map $f \colon E \to F$ mapping τ-compactoid sets into compactoid sets is τ-continuous.*

(7) *$\tau^c = \tau$ iff, for any Banach space F, any linear function $f \colon E \to F$, mapping τ-compactoid sets into compactoid sets, is τ-continuous.*

Proof. (1) and (2) follows easily from the definitions.

(3) It follows from (1) and Proposition 3.1.

(4) It follows from (3) since $\tau \le \tau^c$.

(5) Necessity follows from (1) since images of compactoid sets, under continuous linear mappings, are compactoid. For the sufficiency, let the linear function $f \colon E \to F$ map τ-compactoid sets into compactoid sets and let V be a convex neighborhood of zero in F. Let A be an absolutely convex τ-compactoid in E

and let $|\lambda| > 1$. Since $f(A)$ is compactoid in F, there exists a finite subset T of $\lambda f(A)$ such that $f(A) \subset \mathrm{co}\,(T) + V$. If S is a finite subset of λA with $T = f(S)$, then $A \subset \mathrm{co}\,(S) + f^{-1}(V)$. This proves that $f^{-1}(V) \in \mathbb{B}_\tau$ and so $f^{-1}(V)$ is a τ^c-neighborhood of zero.

(6) Necessity follows from (5). To prove the sufficiency of the condition, it suffices to take $F = (E, \tau^c)$ and consider the identity map from E to F.

(7) Suppose that, for any Banach space F, any linear function from E to F, which maps τ-compactoid sets into compactoid sets, is continuous. Let p be a τ^c-continuous non-Archimedean seminorm on E and consider the Banach space $G = \hat{E}_p$. The canonical mapping $\varphi_p \colon E \to G$ maps τ-compactoid sets into compactoid sets, and so φ_p is continuous, which implies that p is τ-continuous. □

Notation 3.3. We will denote by τ_π^c the finest of all polar topologies τ_1 on E such that every τ-compactoid set is also τ_1-compactoid.

We have the following easily established

Lemma 3.4. *a) τ_π^c is the finest polar topology on E coarser than τ^c.*

b) If τ is polar, then $\tau \le \tau_\pi^c$ and the two topologies τ and τ_π^c have the same compactoid sets and the same bounded sets.

c) Every τ-bounded set is τ_π^c-bounded.

Let us recall next the notion of the Kolmogorov diameters of a bounded set. If p is a non-Archimedean seminorm on E and A a p-bounded set, then for each non-negative integer n the n-th Kolmogorov diameter $\delta_{n,p}(A)$ of A, with respect to p, is the infimum of all $|\mu|$, $\mu \in \mathbb{K}$, for which there exists a subspace F of E, with $\dim F \le n$, such that

$$A \subset F + \mu B_p(0,1),$$

where $B_p(0,1) = \{x \in E : p(x) \le 1\}$ (see [**8**]).
By [**8**], a subset A of E is τ-compactoid iff $\lim_{n\to\infty} \delta_{n,p}(A) = 0$ for each τ-continuous seminorm p on E.

Lemma 3.5. *A non-Archimedean seminorm p on E is τ-bounded, i.e. it is bounded on bounded sets, iff p is bounded on τ-compactoid sets.*

Proof. Assume that there exists a τ-bounded set A such that $\sup_{x\in A} p(x) = \infty$. Given $|\lambda| > 1$, there exists a sequence (x_n) in A with $p(x_n) > |\lambda|^{2n}$. Now, the sequence $(\lambda^{-n}x_n)$ is τ-null and hence τ-compactoid but $\sup_n p(\lambda^{-n}x_n) = \infty$. □

Proposition 3.6. *Let $\mathcal{P}_\tau$ be the family of all τ-bounded non-Archimedean seminorms p on E such that $\lim_{n\to\infty} \delta_{n,p}(A) = 0$ for each τ-compactoid set A. Then:*

a) If $p \in \mathcal{P}_\tau$ and if q is a non-Archimedean seminorm on E with $q \le p$, then $q \in \mathcal{P}_\tau$.

b) If $p_1, p_2 \in \mathcal{P}_\tau$, then $p_1 + p_2$ and $p = \max\{p_1, p_2\}$ are also in $\mathcal{P}_\tau$.

c) If $p \in \mathcal{P}_\tau$, then $|\mu|p \in \mathcal{P}_\tau$ for each $\mu \in \mathbb{K}$.

Proof. Let $p \in \mathcal{P}_\tau$ and let $\mu \neq 0$. Given a τ-compactoid set A, there exists an n such that $\delta_{n,p}(\mu A) < |\mu|$ since μA is τ-compactoid. By [**8**, Proposition 3.2], there are $x_1, \ldots, x_n$ in E such that

$$\mu A \subset \mathrm{co}\,(x_1, \ldots, x_n) + \mu B_p(0,1)$$

and so

$$A \subset \mathrm{co}\,(\mu^{-1}x_1, \ldots, \mu^{-1}x_n) + B_p(0,1)\,.$$

This proves that $B_p(0,1)$ is in $\mathbb{B}_\tau$.
Let now $p_1, p_2 \in \mathcal{P}_\tau$ and $\mu \neq 0$. Choose $\lambda \in \mathbb{K}$ with $0 < |\lambda| < |\mu|/2$. Since both $\lambda B_{p_1}(0,1)$ and $\lambda B_{p_2}(0,1)$ are in $\mathbb{B}_\tau$, the same is true for the set

$$V = [\lambda B_{p_1}(0,1)] \cap [\lambda B_{p_2}(0,1)] = \lambda[B_{p_1}(0,1) \cap B_{p_2}(0,1)]\,.$$

If $p = p_1 + p_2$, then $V \subset \mu B_p(0,1)$ and so $B_p(0,1) \in \mathbb{B}_\tau$. It follows that there are $y_1, \ldots, y_m$ in E such that

$$A \subset \mathrm{co}\,(y_1, \ldots, y_m) + \mu B_p(0,1)$$

and so $\delta_{m,p}(A) \leq |\mu|$. Thus, for $n \geq m$, we have $\delta_{n,p}(A) \leq |\mu|$, which proves that $\delta_{n,p}(A) \to 0$ and so $p \in \mathcal{P}_\tau$. The proofs of the other assertions in the Proposition follow easily from the definitions. □

Proposition 3.7. (1) *A non-Archimedean seminorm p on E is τ^c-continuous iff it belongs to $\mathcal{P}_\tau$.*
(2) *The family of all polar members of $\mathcal{P}_\tau$ generates the topology τ^c_π.*

Proof. (1) If $p \in \mathcal{P}_\tau$, then, as we have seen in the proof of the preceding Proposition, $B_p(0,1)$ belongs to $\mathbb{B}_\tau$ and so p is τ^c-continuous. Conversely, let p be τ^c-continuous. If A is τ-compactoid, then A is τ^c-compactoid and so $\delta_{n,p}(A) \to 0$, which proves that $p \in \mathcal{P}_\tau$.
(2) The proof is analogous to that of (1). □

We have the following easily established

Proposition 3.8. *If a linear map $f\colon (E,\tau) \to (F,\tau_1)$ is continuous, then f is (τ^c, τ_1^c)-continuous and $(\tau^c_\pi, (\tau_1)^c_\pi)$-continuous.*

In view of the preceding Proposition, we get that the category of all locally convex spaces (E,τ), with $\tau^c = \tau$, and continuous linear maps is a full coreflective subcategory of the category of all locally convex spaces.

Corollary 3.9. *If $(E,\tau) = \prod_{\alpha\in I}(E_\alpha, \tau_\alpha)$, then $\tau^c \geq \prod_{\alpha\in I}(\tau_\alpha)^c$ and $\tau^c_\pi \geq \prod_{\alpha\in I}(\tau_\alpha)^c_\pi$.*

The proof of the following Proposition is analogous to that of Proposition 2.7.

Proposition 3.10. *If* $(E,\tau)=\prod_{k=1}^n(E_k,\tau_k)$, *then* $\tau^c=\prod_{k=1}^n\tau_k^c$ *and* $\tau_\pi^c=\prod_{k=1}^n(\tau_k)_\pi^c$.

Proposition 3.11. $(E,\tau^c)'=(E,\tau_\pi^c)'=E^b$.

Proof. If A is τ-compactoid, then A is τ-bounded and so A is $\sigma(E,E^b)$-bounded, which implies A is $\sigma(E,E^b)$-compactoid. Since $\sigma(E,E^b)$ is a polar topology, we have that $\sigma(E,E^b)\le\tau_\pi^c$ and so

$$E^b=(E,\sigma(E,E^b))'\le(E,\tau_\pi^c)'.$$

On the other hand, let $f\in(E,\tau^c)'$ and let A be a τ-bounded set. Then, A is τ^c-bounded and so f is bounded on A which proves that $f\in E^b$. □

We will need the following Proposition which is analogous to the Grothendieck's interchange Theorem. We will say that a family $\mathcal{M}$ of subsets of a vector space G is directed if given $M_1,M_2\in\mathcal{M}$ there exists $M_3\in\mathcal{M}$ containing both M_1 and M_2.

Proposition 3.12. *Let* $\langle E,F\rangle$ *be a dual pair of vector spaces over* $\mathbb{K}$ *and let* $\mathcal{M}$ *(resp.* $\mathcal{N}$*) be a directed family of* $\sigma(E,F)$*-bounded (resp.* $\sigma(F,E)$*-bounded) subsets of* E *(resp.* F*) covering* E *(resp.* F*). On* E *we consider the topology* $\tau_{\mathcal{N}}$ *of uniform convergence on the members of* $\mathcal{N}$ *and on* F *the topology* $\tau_{\mathcal{M}}$ *of uniform convergence on the members of* $\mathcal{M}$*. Then, the following statements are equivalent:*
(1) *Each member of* $\mathcal{M}$ *is* $\tau_{\mathcal{N}}$*-compactoid.*
(2) *Each member of* $\mathcal{N}$ *is* $\tau_{\mathcal{M}}$*-compactoid.*

Proof. (1) ⇒ (2). Without loss of generality, we may assume that all members of $\mathcal{M}$ and $\mathcal{N}$ are absolutely convex. Let $H\in\mathcal{N}$, $M\in\mathcal{M}$ and $\gamma\ne0$. Since $\gamma^{-1}M$ is $\tau_{\mathcal{N}}$-compactoid, given $|\lambda|>1$ there are $x_1,\dots,x_n$ in E such that

$$\gamma^{-1}M\subset\operatorname{co}(x_1,\dots,x_n)+\lambda^{-1}H^0\,.$$

The set

$$D=\{(f(x_1),\dots,f(x_n)):f\in H\}$$

is bounded in $\mathbb{K}^n$. Let $\mu\in\mathbb{K}$ be such that $|f(x_k)|\le|\mu|$ for all $f\in H$ and for $k=1,2,\dots,n$. If $z^{(k)}=(0,0,\dots,\mu,\dots,0)$, where μ is in the k-position, then $D\subset\operatorname{co}(z^{(1)},\dots,z^{(n)})$. By [**3**, Lemma 1.2], there are $f_1,\dots,f_n$ in H such that $D\subset\lambda\operatorname{co}(c^1,\dots,c^n)$, where $c^k=(f_k(x_1),\dots,f_k(x_n))$. Let now $f\in H$. There are $\gamma_1,\dots,\gamma_n$ in $\mathbb{K}$, $|\gamma_k|\le1$ such that

$$f(x_i)=\lambda\sum_{k=1}^n\gamma_kf_k(x_i),\quad i=1,\dots,n\,.$$

Let $h = f - \lambda \sum_{k=1}^{n} \gamma_k f_k$. If $x \in M$, then

$$\gamma^{-1} x = \sum_{i=1}^{n} \lambda_i x_i + z, \quad |\lambda_i| \leq 1, \quad z \in \lambda^{-1} H^0 .$$

We have

$$h\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i \left(f(x_i) - \lambda \sum_{k=1}^{n} \gamma_k f_k(x_i)\right) = 0 .$$

Thus,

$$|h(\gamma^{-1} x)| = \left| f(z) - \lambda \sum_{k=1}^{n} \gamma_k f_k(z) \right| \leq 1$$

since $f, f_k \in H$ and $z \in \lambda^{-1} H^0$. This proves that $h \in \gamma M^0$. Therefore,

$$H \subset \lambda \operatorname{co}(f_1, \dots, f_n) + \gamma M^0,$$

which proves that H is $\tau_{\mathcal{M}}$-compactoid.

(2) $\Rightarrow$ (1). The proof is analogous. □

Let τ^0 denote the topology on E^b of uniform convergence on the τ-compactoid subsets of E. By τ^{00} we will denote the topology on E of uniform convergence on the τ^0-compactoid subsets of E^b.

Proposition 3.13. *If a set $B \subset E^b$ is τ^0-compactoid, then its bipolar B^{00}, with respect to the pair $\langle E, E^b \rangle$, is also τ^0-compactoid.*

Proof. Let $A \subset E$ be τ-compactoid and let A^0 be its polar in E^b. Let $|\lambda| > 1$ and let $f_1, \dots, f_n$ in E^b be such that

$$B \subset \operatorname{co}(f_1, \dots, f_n) + \lambda^{-1} A^0 .$$

By an argument similar to the one used by Schikhof in [**9**, Corollary 5.8], we get that

$$\begin{aligned} B^{00} &\subset [\operatorname{co}(f_1, \dots, f_n) + \lambda^{-1} A^0]^{00} \subset [\operatorname{co}(f_1, \dots, f_n) + \lambda^{-1} A^0]^e \\ &\subset \operatorname{co}(\lambda f_1, \dots, \lambda f_n) + A^0 , \end{aligned}$$

which proves that B^{00} is τ^0-compactoid. □

Lemma 3.14. *If τ is polar, then $\tau \leq \tau^{00}$.*

Proof. Consider the dual pair $\langle E, E' \rangle$, $E' = (E, \tau)'$, and take $\mathcal{M}$ the family of all τ-compactoid subsets of E and $\mathcal{N}$ the family of all τ-equicontinuous subsets of E'. Since τ is polar, we have that $\tau = \tau_{\mathcal{N}}$. Using Proposition 3.12, we get that each $H \in \mathcal{N}$ is $\tau_{\mathcal{M}}$-compactoid and so H is τ^0-compactoid. If now V is a polar τ-neighborhood of zero, we have that $V = V^{00}$ is a τ^{00}-neighborhood of zero, and this proves that $\tau \leq \tau^{00}$. □

Proposition 3.15. *For every locally convex space (E,τ), we have $\tau^{00} = \tau_\pi^c$.*

Proof. Let $\mathcal{M}$ be the family of all τ-compactoid subsets of E and $\mathcal{N}$ the family of all τ^0-compactoid subsets of E^b. Since $\tau_{\mathcal{M}} = \tau^0$ and $\tau_{\mathcal{N}} = \tau^{00}$, it follows from Proposition 3.12 that every τ-compactoid set is τ^{00}-compactoid and so $\tau^{00} \le \tau^c$. Since τ^{00} is a polar topology, we have that $\tau^{00} \le \tau_\pi^c$. On the other hand, if $\tau_1 = \tau_\pi^c$, then every τ-compactoid set is τ_1-compactoid and so every τ-bounded set is τ_1-bounded, which implies that $G = (E,\tau_1)^b \subset E^b$ and $\tau^0|G \le \tau_1^0$. If $H \subset G$ is τ_1^0-compactoid, then H is τ^0-compactoid, and this implies that $\tau_1^{00} \le \tau^{00}$. Since τ_1 is polar, we have (by the preceding Proposition) $\tau_1 \le \tau_1^{00} \le \tau^{00}$ and so $\tau^{00} = \tau_\pi^c$.□

Proposition 3.16. *$\tau^{00} = \tau_\pi^c$ is the finest of all polar topologies on E which agree with $\sigma(E,E^b)$ on τ-compactoid sets.*

Proof. The topology τ^{00} is polar and $(E,\tau^{00})' = E^b$. If $A \subset E$ is τ-compactoid, then A is τ^{00}-compactoid and so $\tau^{00} = \sigma(E,E^b)$ on A by [**9**, Theorem 5.12]. On the other hand, let τ_1 be a polar topology on E agreeing with $\sigma(E,E^b)$ on τ-compactoid sets. Let A be an absolutely convex τ-compactoid set. Then, A is $\sigma(E,E^b)$-bounded and hence A is $\sigma(E,E^b)$-compactoid. Since $\tau_1 = \sigma(E,E^b)$ on A, it follows that A is τ_1-compactoid by [**10**, Proposition 4.5]. Thus, every τ-compactoid set is τ_1-compactoid, which implies that $\tau_1 \le \tau^{00}$ since τ_1 is polar.□

Corollary 3.17. *An absolutely convex subset V of E is a τ_π^c-neighborhood of zero iff for each τ-compactoid set A there are $f_1,\dots,f_n$ in E^b such that*

$$\bigcap_{k=1}^{n} \{x \in E : |f_k(x)| \le 1\} \cap A \subset V\,.$$

Corollary 3.18. *Let τ be a polar topology and assume that $E^b = E'$. Then, τ_π^c is the finest of all polar locally convex topologies on E which agree with τ on τ-compactoid sets.*

Proof. It follows from Proposition 3.16 since $\tau = \sigma(E,E')$ on τ-compactoid sets by [**9**, Theorem 5.12]. □

Proposition 3.19. *For a locally convex space (E,τ), the following are equivalent:*

(1) $\tau_\pi^s \le \tau_\pi^c$.
(2) *Every τ-compactoid set is τ_π^s-compactoid.*
(3) *Every sequentially τ-equicontinuous subset of E^b is τ^0-compactoid.*
(4) *The topologies τ_π^s and $\sigma(E,E^s)$ coincide on τ-compactoid sets.*
(5) *The topologies τ^0 and $\sigma(E^b,E)$ coincide on every sequentially τ-equicontinuous subset of E^b.*

Proof. (1) $\Rightarrow$ (2) It follows from the fact that every τ-compactoid set is τ_π^c-compactoid.

(2) $\Rightarrow$ (1) It is obvious.

(1) $\Rightarrow$ (3) Let $H \subset E^b$ be sequentially τ-equicontinuous. Then, H^0 is a τ_π^s-neighborhood of zero and so H^0 is a τ_π^c-neighborhood of zero. Therefore, there exists a τ^0-compactoid subset B of E^b such that $B^0 \subset H^0$. It follows that $H \subset B^{00}$ and B^{00} is τ^0-compactoid by Proposition 3.13.

(3) $\Rightarrow$ (1) It follows from Propositions 2.15 and 3.15.

(2) $\Rightarrow$ (4) Since $E^s = (E, \tau_\pi^s)'$, the topologies τ_π^s and $\sigma(E, E^s)$ coincide on τ_π^s-compactoid sets (by [**9**, Theorem 5.12]) and so they coincide on τ-compactoid sets.

(4) $\Rightarrow$ (2) Let A be an absolutely convex τ-compactoid. Then, A is τ-bounded and so A is $\sigma(E, E^s)$-bounded, which implies that A is $\sigma(E, E^s)$-compactoid. By [**10**, Proposition 4.5], A is τ_π^s-compactoid.

(3) $\Rightarrow$ (5) Let $H \subset E^b$ be sequentially τ-equicontinuous. Then, H is τ^0-compactoid. Consider the topologies τ^0 and $\sigma(E^b, E)$. The topology τ^0 is finer than $\sigma(E^b, E)$ and it has a base at zero consisting of absolutely convex $\sigma(E^b, E)$-closed sets. By [**10**, Theorem 1.4], $\tau^0 = \sigma(E^b, E)$ on H.

(5) $\Rightarrow$ (3) Let H be a sequentially τ-equicontinuous subset of E^b. Without loss of generality, we may assume that H is absolutely convex. Since H is $\sigma(E^b, E)$-bounded, it is $\sigma(E^b, E)$-compactoid and so H is τ^0-compactoid by [**10**, Proposition 4.5]. □

Proposition 3.20. *For every locally convex space (E, τ), the following are equivalent:*

(1) $\tau_\pi^c \leq \tau_\pi^s$.

(2) $E^s = E^b$.

(3) τ_π^s *coincides with the topology of uniform convergence on the* τ^n*-compactoid subsets of* E^b.

(4) $\tau_\pi^s \geq (\tau_\pi^c)_\pi^s$.

(5) τ_π^s *is finer than the topology of uniform convergence on the* τ^n*-null sequences in* E^b.

Proof. (1) $\Rightarrow$ (2)

$$E^b = (E, \tau_\pi^c)' \subset (E, \tau_\pi^s)' = E^s \subset E^b .$$

(2) $\Rightarrow$ (3) It follows from Proposition 2.15 and 2.16.

(3) $\Rightarrow$ (1) Let τ_1 be the topology of uniform convergence on the τ^n-compactoid subsets of E^b. Since every τ-null sequence is τ-compactoid, we have that $\tau^n \leq \tau^0$ and hence every τ^0-compactoid is also τ^n-compactoid. Therefore, $\tau_1 \geq \tau^{00} = \tau_\pi^c$.

(1) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (1) Since τ_π^c is polar, we have

$$\tau_\pi^c \leq (\tau_\pi^c)_\pi^s \leq \tau_\pi^s .$$

(3) $\Rightarrow$ (5) It follows from the fact that every τ^n-null sequence is τ^n-compactoid.

(5) $\Rightarrow$ (2) Let τ_2 be the topology of uniform convergence on the τ^n-null sequences in E^b. Since τ_2 is finer than $\sigma(E, E^b)$, we have that $E^b \subset (E, \tau_2)'$ and so

$$E^s \subset E^b \subset (E, \tau_2)' \subset (E, \tau_\pi^s)' = E^s .$$

$\square$

Note. If τ is polar, then $\tau \leq \tau_\pi^c$ and so $\tau_\pi^s \leq (\tau_\pi^c)_\pi^s$. Hence, in case of polar τ, in the preceding Proposition (4) may be replaced by $\tau_\pi^s = (\tau_\pi^c)_\pi^s$.

Proposition 3.21. *If (E, τ) is a polar space which is polarly bornological, then $\tau = \tau_\pi^s = \tau_\pi^c$.*

Proof. Let $f \in E^b$. The seminorm $p(x) = |f(x)|$ is polar and it is bounded on τ-bounded sets and so p is τ-continuous, since (E, τ) is polarly bornological. It follows that every $f \in E^b$ is τ-continuous and so $E^b = E^s = E'$. Since τ is polar, we have $\tau \leq \tau_\pi^c$. Thus, by Propositions 2.8 and 3.20, we have $\tau \leq \tau_\pi^c \leq \tau_\pi^s \leq \tau$. $\square$

Examples. Taking as τ either the topology $\sigma(\ell^\infty, c_0)$ or the topology $\sigma(c_0, \ell^\infty)$, we will look at the topologies τ^s, τ_π^s, τ^c, τ_π^c.

Every element y of ℓ^∞ defines a continuous linear functional f_y on c_0 by $f_y(x) = \langle x, y \rangle = \sum_{n=1}^\infty x_n y_n$. Moreover, $\|f_y\| = \|y\|$. Using the principle of uniform boundedness, we get that a subset of ℓ^∞ is norm-bounded iff it is $\sigma(\ell^\infty, c_0)$-bounded. Also, a subset of c_0 is norm-bounded iff it is $\sigma(c_0, \ell^\infty)$-bounded by [**9**, Corollary 7.7].

We will need the following

Proposition 3.22. *Let $\tau = \sigma(\ell^\infty, c_0)$ and let τ_1 be the finest locally convex topology on ℓ^∞ agreeing with τ on norm-bounded (equivalently on τ-bounded) sets. Then:*

1) *The topology τ_1 is polar.*

2) *If τ_2 is the metrizable locally convex topology on ℓ^∞ generated by the countable family of seminorms $\{p_n : n \in \mathbb{N}\}$, $p_n(x) = |x_n|$, then τ_2 coincides with τ on norm-bounded sets and hence $\tau|A$ is metrizable on each τ-bounded set A.*

Proof. 1) Let $B_n = \{x \in \ell^\infty : \|x\| \leq n\}$. By [**5**, Theorem 5.2], τ_1 has a base at zero consisting of sets W of the form

$$(*) \qquad W = W_0 \cap \left(\bigcap_{n=1}^\infty (B_n + W_n) \right),$$

where each W_n is a τ-neighborhood of zero. We may assume that, for each n, there exists a finite subset S_n of c_0 such that $W_n = S_n^0$, where S_n^0 is the polar of S_n in ℓ^∞. If $F = (\ell^\infty, \tau_1)$, then each W_n is a polar set in F. Note that $\tau \leq \tau_1$

and so $c_0 \subset F'$. Since B_n is τ-bounded, it is τ-compactoid and hence, for $|\lambda| > 1$, there exists a finite subset A_n of B_n such that

$$B_n \subset \lambda c_0(A_n) + W_n \,.$$

Taking bipolars with respect to the pair $\langle F, F' \rangle$ and using [**9**, Corollary 5.8], we get

$$\begin{aligned}(B_n + W_n)^{00} &\subset (\lambda c_0(A_n) + W_n)^{00} = (\lambda c_0(A_n) + W_n)^e \\ &\subset \lambda(\lambda c_0(A_n) + W_n \subset \lambda^2(c_0(A_n) + W_n) \subset \lambda^2(B_n + W_n)\end{aligned}$$

and so $W^{00} \subset \lambda^2 W$. This proves that τ_1 has a base at zero consisting of polar sets and so it is a polar topology.

2) Clearly $\tau_2 \leq \tau$. To show that $\tau_2 = \tau$ on norm-bounded sets, let (x^α) be a norm-bounded net in E with $x^\alpha \xrightarrow{\tau_2} 0$. Let $\varepsilon > 0$ and let $d \geq \sup_\alpha \|x^\alpha\|$. Let $y \in c_0$ and choose $\varepsilon_1 > 0$ with $\varepsilon_1(d + \|y\|) < \varepsilon$. Let m be such that $|y_n| < \varepsilon_1$ if $n > m$. If α_0 is such that $|x_k^\alpha| < \varepsilon_1$ for all $\alpha \geq \alpha_0$ and all $k = 1, 2, \ldots, m$, then for $\alpha \geq \alpha_0$ we have

$$|\langle x^\alpha, y \rangle| = \left| \sum_n x_n^\alpha y_n \right| \leq \varepsilon \,.$$

□

The proof of the next Proposition is similar to the one of Proposition 3.22.

Proposition 3.23. 1) *On each norm-bounded subset of c_0, the weak topology $\tau = \sigma(c_0, \ell^\infty)$ coincides with the metrizable locally convex topology generated by the countable family of seminorms $\{p_n : n \in \mathbb{N}\}$, $p_n(x) = |x_n|$.*

2) *If τ_3 is the finest locally convex topology on c_0 agreeing with τ on norm-bounded sets. then τ_3 is polar.*

Example I. *Let $E = \ell^\infty$ and $\tau = \sigma(\ell^\infty, c_0)$. If τ_1 is as in Proposition* 3.22, *then:*

1) *$\tau^s = \tau_\pi^s = \tau_1$ and each of these topologies is strictly coarser than the norm topology $\tau_{\|\cdot\|}$ and strictly finer than τ.*

2) *If $\mathbb{K}$ is not spherically complete, then $\tau_1 = \tau_\pi^s = \tau^s = \tau_\pi^c$.*

Proof. We show first that $\tau = \tau^s$ on each norm-bounded set A. To show that $\tau^s|A \leq \tau|A$, it suffices to prove that the identity map

$$I \colon (A, \tau|A) \to (A, \tau^s|A)$$

is continuous. Since $\tau|A$ is metrizable, it suffices to show that I is sequentially continuous, which is clearly true. Thus $\tau|A = \tau^s|A$ and so $\tau^s \leq \tau_1$ by the definition of τ_1. To prove that $\tau_1 \leq \tau^s$, it is sufficient to show that every τ-null sequence is τ_1-null. So, let (y^n) be a τ-null sequence. Then (y^n) is τ-bounded and hence norm-bounded. Since $\tau = \tau_1$ on norm-bounded sets, we have that (y^n) is a τ_1-null sequence. Thus $\tau_1 = \tau^s$. In view of Proposition 3.2, the topology τ^s is polar and so $\tau^s = \tau_\pi^s$. The topology τ_1 is coarser than the norm topology. In fact, let V be an absolutely convex subset of E which is not a norm-neighborhood of zero. There exists a sequence (y^n) in E, $\|y^n\| \le \frac{1}{n}$, $y_n \notin V$. Since $\tau_1 = \tau$ on norm-bounded sets and since $y^n \xrightarrow{\tau} 0$, it follows that $y^n \xrightarrow{\tau_1} 0$ and this implies that V is not a τ_1-neighborhood of zero. So, $\tau_1 \le \tau_{\|\cdot\|}$. But $\tau_1 \ne \tau_{\|\cdot\|}$. In fact, let $e^n \in \ell^\infty$, $e^n_k = 0$ if $k \ne n$ and $e^n_n = 1$. The sequence (e^n) is τ-null and hence τ_1-null, since (e^n) is norm-bounded, but (e^n) is not a norm-null sequence.

Finally, τ_1 is strictly finer than τ. In fact, let $0 < |\lambda| < 1$. The sequence $(\lambda^n e^n)$ is a sequentially τ-equicontinuous subset of E^b. If $H = \{\lambda^n e^n : n \in \mathbb{N}\}$, then the polar H^0 of H in E is a τ_π^s-neighborhood of zero by Proposition 2.15. But H^0 is not a τ-neighborhood of zero. In fact, assume that there exists a finite subset S of c_0 such that $S^0 \subset H^0$. Using [**11**, Corollary 1.2], we get that the set $[c_0(S)]^e$ is $\sigma(c_0, l^\infty)$-closed. Taking bipolars with respect to the pair $\langle c_0, l^\infty\rangle$, and using [**9**, Proposition 4.10], we get

$$
\begin{aligned}
H \subset H^{00} \subset S^{00} &= [c_0(S)]^{00} \\
&= \left[\overline{c_0(S)}^{\sigma(c_0, l^\infty)}\right]^e \\
&= [c_0(S)]^e \subset \lambda c_0(S)
\end{aligned}
$$

for $|\lambda| > 1$, which cannot hold since H is linearly independent. This proves that τ is strictly coarser than $\tau_\pi^s = \tau_1$.

2) Assume that $\mathbb{K}$ is not spherically complete. Then $(\ell^\infty, \|\cdot\|)' = c_0$. Since a subset of ℓ^∞ is norm-bounded iff it is τ-bounded, it follows that

$$E^s = E^b = c_0 = (\ell^\infty, \|\cdot\|)' = (E, \tau)'.$$

Since a subset A of E is τ-compactoid iff it is τ-bounded and this is true iff A is norm-bounded, it follows from Corollary 3.18 that $\tau_\pi^c = \tau_1$. Also since a subset A of E is τ-compactoid iff it is norm-bounded, it follows that the topology τ^0 on $E^b = c_0$ is the norm topology of c_0. As it is well known, a subset H of c_0 is norm-compactoid iff there exists $z \in c_0$ such that

$$H \subset H_z = \{y \in c_0 : |y_n| \le |z_n| \quad \text{for all } n\}.$$

If p_z is defined on ℓ^∞ by $p_z(x) = \sup_n |z_n x_n|$, then the polar of H_z in ℓ^∞ coincides with the set

$$\{x \in \ell^\infty : p_z(x) \le 1\}.$$

Thus $\tau^{00} = \tau_\pi^c$ is generated by the family of seminorms $\{p_z : z \in c_0\}$. $\square$

Example II. *Let $E = c_0$ and $\tau = \sigma(c_0, \ell^\infty)$. Then*
1) $\tau^s = \tau^s_\pi = \tau_{\|\cdot\|}$.
2) $\tau^c = \tau^c_\pi$ *and this topology is strictly coarser than the norm topology* $\tau_{\|\cdot\|}$ *of* c_0.

Proof. 1) It is well known that a sequence in c_0 is norm-convergent iff it is weakly convergent (see [**12**, p. 158]). It follows from this that the norm topology $\tau_{\|\cdot\|}$ on c_0 is coarser than τ^s_π since the norm topology is polar. On the other hand, every norm-convergent sequence is τ-convergent and hence τ^s-convergent, which implies that $\tau^s \leq \tau_{\|\cdot\|}$. Therefore $\tau^s = \tau^s_\pi = \tau_{\|\cdot\|}$.

2) There are norm-bounded subsets of c_0 which are not norm-compactoid. If A is such a set, then A is τ-compactoid (since it is τ-bounded) and hence τ^c-compactoid. This implies that $\tau^c \neq \tau_{\|\cdot\|}$. If V is an absolutely convex subset of E which is not a norm-neighborhood of zero and if $0 < |\lambda| < 1$, then there exists a sequence (y^n) in E with $y^n \notin V$ and $\|y^n\| \leq |\lambda|^n$. Now the sequence $(z^n) = (\lambda^{-n} y^n)$ is τ-bounded and hence τ^c-bounded. Since $z^n \notin \lambda^{-n} V$ for all n, it follows that V is not a τ^c-neighborhood of zero. This proves that τ^c is strictly coarser that the norm topology. To prove that $\tau^c = \tau^c_\pi$, we consider the topology τ_3 defined in Proposition 3.23. Let A be a τ-bounded set and consider the identity map

$$I : (A, \tau|A) \to (A, \tau^c|A).$$

If (y^n) is a sequence in A with $y^n \xrightarrow{\tau} x$, then $y^n \to x$ in the norm topology, which implies that $y^n \xrightarrow{\tau^c} x$ since τ^c is coarser than $\tau_{\|\cdot\|}$. Thus I is sequentially continuous and hence I is continuous since $\tau|A$ is metrizable by Proposition 3.23. It follows that $\tau^c|A = \tau|A$, for each τ-bounded set A, and so $\tau^c \leq \tau_3$. It is also clear that $E^b = c_0' = \ell^\infty$ since the τ-bounded subsets of E coincide with the norm-bounded sets. Thus $E^b = (E, \tau)'$. Since τ and τ_3 are polar topologies, we have that $\tau_3 = \tau^c_\pi$ and so $\tau^c = \tau^c_\pi = \tau_3$. □

Remark 3.24. Let F be a subspace of a locally convex space (E, τ) and let $\tau_1 = \tau|F$. It is easy to see that $\tau^c|F \leq \tau_1^c$, $\tau^c_\pi|F \leq (\tau_1)^c_\pi$, $\tau^s|F \leq \tau_1^s$, $\tau^s_\pi|F \leq (\tau_1)^s_\pi$. The following is an example where $\tau^c_\pi|F \neq (\tau_1)^c_\pi$, $\tau^s|F \neq \tau_1^s$, $\tau^s_\pi|F \neq (\tau_1)^s_\pi$.

Example. Assume that $\mathbb{K}$ is not spherically complete and take $E = \ell^\infty$, $\tau = \sigma(\ell^\infty, c_0)$. As we have seen $\tau^c_\pi = \tau^s = \tau^s_\pi$. Let $F = c_0$, $\tau_1 = \tau|F$. Since a subset of F is τ_1-bounded iff it is norm-bounded, it follows that $(F, \tau_1)^b = c_0' = \ell^\infty$. Let $z \in \ell^\infty \setminus c_0$. The set

$$V = \{x \in c_0 : |\langle x, z \rangle| \leq 1\}$$

is a $\tau_1^{00} = (\tau_1)^c_\pi$ neighborhood of zero. But V is not a neighborhood of zero with respect to the topology $\tau_2 = \tau^c_\pi|F$. In fact, if V were a τ_2-neighborhood of zero, then there would exist $y \in c_0$ such that

$$(*) \qquad W = \{x \in c_0 : p_y(x) \leq 1\} \subset V\,.$$

It follows easily from (*) that $|z_n| \le |y_n|$, for all n, and so $z \in c_0$, a contradiction. Thus τ_2 is strictly coarser than $(\tau_1)^c_\pi$. Since V is clearly a convex sequential τ_1-neighborhood of zero, it follows that $\tau^s|F$ is strictly coarser than τ_1^s and that $\tau_\pi^s|F$ is strictly coarser than $(\tau_1)^s_\pi$.

4. Sequential Spaces of Continuous Functions

Let X be a zero-dimensional Hausdorff topological space and let $\beta_0 X$ be its Banaschewski compactification (see [**1**]). As in [**1**], $\upsilon_0 X$ is the set of all $x \in \beta_0 X$ with the following property: For every sequence (V_n) of neighborhoods of x in $\beta_0 X$ we have $\cap_{n=1}^\infty V_n \cap X \neq \emptyset$. By [**1**, Theorem 9], $\upsilon_0 X$ is the $\mathbb{N}$-repletion $\upsilon_{\mathbb{N}} X$ of X. Let E be a Haudorff locally convex space over $\mathbb{K}$ and let $C(X,E)$ be the space of all continuous E-valued functions on X. We will denote by $C_s(X,E)$ (resp. $C_c(X,E)$) the space $C(X,E)$ equipped with the topology of simple convergence (resp. of compact convergence). If f is a function from X to E, p a seminorm on E and A a subset of X, we define

$$\|f\|_{A,p} = \sup\{p(f(x)) : x \in A\}.$$

Let now p be a non-zero non-Archimedean continuous seminorm on E and set $G^p = \{p(s) : s \in E\}$. On G^p we consider the ultrametric d defined by

$$d(a,b) = \begin{cases} 0, & \text{if } a = b \\ \max\{a,b\}, & \text{if } a \neq b. \end{cases}$$

Under this metric, G^p becomes a real-compact, strongly ultraregular, non-compact topological space and so $u_{G^p} X = \upsilon_0 X = \upsilon_{\mathbb{N}} X$ (see [**1**, Theorem 9]). If f is in $C(X,E)$, then the function

$$f_p \colon X \to G^p, \quad f_p(x) = p(f(x)),$$

is continuous and so it has a continuous extension $\tilde{f}_p$ to all of $\upsilon_0 X$.

Proposition 4.1. *If $C_s(X,E)$ or $C_c(X,E)$ is sequential, then E is sequential and X is $\mathbb{N}$-replete.*

Proof. Let W be a convex sequential neighborhood of zero in E. Let $F = C_s(X,E)$ (resp. $F = C_c(X,E)$) and suppose that F is sequential. Let $x_0 \in X$. The set

$$V = \{f \in F : f(x_0) \in V\}$$

is a convex sequential neighborhood of zero in F. Since F is sequential, there exists a finite (resp. compact) subset D of X and a continuous seminorm q on E such that

$$(*) \qquad \{f \in F : \|f\|_{D,q} \le 1\} \subset V\,.$$

It follows from (*) that
$$\{s \in E : q(s) \leq 1\} \subset V$$
and so V is a neighborhood of zero in E.
To prove that X is $\mathbb{N}$-replete, suppose that there exists $y_0 \in v_0X \setminus X$. Let p be a continuous non-zero seminorm on E. We define q on F by $q(f) = \tilde{f}_p(y_0)$. It is easy to see that q is a non-Archimedean seminorm on F. Moreover, q is sequentially continuous.
In fact, let $f_n \to 0$ in F and suppose that $q(f_n) \not\to 0$. Going to a subsequence if necessary, we may assume that $q(f_n) > \varepsilon > 0$ for all n. Set
$$V_n = \{y \in u_0X : (\tilde{f}_n)_p(y) > \varepsilon\}.$$
Each V_n is a neighborhood of y_0 in v_0X and so $\cap_{n=1}^{\infty} V_n \cap X \neq \emptyset$. Let $z \in V_n \cap X$ for all n. For this z, we have $p(f_n(z)) > \varepsilon$ for all n, which contradicts the fact that $f_n \to 0$ in F.

Proposition 4.2. *If E is metrizable, then the following assertions are equivalent:*

(1) $C_s(X,E)$ *is bornological.*
(2) $C_s(X,E)$ *is sequential.*
(3) $C_c(X,E)$ *is bornological.*
(4) $C_c(X,E)$ *is sequential.*
(5) $C_s(X,\mathbb{K})$ *is bornological.*
(6) $C_s(X,\mathbb{K})$ *is sequential.*
(7) $C_c(X,\mathbb{K})$ *is bornological.*
(8) $C_c(X,\mathbb{K})$ *is sequential.*
(9) X *is* $\mathbb{N}$-*replete.*

Proof. Since every bornological space is sequential, it follows from the preceding Proposition that each of the (1)–(8) implies (9). Also, by [**6**, Theorem 2.9], (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (9) and (5) $\Leftrightarrow$ (7) $\Leftrightarrow$ (9). Thus the result follows. $\square$

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