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ON THE NILPOTENCY OF THE JACOBSON RADICAL OF SEMIGROUP RINGS

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Munn [**11**] proved that the Jacobson radical of a commutative semigroup ring is nil provided that the radical of the coefficient ring is nil. This was generalized, for semigroup algebras satisfying polynomial identities, by Okniński [**14**] (cf. [**15**, Chapter 21]), and for semigroup rings of commutative semigroups with Noetherian rings of coefficients, by Jespers [**4**]. It would be interesting to obtain similar results concerning rings with nilpotent Jacobson radical. For band rings this was accomplished in [**12**], and for special band-graded rings in [**13**, §6]. However, for commutative semigroup rings analogous implication concerning the nilpotency of the radicals is not true: it follows from [**7**, Theorems 44.1 and 44.2], that if F is a field with $\operatorname{char} F = p$ and G is an infinite abelian p-group, then the Jacobson radical $J(FG)$ is nil but not nilpotent.

On the other hand, Braun [**1**] proved that the Jacobson radical of every finitely generated PI-algebra over a Noetherian ring is nilpotent. This famous result has several important corollaries (cf. [**9**], [**19**]). It shows that the existence of a finite generating set is a natural condition which may influence the nilpotency of the Jacobson radical of a ring. We shall prove the following

Theorem 1. *Let S be a finitely generated commutative semigroup, R a ring. If $J(R)$ is nilpotent, then $J(RS)$ is nilpotent, too.*

Note that the ring of coefficients is not necessarily commutative, and so RS may be not a PI-ring. Besides, RS may have no finite generating sets, although S is finitely generated. The commutativity of S cannot be removed from Theorem 1. Indeed, there exists a finitely generated solvable group G and a field F such that the Jacobson radical $J(FG)$ is nil but is not nilpotent (cf. [**7**, Theorem 46.32], and [**17**, Lemma 8.1.16]).

Our second theorem characterizes all commutative semigroups satisfying the property we are concerned with. First, we need a few definitions. A semigroup Y is called a **semilattice** if it entirely consists of idempotents. A semigroup S is said to be a **semilattice Y of its subsemigroups S_y**, $y \in Y$, if and only if

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$S = \cup_{y \in Y} S_y$, $S_y \cap S_z = \emptyset$ whenever $y \neq z$, and $S_y S_z \subseteq S_{yz}$ for all $y, z \in Y$. By Theorem 4.13 of [**3**] each commutative semigroup S can be uniquely represented as a semilattice of its Archimedean subsemigroups S_y, $y \in Y$. Then semigroups S_y are called the **Archimedean components** of S.

Let R be a ring, p a prime number. A commutative semigroup S is said to be **separative** (p-**separative**) if, for any $s, t \in S$, the equality $s^2 = st = t^2$ ($s^p = t^p$) implies $s = t$. The least separative (p-separative) congruence on S is denoted by ζ (resectively, ζ_p). Explicitly

$$\zeta = \{\, (s,t) \mid \exists\, n : st^n = t^{n+1} \ \text{ and } \ s^n t = s^{n+1} \,\},$$
$$\zeta_p = \{\, (s,t) \mid \exists\, n : s^{p^n} = t^{p^n} \,\}.$$

Let ρ be a congruence on S. Then $I(R, S, \rho)$ denotes the ideal

$$\left\{ \sum_i r_i(s_i - t_i) \mid r_i \in R, s_i, t_i \in S,\, (s_i, t_i) \in \rho \right\}$$

of RS. If T is separative, then all Archimedean components of T are cancellative by [**3**, Theorem 4.16].

Theorem 2. *Let R be an associative ring, S a commutative semigroup, $\mathbf{Z}$ the ring of integers, $T = S/\zeta$. Denote by T_y, $y \in Y$, the Archimedean components of T. Put $Q = \cup_{y \in Y} Q_y$, where Q_y is the group of quotients of T_y. Then the following are equivalent:*

(1) *$J(R)$ nilpotent implies $J(RS)$ nilpotent;*
(2) *$I(\mathbf{Z}, S, \zeta)$ is nilpotent and there exists a positive integer n such that every finite subgroup of Q has $\leq n$ elements.*

Now we shall give an example which shows that it is difficult to describe semigroups S with nilpotent $I(\mathbf{Z}, S, \zeta)$ in terms of the Archimedean components of S.

Example 3. Let S be the semigroup with generators $x_1, x_2, \ldots, 0_1, 0_2, \ldots$ subjected to relations $0_n x_m = x_m 0_n = 0_m 0_n = 0_n 0_m = 0_m x_n = x_n 0_m = 0_m$ whenever $m \geq n \geq 1$; and $x_1^{\alpha_1} \ldots x_k^{\alpha_k} = 0_k$ whenever $\alpha_k \geq 1$, $\alpha_i \geq 0$ for $1 \leq i \leq k$, and $\alpha_i \geq 2$ for some i. Put

$$S_i = \{x_1^{\alpha_1} \ldots x_i^{\alpha_i} \mid \alpha_1, \ldots, \alpha_i \geq 0; \alpha_i \geq 1\} \cup \{0_i\}.$$

Denote by Y the semilattice of all positive integers with multiplication $mn = \max\{m, n\}$. Then $S = \cup_{y \in Y} S_y$ is a semilattice of semigroups. Clearly $S_y^2 = 0_y$ for every $y \in Y$. Therefore $S/\zeta \cong Y$. For $y \in Y$, consider elements $r_y = x_y - 0_y$ of the semigroup ring $\mathbf{Z}S$. Obviously, all of them belong to $I(\mathbf{Z}, S, \zeta)$. Besides

$r_1r_2\dots r_k = x_1x_2\dots x_k - 0_k \neq 0$. Thus $I(\mathbf{Z}, S, \zeta)$ is not nilpotent, though S is a semilattice of semigroups with zero multiplication.

Throughout S will be a commutative semigroup. For the previous results on the Jacobson radical of RS we refer to [**5**] and [**8**]. Let $\mathbf{P}$ be the set of all prime numbers. For any positive integer n, we put $J_n(R) = \{r \in R \mid nr \in J(R)\}$. We shall use the following

Lemma 1 ([**16**]). *If R is a ring with nilpotent Jacobson radical, then*

$$J(RS) = J(R)S + I(R, S, \zeta) + \sum_{p\in\mathbf{P}} I(J_p(R), S, \zeta_p).$$

Lemma 2 ([**2**]). *Let Y be a finite semilattice, S a semilattice Y of semigroups S_y. If $J(RS_y)$ is nilpotent for every $y \in Y$, then $J(RS)$ is nilpotent.*

In fact in [**2**] a much more general result is obtained. In our special case the proof also easily follows from [**20**], the proof of Theorem 1, by induction on $|Y|$. For the sake of completeness we include this proof.

Proof. The case where $|Y| = 1$ is trivial. Assume that $|Y| > 1$ and that the claim has been proved for all finite semilattices V with $|V| < |Y|$. Consider the partial order $\leq$ defined on Y by $y \leq z \Leftrightarrow yz = y$. Let m be a maximal element of Y. Then $V = Y\backslash\{m\}$ is a subsemilattice of Y, and $T = \cup_{y\in V} S_y$ is a semilattice V of the S_y. Put $I = J(RS)$. Denote by I_m the natural projection of I on RS_m. It follows from [**20**], the proof of Theorem 1, that $I_m \subseteq J(RS_m)$. Therefore $I_m^n = 0$ for some $n > 0$. Hence $I^n \subseteq J(RT)$. Since $|V| < |Y|$, the induction assumption completes the proof. □

Lemma 3. *If R is a ring with nilpotent Jacobson radical, G an abelian group, S a subsemigroup of G, then $J(RS) = RS \cap J(RG)$.*

Proof. Obviously $J(R)S = RS \cap J(R)G$. In view of Lemma 1 we may factor out $J(R)G$ from RG and assume that $J(R) = 0$. Further, given that G is a group, it easily follows that $I(R, G, \zeta) = 0$, and so $I(R, S, \zeta) = 0$. For $p \in \mathbf{P}$ put $R_p = \{r \in R \mid pr = 0\}$. Then

$$J(RG) = \sum_{p\in\mathbf{P}} I(R_p, G, \zeta_p),$$
$$J(RS) = \sum_{p\in\mathbf{P}} I(R_p, S, \zeta_p)$$

by Lemma 1. Put $T = \oplus_{p\in\mathbf{P}} R_p$. We get $J(RG) = J(TG)$ and $J(RS) = J(TS)$. Since T is the direct sum of the R_p, to simplify the notation we may assume that $R = R_p$ from the very beginning. Then $J(RG) = I(R, G, \zeta_p)$ and $J(RS) =$

$I(R_p, S, \zeta_p)$. The inclusion $I(R_p, S, \zeta_p) \subseteq I(R_p, G, \zeta_p)$ immediately follows from the definition of these ideals. Therefore $J(RS) \subseteq RS \cap J(RG)$.

Now take any $x \in RS \cap J(RG)$, say $x = \sum_{i=1}^{n} r_i(s_i - t_i)$ where $r_i \in R$, $(s_i, t_i) \in \zeta_p$. Suppose that n is the minimal possible number. Then we claim that all s_i, t_i belong to S.

Suppose to the contrary that s_1 is not in S. Since $x \in RS$, the summand $r_1 s_1$ must be cancelled, and so s_1 occurs in some other summands. Let $s_1 = s_2 = \ldots = s_k$ and let all $s_{k+1}, \ldots, s_n, t_1, \ldots, t_n$ be distinct from s_1. Then $\sum_{i=1}^{k} r_i = 0$. By the transitivity of ζ_p we can rewrite x as a sum of $(n-1)$ summands:

$$x = r_1(t_2 - t_1) + (r_2 + r_1)(t_3 - t_2) + \cdots + (r_{k-1} + \cdots + r_1)(t_k - t_{k-1}) + \sum_{i=k+1}^{n} r_i(s_i - t_i).$$

The contradiction with the minimality of n shows that $x \in I(R, S, \zeta_p)$. Thus $J(RS) \supseteq RS \cap J(RG)$, which completes the proof. □

It was proved in [**10**] (cf. [**16**]) that $I(R, S, \zeta)$ is a sum of nilpotent ideals of RS. Now we shall show that more can be said for finitely generated S.

Lemma 4. *If S is finitely generated, then the ideal $I(R, S, \zeta)$ is nilpotent.*

Proof. Let $\mathbf{Q}$ be the field of rational numbers. It follows from Braun's Theorem (cf. [**1**]) that $J(\mathbf{Q}S)$ is nilpotent. Lemma 1 shows that $I(\mathbf{Q}, S, \zeta)^n = 0$ for some $n \geq 1$. Hence $I(\mathbf{Z}, S, \zeta)^n = 0$, where $\mathbf{Z}$ stands for the ring of integers. The definition of $I(R, S, \zeta)$ implies that every element of this ideal is a sum of several summands of the form $r_i u_i$, where $r_i \in R$, $u_i \in I(\mathbf{Z}, S, \zeta)$. Therefore $I(R, S, \zeta)^n = 0$. □

Lemma 5. *If G is a finitely generated abelian group and R is a ring with nilpotent Jacobson radical, then $J(RG)$ is nilpotent.*

Proof. By [**6**, Theorem 8.1.2], G is a direct product of a finite group T and a torsion-free group H. The radical $J(RT)$ is nilpotent by [**14**, Lemma 1.1]. Let $J(RT)^n = 0$. Since $R(T \times H) = (RT)H$, and H is torsion-free, Lemma 1 yields $J(RG) = J(RT)H$. Hence $J(RG)^n = 0$, as required. □

Lemma 5 also follows from [**7**, Theorem 43.6].

Proof of Theorem 1. Let S be a semilattice Y of its Archimedean subsemigroups S_y. Lemma 4 implies that $I(R, S, \zeta)$ is nilpotent. Since $RS/I(R, S, \zeta) \cong R(S/\zeta)$, we can replace S by S/ζ and RS by $R(S/\zeta)$ without affecting the hypothesis or conclusion of the theorem. Thus it remains to consider the case when S is separative.

By [**3**, Theorem 4.16], all S_y are cancellative. Although some of the S_y may be not finitely generated, we shall check that each S_y is contained in a finitely generated abelian group. Indeed, each S_y has a group of quostients Q_y. Let e_y be the identity element of Q_y. Put $Q = \cup_{y\in Y} Q_y$. The multiplication of S can be easily extended to the whole Q so that $e_y e_z = e_{yz}$. Then Q is a **strong semilattice** of the groups Q_y, $y \in Y$ (cf. [**18**]). Fix any $z \in Y$. Given that S is finitely generated, it is easily seen that $V = \cup_{y\geq z} S_y$ is a finitely generated subsemigroup of S. Since the mapping $f\colon s \longmapsto se_z$ is a homomorphism from V into Q_z, it follows that S_z is contained in the finitely generated subsemigroup $f(V)$. Therefore Q_z is finitely generated. Now Lemmas 3 and 5 imply that $J(RS_y)$ is nilpotent for every $y \in Y$. This and Lemma 2 complete the proof. □

Proof of Theorem 2. As in the proof of Theorem 1,in view of Lemma 1 and the fact that $RS/I(R, S, \zeta) \cong R(S/\zeta)$, it suffices to prove the theorem for a separative semigroup S.

(1) ⇒ (2): Suppose to the contrary that (1) holds but Q contains arbitrarily large finite subgroups. Then, for any positive integer m, there exist a prime p and a finite p-subgroup G of S with $|G| > m$. Let D denote the direct sum of all simple fields $F_p = GF(p)$ for all prime p. By (1) $J(DS)^n = 0$ for some $n \geq 1$.

Take any prime p and a finite p-subgroup G of Q. Let $|G| = p^m$. Then G is the direct product of cyclic groups: $G = G_1 \times \ldots \times G_k$. Denote by g_i the generator of G_i and let $|G_i| = p^{m_i}$ where $i = 1, \ldots, k$. There exists $y \in Y$ such that $G \subseteq Q_y$. Keeping in mind that Q_y is the group of quotients of S_y, denote by s the product of the denominators of all elements of G. Then $sG \subseteq S$. Consider elements $h_i = s - sg_i$ of F_pS, for $i = 1, \ldots, k$. Lemma 1 yields $h_1, \ldots, h_k \in J(F_pS) \subseteq J(DS)$. Put $q_i = p^{m_i} - 1$ for $i = 1, \ldots, k$. Streightforward (although lengthy) calculations show that

$$h_i^{q_i} = s^{q_i} \sum_{g\in G_i} g,$$

$$h_1^{q_1} \ldots h_k^{q_k} = s^{q_1+\ldots+q_k} \sum_{g\in G} g.$$

Therefore $n \geq p^{m_1} + \ldots + p^{m_k} - k \geq m_1 + \ldots + m_k - k$. However, the right hand side can be made greater than n, if we choose $m = m_1 + \cdots + m_k$ sufficiently large. This contradiction shows that (1) implies (2).

(2) ⇒ (1): Take any ring R with nilpotent Jacobson radical. In view of Lemma 1 we may pass to the quotient ring $R/J(R)$ and assume that $J(R) = 0$. Put $P = \oplus_{p\in\mathbf{P}} R_p$, where $R_p = \{r \in R \mid pr = 0\}$. By Lemma 1 we get $J(RS) = J(PS)$. Since P is the direct sum of R_p, $p \in \mathbf{P}$, it remains to show that there exists a positive integer m such that $J(R_pS)^m = 0$ for all $p \in \mathbf{P}$. To simplify the notation we fix a prime p and assume that $R = R_p$ is a semisimple algebra over the field F_p. Put $m = n$, where n is taken from (2). We claim that $J(RS)^m = 0$.

Lemma 1 easily shows that $J(RS) \subseteq J(RQ)$ and so it suffices to prove that $J(RQ)^m = 0$. A standard verification using Lemma 1 gives us $J(RQ) =$

$\oplus_{y\in Y} J(RQ_y)$. Take any m elements $r_1 \in J(RQ_{y_1}), \dots, r_m \in J(RQ_{y_m})$. We need to show that $r_1 \dots r_m = 0$.

Put $y = y_1 \dots y_m$ and denote by e the identity element of Q_y. Let $z_i = ey_i$ for $i = 1, \dots, m$. It is routine to verify with Lemma 1 that $z_1, \dots, z_m \in J(RQ_y)$. Denote by T the torsion part of Q_y. Obviously, $|T| \leq n$. By Lemma 1 we get $J(RQ_\gamma) = J(RT)RQ_\gamma$ and the nilpotency index of $J(RQ_\gamma)$ is equal to the nilpotency index of $J(RT)$ (see [**7**, Proposition 52.1]). Since $J(RT)^{|T|} = 0$ by [**7**, Theorem 30.34], and $|T| \leq m$, we get $z_1 \dots z_m = 0$. This completes the proof.

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