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STATISTICAL MODELLING OF DEFORMATION MEASUREMENTS

L. KUBÁČEK

1. Introduction

Many engineer's construction works (e.g., bridges, dams, gas holders, crane runways) before putting into operation and during operation must be tested as far as the deformations caused by their loading are concerned.

In order to verify that the actual deformations lie within safety limits determined in advance (e.g., by static experts) replicated measurements have to be carried out in a network of suitably chosen points whose positions are studied.

The statistical problems of processing this kind of measurements (problems of determining the first order and second order parameters and testing hypotheses on existence or non-existence of deformations) are solved in the paper.

2. Notations and Auxiliary Statements

Let β_i, $i = 1, \dots, k$, denote parameters whose values must be determined by deformation measurements. The results of measurements are given by a realization y of an n-dimensional random vector Y. The class of distribution functions connected with Y is $\mathcal{F} = \{F(\cdot, \beta, \vartheta) \colon \beta \in R^k, \vartheta \in \underline{\vartheta}\}$, where $\beta = (\beta_1, \dots, \beta_k)'$ ($'$ denotes the transposition of a matrix) $\in R^k$ (k-dimensional Euclidean space), $\vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta}$ (an open set in R^p); the parameter ϑ, which characterizes the accuracy of measurement techniques, is also a priori unknown and must be determined on the basis of the vector Y. The class $\mathcal{F}$ is supposed to have the following two properties

$$\int_{R^n} u dF(u, \beta, \vartheta) = X\beta, \ \beta \in R^k,$$

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where X is an $n \times k$ matrix which is known (so called design matrix, cf. [**Ká1**, chpt. 5]) and

$$\int_{R^n} (u - X\beta)(u - X\beta)' dF(u, \beta, \vartheta) = \sum_{i=1}^{p} \vartheta_i V_i, \ \beta \in R^k, \ \vartheta \in \underline{\vartheta},$$

where $V_1, \dots, V_p$ are known symmetric matrices.

In the following, for the sake of simplicity, the rank of the matrix X is supposed to be $r(X) = k < n$ and the set $\underline{\vartheta}$ to have the property

$$\vartheta \in \underline{\vartheta} \ \Rightarrow \ \Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i \quad \text{is p.d. (positive definite).}$$

These assumptions are realistic and have the following meaning. The measurements are not affected by a systematic influence (cf. [**K1**]) and thus the mean value of Y is a known vector function of β only and this fuction can be linearized; further this function does not depend on ϑ which characterizes the accuracy of the measurement devices used. The assumption that the covariance matrix $\Sigma(\vartheta)$ does not depend on the measured parameter β is sometimes not satisfied (cf. [**W**]). Nevertheless, this assumption is accepted because of the following two reasons: In many cases a deviation of the reality from this assumption has no essential influence on the estimate of β and further the problems connected with respecting the fact that the covariance matrix depends also on β are difficult and overcome the framework of this paper.

The just described statistical model is denoted as

$$(Y, X\beta, \sum_{i=1}^{p} \vartheta_i V_i), \ \beta \in R^k, \ \vartheta \in \underline{\vartheta}$$

and, in view of the assumptions $r(X) = k < n$, and $\Sigma(\vartheta)$ is p.d. for $\vartheta \in \underline{\vartheta}$, it is called regular. In this model two kinds of estimators are considered: linear estimators $L'Y$ of a given linear function $f(\beta) = f'\beta$, $\beta \in R^k$, and quadratic estimators $Y'AY$ (A a symmetric matrix) of a given linear function $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$.

Definition 2.1. The ϑ_0-LBLUE (locally best linear unbiased estimator) of a function

$$f(\beta) = f'\beta, \quad \beta \in R^k,$$

in the model $(Y, X\beta, \sum_{i=1}^{p} \vartheta_i V_i)$, $\beta \in R^k$, $\vartheta \in \underline{\vartheta}$, is a statistic $L'Y$ such that

(i) $\forall\{\beta \in R^k, \vartheta \in \underline{\vartheta}\}$ $E(L'Y|\beta) = \int_{R^n} L'u dF(u, \beta, \vartheta) = f'\beta$

and

(ii) $\forall\{L_1 \in R^n$, L_1 satisfying (i)$\}$

$$\mathrm{Var}(L'Y|\vartheta_0) = \int_{R^n} (L'u - f'\beta)^2 dF(u, \beta, \vartheta_0) \leq \mathrm{Var}(L_1'Y|\vartheta_0).$$

Definition 2.2. The ϑ_0-MINQUE (minimum norm quadratic unbiased estimator; cf. [**R1**, p. 304]) of a function $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$, in the model $(Y, X\beta, \sum_{i=1}^{p} \vartheta_i V_i)$, $\beta \in R^k$, $\vartheta \in \underline{\vartheta}$, is a statistic $Y'AY$ possesing the properties

(i) $\forall\{\beta \in R^k,\ \vartheta \in \underline{\vartheta}\} E(Y'AY|\beta,\vartheta) = g'\vartheta$,

(ii) $\forall\{\delta \in R^k\}(Y + X\delta)'A(Y + X\delta) = Y'AY$

and

(iii) $\forall\{\overline{A} : \overline{A}$ satysfying (i) and (ii), $\overline{A} = \overline{A}'\}$

$$\operatorname{tr}(A\Sigma_0 A\Sigma_0) \leq \operatorname{tr}(\overline{A}\Sigma_0\overline{A}\Sigma_0).$$

Here $\Sigma_0 = \Sigma(\vartheta_0)$.

The motivation of Definition 2.2 lies in two facts: the defined estimator is closely connected with natural estimators (cf. [**R1**, p. 303] or [**R3**]) and in the case of normality of the vector Y ($Y \sim N_n(X\beta, \Sigma(\vartheta))$) it represents the ϑ_0-LMVIQUE (locally minimum variance invariant quadratic unbiased estimator). The invariance is characterized by (ii) in Definition 2.2.

Lemma 2.3. *In the model*

$$(Y_{(n,1)}, X_{(n,k)}\beta_{(k,1)}, \Sigma(\vartheta)),\ \beta \in R^k,\ \vartheta \in \underline{\vartheta},$$

(i) *an unbiased linear estimator of a function* $f(\beta) = f'\beta$, $\beta \in R^k$, *exists iff* $f \in M(X') = \{X'u :\ u \in R^n\}$;

(ii) *the* ϑ_0*-LBLUE of the function* $f(\cdot)$ *in the regular model is*

$$\widehat{f'\beta} = f'(X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}Y,$$

and its dispersion at ϑ_0 *is*

$$\operatorname{Var}(\widehat{f'\beta}|\vartheta_0) = f'(X'\Sigma_0^{-1}X)^{-1}f.$$

Proof. Cf. [**R1**]. □

Lemma 2.4. *The* ϑ_0*-MINQUE of a function* $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$,

(i) *exists iff* $g \in M(C^{(I)})$, *where*

$$\{C^{(I)}\}_{i,j} = \operatorname{tr}(M_X V_i M_X V_j),\ i,j = 1,\dots,p,$$

$M_X = I - X(X'X)^{-1}X'$, *I is the identity matrix, and*

(ii) *if* $g \in M(C^{(I)})$, *the* ϑ_0*-MINQUE is*

$$\widehat{g'\vartheta} = Y'AY = \sum_{i=1}^{p} \lambda_i Y'(M_X\Sigma_0 M_X)^+ V_i (M_X\Sigma_0 M_X)^+ Y,$$

with $(M_X\Sigma_0M_X)^+ = \Sigma_0^{-1} - \Sigma_0^{-1}X(X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}$, *where* $\lambda = (\lambda_1, \dots, \lambda_p)'$ *is a solution of the equation*

$$S_{(M_X\Sigma_0M_X)^+}\lambda = g;$$

here

$$\{S_{(M_X\Sigma_0M_X)^+}\}_{i,j} = \mathrm{tr}[(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+V_j], \ i,j = 1, \dots, p,$$

and $(\cdot)^+$ *denotes the Moore-Penrose inverse (cf.* [**R2**]*) of the matrix in the brackets;*
(iii) *if* $Y \sim N_n(X\beta, \Sigma_0)$, *the dispersion of the estimator* $\widehat{g'\vartheta}$ *at* ϑ_0 *is*

$$\mathrm{Var}(\widehat{g'\vartheta}|\vartheta_0) = 2g'S^-_{(M_X\Sigma_0M_X)^+}g;$$

the expression for the variance does not depend on the choice of the generalized inverse $S^-_{(M_X\Sigma_0M_X)^+}$ *of the matrix* $S_{(M_X\Sigma_0M_X)^+}$.

Proof. Cf. [**R3**] or [**Se2**]. □

Lemma 2.5. *Let* $Y \sim N_n(X_{(n,k)}\beta, \Sigma)$, $\beta \in R^k$. *Let* Σ *be known;* $r(X) = k$ *and* $r(\Sigma) = n$. *A null-hypothesis* $A\beta = a$ *on* β, *where* A *is a* $q \times k$ *matrix with the rank* $r(A) = q < k$, *and* a *is a given vector, can be tested against* $A\beta \neq a$ *using the statistic*

$$(A\hat{\beta} - a)'[A(X'\Sigma^{-1}X)^{-1}A']^{-1}(A\hat{\beta} - a),$$

which has, if the null hypothesis is true, the central chi-square distribution with q *degrees of freedom;* $\hat{\beta}$ *is the* Σ*-LBLUE of* β.

Proof. Cf. [**R1**]. □

3. Models with Stable and Variable Parameters

One possibility how to design an experiment of deformation measurements is to decompose the vector β into two parts $\beta_1 \in R^{k_1}$ and $\beta_2 \in R^{k_2}$, $k_1 + k_2 = k$, in such a way that β_1 is connected with stable points and β_2 with variable ones. The following example can serve as explanation of the situation.

Example. Let $(0,0,0)$, $(\gamma_1, 0, \gamma_2)$, $(\gamma_3, \gamma_4, \gamma_5)$ be cartesian coordinates of three points P_1, P_2, and P_3 in R^3, respectively, which are located on the river-side in the neighbourhood of a bridge the deformation of which is investigated; they can be considered to be stable because a loading of the bridge cannot influence the coordinates $\gamma_1, \dots, \gamma_5$. Let

$$(\gamma_6, \gamma_7, \gamma_8), \ (\gamma_9, \gamma_{10}, \gamma_{11}), \ (\gamma_{12}, \gamma_{13}, \gamma_{14})$$

be coordinates of points A, B, C which are located on the construction of the bridge at positions determined by a static expert. The problem is to determine the values of parameters $\gamma_6, \dots, \gamma_{14}$, influenced by the given loading. If the measurements are performed in such a way that the horizontal distances among the points P_1, P_2, P_3, A, B, C and horizontal and vertical angles in different triangles given by these points are measured, the parameters $\gamma_6, \dots, \gamma_{14}$ can be estimated only on the basis of $\gamma_1, \dots, \gamma_5$.

If $\beta_1 = (\gamma_1, \dots, \gamma_5)'$ and $\beta_2 = (\gamma_6, \dots, \gamma_{14})'$, then in the most simple case the model of these measurements is

$$\left(Y, (X_1, X_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \vartheta_1 V_1 + \vartheta_2 V_2\right),$$

where ϑ_1 is the dispersion of the angle measurements and ϑ_2 is the dispersion of the distance measurements. Of course this experiment must be replicated for different ways of the loading of the bridge; it is performed in several epochs. During these epochs the parameters ϑ_1 and ϑ_2 can be sometimes considered as stable. Therefore in the following this simple case is considered only.

Definition 3.1. An m-epoch model with stable and variable parameters with the same design in each epoch is

$$\left(\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix}, (\mathbf{1}\otimes X_1, I\otimes X_2)\begin{pmatrix}\beta_1\\ \beta_2^{(\cdot)}\end{pmatrix}, \sum_{i=1}^{p}\vartheta_i(I\otimes V_i)\right),$$

$$\beta_1 \in R^{k_1},\ \beta_2^{(\cdot)} \in R^{mk_2},\ \vartheta \in \underline{\vartheta}.$$

Here $Y_1, \dots, Y_m$ are stochastically independent n-dimensional random vectors, $\mathbf{1} = (1, \dots, 1)' \in R^m$, I is the $m \times m$ identity matrix,

$$\beta_2^{(\cdot)} = (\beta_2^{(1)'}, \dots, \beta_2^{(m)'})',$$

$\beta_2^{(j)}$ is the value of the parameter β_2 in the jth epoch, $j = 1, \dots, m$ and $\otimes$ denotes the Kronecker product of matrices.

For the following let us remind the assumption on the regularity of the model (before Definition 2.1), i.e., in the model from Definition 3.1 it holds $r(X_1, X_2) = k = k_1 + k_2$ and $r(\sum_{i=1}^{p}\vartheta_i V_i) = n$.

Theorem 3.2. *In the model from Definition* 3.1

(i) *the ϑ_0-LBLUE of the vector $(\beta_1', \beta_2^{(\cdot)'})'$ is*

$$\begin{pmatrix}\hat\beta_1(\underline{Y})\\ \hat\beta_2^{(\cdot)}(\underline{Y})\end{pmatrix} = \begin{pmatrix}[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}X_1'(M_{X_2}\Sigma_0 M_{X_2})^+\overline{Y}\\ (I\otimes B)v_1 + \mathbf{1}\otimes v_2\end{pmatrix},$$

where $\underline{Y} = (Y_1', \dots, Y_m')'$, $\overline{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_j$, $B = (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}$,

$$v_1 = [(Y_1 - \overline{Y})', \dots, (Y_m - \overline{Y})']',$$
$$v_2 = [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1}X_2'(M_{X_1}\Sigma_0 M_{X_1})^+\overline{Y},$$

$M_{X_1} = I - X_1(X_1'X_1)^{-1}X_1'$, $M_{X_2} = I - X_2(X_2'X_2)^{-1}X_2'$;

(ii)

$$\operatorname{Var}\left[\begin{pmatrix} \hat{\beta}_1(\underline{Y}) \\ \hat{\beta}_2^{(\cdot)}(\underline{Y}) \end{pmatrix} \Big| \Sigma_0\right] = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix},$$

$$\begin{aligned}
\boxed{11} &= \frac{1}{m}[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1},\\
\boxed{12} &= -\frac{\mathbf{1}'}{m} \otimes (X_1'\Sigma_0^{-1}X_1)^{-1}X_1'\Sigma_0^{-1}X_2[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}\\
&= -\frac{\mathbf{1}'}{m} \otimes [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}\\
&= \boxed{21}\,',\\
\boxed{22} &= M_m \otimes (X_2'\Sigma_0^{-1}X_2)^{-1} + P_m \otimes [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1},
\end{aligned}$$

where $P_m = \mathbf{1}\mathbf{1}'/m = P_m^2$, $M_m = I - P_m = M_m^2$ *with* $P_m M_m = 0 = M_m P_m$.

Proof. With respect to Lemma 2.3 and Definition 3.1 the ϑ_0-LBLUE of the vector $(\beta_1', \beta_2^{(\cdot)'})'$ is

$$\left[\begin{pmatrix} \mathbf{1}' \otimes X_1' \\ I \otimes X_2' \end{pmatrix} (I \otimes \Sigma_0^{-1})(\mathbf{1} \otimes X_1, I \otimes X_2)\right]^{-1} \begin{pmatrix} \mathbf{1}' \otimes X_1' \\ I \otimes X_2' \end{pmatrix} (I \otimes \Sigma_0^{-1})\underline{Y}.$$

Regarding the assumptions that $r(X) = k = k_1 + k_2 = r(X_1, X_2)$ and $\Sigma_0 = \Sigma(\vartheta_0)$ is p.d., the matrix

$$\begin{pmatrix} \mathbf{1}' \otimes X_1' \\ I \otimes X_2' \end{pmatrix} (I \otimes \Sigma_0^{-1})(\mathbf{1} \otimes X_1, I \otimes X_2) = \begin{pmatrix} m \otimes X_1'\Sigma_0^{-1}X_1, & \mathbf{1}' \otimes X_1'\Sigma_0^{-1}X_2 \\ \mathbf{1} \otimes X_2'\Sigma_0^{-1}X_1, & I \otimes X_2'\Sigma_0^{-1}X_2 \end{pmatrix}$$

is regular, thus its inverse exists. It can be verified that this inverse is

$$\begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}.$$

Here the equalities

$$\begin{aligned}
&\begin{pmatrix} A, & B \\ B', & C \end{pmatrix}^{-1} =\\
&= \begin{pmatrix} A^{-1} + A^{-1}B(C - B'A^{-1}B)^{-1}B'A^{-1}, & -A^{-1}B(C - B'A^{-1}B)^{-1} \\ -(C - B'A^{-1}B)^{-1}B'A^{-1}, & (C - B'A^{-1}B)^{-1} \end{pmatrix}\\
&= \begin{pmatrix} (A - BC^{-1}B')^{-1}, & -(A - BC^{-1}B')^{-1}BC^{-1} \\ -C^{-1}B'(A - BC^{-1}B')^{-1}, & C^{-1} + C^{-1}B'(A - BC^{-1}B')^{-1}BC^{-1} \end{pmatrix}
\end{aligned}$$

valid for a p.d. matrix $\begin{pmatrix} A, & B \\ B', & C \end{pmatrix}$ are utilized. In our case it is also necessary to use the equalities

$$(M_{X_i}\Sigma_0 M_{X_i})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}X_i(X_i'\Sigma_0^{-1}X_i)^{-1}X_i'\Sigma_0^{-1},\ i = 1,2,$$

and

$$\begin{gathered}[M_m \otimes X_2'\Sigma_0^{-1}X_2 + P_m \otimes X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1} = \\ = M_m \otimes (X_2'\Sigma_0 X_2)^{-1} + P_m \otimes [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}\end{gathered}$$

which can be directly verified.

The assertion (ii) is a direct consequence of Lemma 2.3 and the assertion (i).□

Theorem 3.3. *The ϑ_0-LBLUE of $(\beta_1', \beta_2^{(j)'})'$, based on the vector Y_j only, is*

$$\begin{pmatrix} \hat\beta_1(Y_j) \\ \hat\beta_2^{(j)}(Y_j) \end{pmatrix} = \begin{pmatrix} [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}X_1'(M_{X_2}\Sigma_0 M_{X_2})^+Y_j \\ [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}X_2'(M_{X_1}\Sigma_0 M_{X_1})^+Y_j \end{pmatrix}$$

and its covariance matrix is

$$\operatorname{Var}\left[\begin{pmatrix} \hat\beta_1(Y_j) \\ \hat\beta_2^{(j)}(Y_j) \end{pmatrix}\Bigg|\Sigma_0\right] = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix},$$

where

$$\begin{aligned}
A &= [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}, \\
B &= -[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1} = C', \\
C &= -[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}X_2'\Sigma_0^{-1}X_1(X_1'\Sigma_0^{-1}X_1)^{-1} = B', \\
D &= [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1}.
\end{aligned}$$

Proof. As $E(Y^{(j)}|\beta_1, \beta_2^{(j)}) = (X_1, X_2)\begin{pmatrix} \beta_1 \\ \beta_2^{(j)} \end{pmatrix}$, the ϑ_0-LBLUE of the vector $(\beta_1', \beta_2^{(j)'})'$ is

$$\left[\begin{pmatrix} X_1' \\ X_2' \end{pmatrix}\Sigma_0^{-1}(X_1, X_2)\right]^{-1}\begin{pmatrix} X_1' \\ X_2' \end{pmatrix}\Sigma_0^{-1}Y^{(j)}.$$

The proof can be easily finished using the analogous equalities as in proof of Theorem 3.2. □

Corollary 3.4. *Theorems* 3.2 *and* 3.3 *form a good basis for comparing results from separate epochs with results from the experiment in the whole. If the normality of the vector Y is presupposed and points connected with β_1 can be considered to be stable, then*

$$\hat{\beta}_1(Y_j) - \hat{\beta}_1(Y_1, \dots, Y_{j-1}) \sim N_{k_1}\{0, [(j)/(j-1)][X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}\}.$$

Thus the statistic

$$[\hat{\beta}_1(Y_j) - \hat{\beta}_1(Y_1, \dots, Y_{j-1})]' \frac{j-1}{j} [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1] \cdot [\hat{\beta}_1(Y_j) - \hat{\beta}_1(Y_1, \dots, Y_{j-1})] \sim \chi^2_{k_1}$$

(distributed as central chi-square with k_1 degrees of freedom) can be used for testing the stability after each epoch. Another possibility is given by the following theorem and corollary.

Theorem 3.5. *Under the normality of the vector $(Y_1', \dots, Y_j')'$, $j \le m$, in the model from Definition* 3.1 *the joint distribution of the ϑ_0-LBLUE of $\beta_2^{(j)}$ based on Y_j and $(Y_1', \dots, Y_j')'$, respectively, is*

$$\begin{pmatrix} \hat{\beta}_2^{(j)}(Y_j) \\ \hat{\beta}_2^{(j)}(Y_1, \dots, Y_j) \end{pmatrix} \sim N_{2k_2}\left[\begin{pmatrix} \beta_2^{(j)} \\ \beta_2^{(j)} \end{pmatrix}, \begin{pmatrix} W_{1,1}, & W_{1,2} \\ W_{2,1}, & W_{2,2} \end{pmatrix}\right],$$

where

$$\begin{aligned} W_{1,1} &= [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1}, \\ W_{1,2} &= \frac{1}{j}[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} + (1 - \frac{1}{j})(X_2'\Sigma_0^{-1}X_2)^{-1} = W_{2,1}', \\ W_{2,2} &= \frac{1}{j}[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} + (1 - \frac{1}{j})(X_2'\Sigma_0^{-1}X_2)^{-1} = W_{1,2} = W_{2,1}. \end{aligned}$$

Proof. Let

$$\begin{aligned} A &= [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} X_2'(M_{X_1}\Sigma_0 M_{X_1})^+, \\ B &= (X_2'\Sigma_0^{-1}X_2)^{-1} X_2'\Sigma_0^{-1}. \end{aligned}$$

Then, with respect to Theorems 3.3 and 3.2,

$$\begin{aligned} \hat{\beta}_2^{(j)}(Y_j) &= A(e_j' \otimes I)(Y_1', \dots Y_j')', \\ \hat{\beta}_2^{(j)}(Y_1, \dots, Y_j) &= \left[A\left(\frac{1' \otimes I)}{j}\right) + B\left(e_j' \otimes I - \frac{1' \otimes I}{j}\right)\right](Y_1,', \dots, Y_j')'. \end{aligned}$$

Now, if the equalities

$$(M_{X_1}\Sigma_0 M_{X_1})^+\Sigma_0(M_{X_1}\Sigma_0 M_{X_1})^+ = (M_{X_1}\Sigma_0 M_{X_1})^+,$$
$$A\Sigma_0 A' = [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1}$$

and

$$A\Sigma_0 B' = (X_2'\Sigma_0^{-1}X_2)^{-1} = B\Sigma_0 B'$$

are taken into account, we obtain

$$W_{1,1} = A\Sigma_0 A', \qquad W_{1,2} = (1-\frac{1}{j})A\Sigma_0 B' + \frac{1}{j}A\Sigma_0 A',$$
$$W_{2,1} = W_{1,2}', \qquad W_{2,2} = (1-\frac{1}{j})B\Sigma_0 B' + \frac{1}{j}A\Sigma_0 A'.$$

Now it is elementary to finish the proof. □

Corollary 3.6. *If the vector* $(Y_1', \dots, Y_m')'$ *is normally distributed, then*

$$\hat\beta_2^{(j)}(Y_j) - \hat\beta_2^{(j)}(Y_1,\dots,Y_j) \sim N_{k_2}(0,K),$$

where

$$K = (1-\frac{1}{j})(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}$$
$$\cdot X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}.$$

Proof. With respect to Theorem 3.5

$$\mathrm{Var}\left\{\left[\hat\beta_2^{(j)}(Y_j) - \hat\beta_2^{(j)}(Y_1,\dots,Y_j)\right]|\Sigma_0\right\} = W_{1,1} + W_{2,2} - 2W_{1,2}$$
$$= (1-\frac{1}{j})\left\{[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1} - (X_2'\Sigma_0^{-1}X_2)^{-1}\right\}.$$

With respect to the equality

$$(M_{X_1}\Sigma_0 M_{X_1})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}X_1(X_1'\Sigma_0^{-1}X_1)^+X_1'\Sigma_0^{-1},$$

we can write

$$[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1} = [X_2'\Sigma_0^{-1}X_2 - X_2'\Sigma_0^{-1}X_1(X_1'\Sigma_0^{-1}X_1)^{-1}X_1'\Sigma_0^{-1}X_2]^{-1}.$$

Now, using the equality

$$(S_1 - AS_2A')^{-1} = S_1^{-1} + S_1^{-1}A(S_2^{-1} - A'S_1^{-1}A)^{-1}A'S_1^{-1},$$

where $S_1 = X_2'\Sigma_0^{-1}X_2$, $A = X_2'\Sigma_0^{-1}X_1$, $S_2 = (X_1'\Sigma_0^{-1}X_1)^{-1}$, we obtain

$$[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+X_2]^{-1} = (X_2'\Sigma_0^{-1}X_2)^{-1} + (X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1$$
$$\cdot [X_1'\Sigma_0^{-1}X_1 - X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1]^{-1}X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}.$$

As

$$[X_1'\Sigma_0^{-1}X_1 - X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1]^{-1} = [X_1'(M_{X_2}\Sigma_0 M_{X_2})^+X_1]^{-1}$$

the proof is finished. □

Corollary 3.7. *With respect to Theorem* 3.3

$$\operatorname{Var}\left[\hat{\beta}_2^{(j)}(Y_j)|\Sigma_0\right] = [X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1}$$

and with respect to Theorem 3.5,

$$\operatorname{Var}\left[\hat{\beta}_2^{(j)}(Y_1,\dots,Y_j)|\Sigma_0\right] = (1-\frac{1}{j})(X_2'\Sigma_0^{-1}X_2)^{-1} + \frac{1}{j}[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1}.$$

Thus

$$\operatorname{Var}\left[\hat{\beta}_2^{(j)}(Y_j)|\Sigma_0\right] = \operatorname{Var}\left[\hat{\beta}_2^{(j)}(Y_1,\dots,Y_j)|\Sigma_0\right]$$
$$+ (1-\frac{1}{j})\left\{[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} - (X_2'\Sigma_0^{-1}X_2)^{-1}\right\}.$$

The expression $[X_2'(M_{X_1}\Sigma_0 M_{X_1})^+ X_2]^{-1} - (X_2'\Sigma_0^{-1}X_2)^{-1}$ *is given in Corollary* 3.6*; it shows the effect of the measurements of the preceding epochs on the estimator of* $\beta_2^{(j)}$*. From this point of view the relationship*

$$\operatorname{Var}\left[\hat{\beta}_2^{(j)}(Y_1,\dots,Y_j)|\Sigma_0\right] = (X_2'\Sigma_0^{-1}X_2)^{-1}$$
$$+ \frac{1}{j}(X_2'\Sigma_0^{-1}X_2)^{-1}X_2'\Sigma_0^{-1}X_1[X_1'(M_{X_2}\Sigma_0 M_{X_2})^+ X_1]^{-1}X_1'\Sigma_0^{-1}X_2(X_2'\Sigma_0^{-1}X_2)^{-1}$$

is instructive.

Theorem 3.8. *In the model from Definition* 3.1
(i) *the* ϑ_0*-MINQUE of a function* $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$, *exists iff*

$$g \in M\left[(m-1)S_{M_{X_2}} + S_{M_{(X_1,X_2)}}\right],$$

where

$$\{S_{M_{X_2}}\}_{i,j} = \operatorname{tr}(M_{X_2}V_i M_{X_2}V_j),\ i,j = 1,\dots,p,$$
$$\{S_{M_{(X_1,X_2)}}\}_{i,j} = \operatorname{tr}(M_{(X_1,X_2)}V_i M_{(X_1,X_2)}V_j),\ i,j = 1,\dots,p$$

and

$$M_{X_2} = I - X_2(X_2'X_2)^{-1}X_2,$$
$$M_{(X_1,X_2)} = I - (X_1,X_2)[(X_1,X_2)'(X_1,X_2)]^{-1}(X_1,X_2)',$$

(ii) *if* $g \in M\left[(m-1)S_{M_{X_2}} + S_{M_{(X_1,X_2)}}\right]$, *then* ϑ_0*-MINQUE of the function* $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$, *is*

$$\tau_g(\underline{Y}) = \sum_{i=1}^{p}\lambda_i\{\operatorname{tr}[A_i\sum_{j=1}^{m}(Y_j-\overline{Y})(Y_j-\overline{Y})'] + m\overline{Y}'B_i\overline{Y}\},$$

where

$$\begin{aligned}
\underline{Y} &= (Y_1', \dots, Y_m')',\\
A_i &= (M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+,\\
B_i &= (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_i (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+
\end{aligned}$$

and $\lambda = (\lambda_1, \dots, \lambda_p)'$ *is a solution of the equation*

$$\left((m-1)S_{(M_{X_2}\Sigma_0 M_{X_2})^+} + S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right)\lambda = g,$$

(iii) *if the observation vector* Y *is normally distributed, then the variance of* $\tau_g(\underline{Y})$ *at* ϑ_0 *is*

$$\operatorname{Var}[\tau_g(\underline{Y})|\vartheta_0] = 2g'\left((m-1)S_{(M_{X_2}\Sigma_0 M_{X_2})^+} + S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right)^- g$$

(it does not depend on the choice of the generalized inverse of the matrix in the last expression).

Proof. Applying in Lemma 2.4 the design matrix and the covariance matrix of the model from Definition 3.1 we obtain

(i)
$$\{C^{(I)}\}_{i,j} = \operatorname{tr}\left[M_{(1\otimes X_1, I\otimes X_2)}(I\otimes V_i)M_{(1\otimes X_1, I\otimes X_2)}(I\otimes V_j)\right],$$

$i,j = 1,\dots p,$

$$M_{(1\otimes X_1, I\otimes X_2)} = (M_m\otimes I + P_m\otimes I) - (\mathbf{1}\otimes X_1, I\otimes X_2)\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\mathbf{1}'\otimes X_1'\\ I\otimes X_2'\end{pmatrix},$$

where (cf. Theorem 3.2)

$$\begin{aligned}
\boxed{11} &= \frac{1}{m}(X_1' M_{X_2} X_1)^{-1},\\
\boxed{12} &= -\frac{\mathbf{1}'}{m}\otimes (X_1'X_1)^{-1}X_1'X_2(X_2'M_{X_1}X_2)^{-1}\\
&= -\frac{\mathbf{1}'}{m}\otimes (X_1'M_{X_2}X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}\\
&= \boxed{21}\,',\\
\boxed{22} &= M_m\otimes (X_2'X_2)^{-1} + P_m\otimes (X_2'M_{X_1}X_2)^{-1}.
\end{aligned}$$

Thus

$$(\mathbf{1}\otimes X_1, I\otimes X_2)\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\mathbf{1}'\otimes X_1'\\ I\otimes X_2'\end{pmatrix} = M_m\otimes P_{X_2} + P_m\otimes P_{(X_1,X_2)}.$$

Here the equality $P_{(X_1,X_2)} = P_{X_1}^{M_{X_2}} + P_{X_2}^{M_{X_1}}$, which can be easily verified, is used. We obtain

$$M_{(1\otimes X_1, I\otimes X_2)} = M_m \otimes M_{X_2} + P_m \otimes M_{(X_1,X_2)},$$

what implies

$$\{C^{(I)}\}_{i,j} = (m-1)\operatorname{tr}(M_{X_2}V_iM_{X_2}V_j) + \operatorname{tr}(M_{(X_1,X_2)}V_iM_{(X_1,X_2)}V_j).$$

(ii) Taking into account the equality

$$\begin{aligned}&\big(M_{(1\otimes X_1, I\otimes X_2)}(I\otimes\Sigma_0)M_{(1\otimes X_1, I\otimes X_2)}\big)^+ =\\ &= M_m\otimes\Sigma_0^{-1} + P_m\otimes\Sigma_0^{-1} - (M_m\otimes\Sigma_0^{-1} + P_m\otimes\Sigma_0^{-1})(\mathbf{1}\otimes X_1, I\otimes X_2)\\ &\cdot\begin{pmatrix} m\otimes X_1'\Sigma_0^{-1}X_1, & \mathbf{1}'\otimes X_1'\Sigma_0^{-1}X_2\\ \mathbf{1}\otimes X_2'\Sigma_0^{-1}X_1, & I\otimes X_2'\Sigma_0^{-1}X_2\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{1}'\otimes X_1'\\ I\otimes X_2'\end{pmatrix}(M_m\otimes\Sigma_0^{-1} + P_m\otimes\Sigma_0^{-1})\end{aligned}$$

and expressions $\boxed{11}$, $\boxed{12}$, $\boxed{21}$, $\boxed{22}$ from Theorem 3.2 we obtain

$$\begin{aligned}&\big(M_{(1\otimes X_1, I\otimes X_2)}\ (I\otimes\Sigma_0)M_{(1\otimes X_1, I\otimes X_2)}\big)^+ =\\ &\qquad = M_m\otimes(M_{X_2}\Sigma_0M_{X_2})^+ + P_m\otimes(M_{(X_1,X_2)}\Sigma_0M_{(X_1,X_2)})^+\end{aligned}$$

in an analogous way as in (i). Considering the last expression and the equality

$$\begin{aligned}(Y_1',\dots,Y_m')(M_m\otimes A + P_m\otimes B)\begin{pmatrix}Y_1\\ \vdots\\ Y_m\end{pmatrix} &=\\ &= \operatorname{tr}\left[A\sum_{i=1}^m (Y_i-\overline{Y})(Y_i-\overline{Y})'\right] + m\overline{Y}'B\overline{Y}\end{aligned}$$

we obtain easily the assertion from (ii).

(iii) is a direct consequence of (ii), the assumption of normality and (iii) of Lemma 2.4. □

Remark 3.9. In the following the matrix $S_{(M_{(X_1,X_2)}\Sigma_0M_{(X_1,X_2)})^+}$ will be assumed to be p.d.

This matrix is always at least p.s.d. since it is the Gram matrix of the p-tuple

$$\{[(M_{(X_1,X_2)}\Sigma_0M_{(X_1,X_2)})^+]^{1/2}V_i[(M_{(X_1,X_2)}\Sigma_0M_{(X_1,X_2)})^+]^{1/2}\}_{i=1}^p,$$

in the Hilbert space $\mathcal{M}_{n,n}$ of $n\times n$ matrices with the inner product

$$\langle A,B\rangle = \operatorname{tr}(AB'),\quad A,B\in\mathcal{M}_{n,n}.$$

An important consequence of this assumption consists in the fact that the whole vector ϑ can be estimated by the ϑ_0-MINQUE in each separate epoch; for this reason it must be fulfilled always in practice. For our purposes this assumption enables us to compare easily MINQUEs from a separate epoch and MINQUEs from several epochs.

Corollary 3.10. *The ϑ_0-MINQUE of the function $g(\cdot)$ from Theorem 3.8, based on the observation vector Y_j from the j-th epoch only, is*

$$\hat{\vartheta}(Y_j) = S^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\hat{\gamma},$$

where $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_p)'$,

$$\hat{\gamma}_i = Y_j'(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_i (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ Y_j,$$

$i = 1, \dots, p$, *and if the vector Y is normal,*

$$\operatorname{Var}(\hat{\vartheta}(Y_j)|\Sigma_0) = 2S^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}.$$

Corollary 3.10 and Theorem 3.8 give us a tool for investigating a behaviour of the estimators

$$\hat{\vartheta}(Y_1),\ \hat{\vartheta}(Y_2), \dots,$$

the estimators

$$\hat{\vartheta}(Y_1, Y_2),\ \hat{\vartheta}(Y_1, Y_2, Y_3), \dots$$

and the relations among them.

Remark 3.11. With respect to Corollary 3.10, Theorem 3.8 and Lemma 3.12, three types of estimators of the parameter ϑ can be calculated after m epochs of measurements:

(i)

$$\hat{\vartheta}_1(Y_1, \dots, Y_m) = \left[(m-1)S_{(M_{X_2}\Sigma_0 M_{X_2})^+} + S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right]^{-1}\hat{\gamma},$$

where $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_p)'$,

$$\hat{\gamma}_i = \operatorname{tr}\left[(M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+ \sum_{j=1}^m (Y_j - \overline{Y})(Y_j - \overline{Y})'\right] + m\overline{Y}'(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_i (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ \overline{Y},$$

(ii)

$$\hat{\vartheta}_2(Y_1, \dots, Y_m) = \frac{1}{m-1} S^{-1}_{(M_{X_2}\Sigma_0 M_{X_2})^+}\hat{\kappa},$$

where $\hat{\kappa} = (\hat{\kappa}_1, \dots, \hat{\kappa}_p)'$,

$$\hat{\kappa}_i = \operatorname{tr}\left[(M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+ \sum_{j=1}^m (Y_j - \overline{Y})(Y_j - \overline{Y})'\right]$$

and

(iii)

$$\hat{\vartheta}_3(Y_1,\dots,Y_m) = \frac{1}{m}\sum_{j=1}^{m}\hat{\vartheta}(Y_j),$$

where

$$\hat{\vartheta}(Y_j) = S^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\hat{\omega}, \qquad \hat{\omega} = (\hat{\omega}_1,\dots,\hat{\omega}_p)',$$
$$\hat{\omega}_i = Y_j'(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ V_i (M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+ Y_j,$$

$i = 1,\dots,p;\ j = 1,\dots,m.$

The estimator $\hat{\vartheta}_2(Y_1,\dots,Y_m)$ is based on the following lemma.

Lemma 3.12. *In the model from Definition* 3.1 *for any $n\times n$ symmetric matrix A*

$$E\left\{\operatorname{tr}\left[A\sum_{j=1}^{m}(Y_j-\overline{Y})(Y_j-\overline{Y})'\right]\Big|\Sigma\right\} =$$
$$= (m-1)\operatorname{tr}(A\Sigma) + \sum_{j=1}^{m}(\beta_2^{(j)} - \overline{\beta_2^{(\cdot)}})'X_2'AX_2(\beta_2^{(j)} - \overline{\beta_2^{(\cdot)}}),$$

where $\overline{\beta_2^{(\cdot)}} = \frac{1}{m}\sum_{j=1}^{m}\beta_2^{(j)}$.

Proof. The proof can be made in a straightforward way and therefore it is omitted. □

In our case

$$A = (M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+, \quad i = 1,\dots,p;$$

because of $(M_{X_2}\Sigma_0 M_{X_2})^+ X_2 = 0$,

$$E(\hat{\kappa}_i|\Sigma) = (m-1)\operatorname{tr}\left[(M_{X_2}\Sigma_0 M_{X_2})^+ V_i (M_{X_2}\Sigma_0 M_{X_2})^+ \sum_{j=1}^{m}\vartheta_j V_j\right]$$
$$= (m-1)\left\{S_{(M_{X_2}\Sigma_0 M_{X_2})^+}\right\}_{i.}\vartheta$$

and the last relationship enables us to establish easily the estimator $\hat{\vartheta}_2$.

Remark 3.13. In general it is relatively difficult to compare the variances of the estimators $\hat{\vartheta}_1$, $\hat{\vartheta}_2$ and $\hat{\vartheta}_3$; nevertheless in the case of normality obviously

$$\operatorname{Var}[\hat{\vartheta}_1(Y_1,\dots,Y_m)|\Sigma_0] =$$
$$= 2\left\{(m-1)S_{(M_{X_2}\Sigma_0 M_{X_2})^+} + S_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\right\}^{-1}$$
$$\le \operatorname{Var}[\hat{\vartheta}_2(Y_1,\dots,Y_m)|\Sigma_0] = \frac{2}{m-1}S^{-1}_{(M_{X_2}\Sigma_0 M_{X_2})^+}$$

and

$$\begin{aligned}\operatorname{Var}[\hat{\vartheta}_1(Y_1,\dots,Y_m)|\Sigma_0] &\le \operatorname{Var}[\hat{\vartheta}_3(Y_1,\dots,Y_m)|\Sigma_0]\\ &= (2/m)S^{-1}_{(M_{(X_1,X_2)}\Sigma_0 M_{(X_1,X_2)})^+}\end{aligned}$$

($\le$ means the Loewner ordering of p.s.d. matrices).

Further

$$\lim_{m\to\infty}\operatorname{Var}[\hat{\vartheta}_1(Y_1,\dots,Y_m)|\Sigma_0]\left\{\operatorname{Var}[\hat{\vartheta}_2(Y_1,\dots,Y_m)|\Sigma_0]\right\}^{-1} = I_{(p,p)};$$

thus the estimator $\hat{\vartheta}_2$ is to be preferred to $\hat{\vartheta}_3$ if the number of epochs is sufficiently large. It is rather surprising; $\hat{\vartheta}_1$ has been suggested already by B. Schaffrin ([**Sch2**], Chapter 2.1), and similarly by J. Kleffe ([**Kl**]) and others, though in more general form. It seems that even in the case of a relatively small number of epochs the estimator $\hat{\vartheta}_3$ is significantly worse than $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$, respectively, c.f. Example in [**Ká4**].

It is rather unpleasant when the vector ϑ is unknown and must be estimated. Therefore it may be of some interest to know a class of such linear functions of parameteres β_1 and $\beta_2^{(\cdot)}$ which can be estimated by UBLUEs (uniformly best linear unbiased estimators). In solving this problem we can start from Theorem 5.7.2 in [**K2**]. In our case the following theorem can be stated.

Theorem 3.14. *The class of linear functions of* β_1 *and* $\beta_2^{(\cdot)}$

$$f(\beta_1,\beta_2^{(\cdot)}) = f_1'\beta_1 + f_2^{(1)'}\beta_2^{(1)} + \dots + f_2^{(m)'}\beta_2^{(m)},\ \beta_1\in R^{k_1},\ \beta_2^{(\cdot)}\in R^{mk_2},$$

estimable by UBLUEs is

$$\begin{aligned}\mathcal{F} = \Bigg\{ \begin{pmatrix} f_1\\ f_2^{(1)}\\ \vdots\\ f_2^{(m)}\end{pmatrix} &: f_1 = mX_1'[I-(\sum_{i=1}^p V_i M_{(X_1,X_2)}V_i)^-(\sum_{i=1}^p V_i M_{(X_1,X_2)}V_i)]\overline{L}_\cdot,\\ f_2^{(j)} &= [I-(\sum_{i=1}^p V_i M_{X_2}V_i)^-(\sum_{i=1}^p V_i M_{X_2}V_i)](L_j-\overline{L}_\cdot),\\ &j=1,\dots,m,\ \overline{L}_\cdot = \frac{1}{m}\sum_{j=1}^m L_j,\ L=(L_1',\dots,L_m')'\in R^{mn}\Bigg\}.\end{aligned}$$

Proof. Regarding Theorem 5.7.2 from [**K2**],

$$\mathcal{F} = \mathcal{M}\left\{\begin{pmatrix}1'\otimes X_1'\\ I\otimes X_2'\end{pmatrix}\mathcal{K}er\left[\sum_{i=1}^p (I\otimes V_i)M_{(1\otimes X_1, I\otimes X_2)}(I\otimes V_i)\right]\right\}.$$

As $M_{(1\otimes X_1, I\otimes X_2)} = M_m \otimes M_{X_2} + P_m \otimes M_{(X_1,X_2)}$ (cf. proof of Theorem 3.8) and

$$\sum_{i=1}^{p}(I\otimes V_i)M_{(1\otimes X_1,I\otimes X_2)}(I\otimes V_i) = M_m \otimes \sum_{i=1}^{p} V_i M_{X_2} V_i + P_m \otimes \sum_{i=1}^{p} V_i M_{(X_1,X_2)} V_i,$$

we obtain

$$\begin{aligned}\mathcal{K}er&\left[\sum_{i=1}^{p}(I\otimes V_i)M_{(1\otimes X_1,I\otimes X_2)}(I\otimes V_i)\right] = \\ &= \mathcal{M}\left\{M_m \otimes \left[I - (\sum_{i=1}^{p} V_i M_{X_2} V_i)^{-}(\sum_{i=1}^{p} V_i M_{X_2} V_i)\right]\right. \\ &\quad \left. + P_m \otimes \left[I - (\sum_{i=1}^{p} V_i M_{(X_1,X_2)} V_i)^{-}(\sum_{i=1}^{p} V_i M_{(X_1,X_2)} V_i)\right]\right\}.\end{aligned}$$

Now we can easily finish the proof. □

Another important representant of the class of m-epoch models with stable and variable parameters is

$$\left(\underline{Y}, (\mathbf{1}\otimes X_1, I\otimes X_2)\begin{pmatrix}\beta_1\\ \beta_2^{(\cdot)}\end{pmatrix}, D\otimes V\right), \tag{3.1}$$

where $D = \mathrm{Diag}(\sigma_1^2, \dots, \sigma_m^2)$ is an unknown diagonal matrix, V is a given $n\times n$ p.d. matrix and the other notations have the same meaning as in Definition 3.1. In the following the ratio $\sigma_1^2 : \sigma_2^2 : \dots : \sigma_m^2$ is supposed to be unknown.

Theorem 3.15. *In the model* (3.1)

(i) *the ϑ_0-LBLUE of β_1 based on the observation vector $\underline{Y} = (Y_1', \dots, Y_m')'$ is*

$$\hat{\beta}_1(Y_1,\dots,Y_m) = \left(\sum_{s=1}^{m}\frac{1}{\sigma_{s0}^2}\right)^{-1}\sum_{s=1}^{m}\frac{1}{\sigma_{s0}^2}\hat{\beta}_1(Y_j),$$

where $\vartheta_0 = (\sigma_{10}^2, \dots, \sigma_{m0}^2)'$ and

$$\hat{\beta}_1(Y_j) = [X_1'(M_{X_2}VM_{X_2})^{+}X_1]^{-1}X_1'(M_{X_2}VM_{X_2})^{+}Y_j,\ j=1,\dots,m,$$

(ii) *the ϑ_0-LBLUE of $\beta_2^{(j)}$ based on $\underline{Y}$ is*

$$\hat{\beta}_2^{(j)}(Y_1,\dots,Y_m) = (X_2'V^{-1}X_2)^{-1}X_2'V^{-1}\left\{Y_j - X_1\hat{\beta}_1(Y_1,\dots,Y_m)\right\}$$

and

(iii) *the ϑ_0-MINQUE of the function $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$, is*

$$\widehat{g'\vartheta} = \sum_{i=1}^{m} \lambda_i \frac{1}{\sigma_{i0}^4} [Y_i - X_i \hat{\beta}_1(Y_1, \dots, Y_m)]'(M_{X_2} V M_{X_2})^+ [Y_i - X_i \hat{\beta}_1(Y_1, \dots, Y_m)]$$

if the system

$$S_{(*)}\lambda = g$$

is consistent. Here $\lambda = (\lambda_1, \dots, \lambda_m)'$ and

$$\begin{aligned} S_{(*)} &= (n - k_2) D_0^{-2} + k_1 \left(\sum_{s=1}^{m} \frac{1}{\sigma_{s0}^2} \right)^{-2} \begin{pmatrix} 1/\sigma_{10}^4 \\ \vdots \\ 1/\sigma_{m0}^4 \end{pmatrix} (1/\sigma_{10}^4, \dots, 1/\sigma_{m0}^4) \\ &\quad - 2k_1 \left(\sum_{s=1}^{m} \frac{1}{\sigma_{s0}^2} \right)^{-1} D_0^{-3}; \\ D_0 &= \mathrm{Diag}(\sigma_{10}^2, \dots, \sigma_{m0}^2). \end{aligned}$$

Proof. Let the notation

$$\underline{V}_i = e_i^{(m)} e_i^{(m)'} \otimes V_i, \; i = 1, \dots, m,$$

be used. Then

$$D \otimes V = \sum_{i=1}^{m} \sigma_i^2 \underline{V}_i = \sum_{i=1}^{m} \sigma_i^2 (e_i^{(m)} e_i^{(m)'} \otimes V_i).$$

Now, regarding Lemma 2.3, it is sufficient to reestablish the expression

$$(\underline{X}' \underline{\Sigma}_0^{-1} \underline{X})^{-1} \underline{X}' \underline{\Sigma}_0^{-1} \underline{Y},$$

where

$$\underline{X} = (1 \otimes X_1, I \otimes X_2) \quad \text{and} \quad \underline{\Sigma}_0 = D_0 \otimes V = \sum_{i=1}^{m} \sigma_{i0}^2 (e_i^{(m)} e_i^{(m)'} \otimes V).$$

The formulae given in the proof of Theorem 3.2 must be used in order to obtain the expressions from (i) and (ii).

Regarding Lemma 2.4 and the relationships

$$\begin{aligned} (M_{\underline{X}} \underline{\Sigma}_0 M_{\underline{X}})^+ = D_0^{-1} \otimes (M_{X_2} V M_{X_2})^+ - \frac{D_0^{-1} 1 1' D_0^{-1}}{1' D_0^{-1} 1} \otimes (M_{X_2} V M_{X_2})^+ X_1 \\ \cdot [X_1' (M_{X_2} V M_{X_2})^+ X_1]^{-1} X_1' (M_{X_2} V M_{X_2})^+, \end{aligned}$$

and

$$(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+\underline{Y} = (v_1', \dots, v_m')',$$

where

$$v_i = \frac{1}{\sigma_{i0}^2}(M_{X_2}VM_{X_2})^+Y_i - \frac{1/\sigma_{i0}^2}{\sum_{j=1}^m 1/\sigma_{j0}^2}(M_{X_2}VM_{X_2})^+X_1 \cdot \sum_{j=1}^m (1/\sigma_{j0}^2)[X_1'(M_{X_2}VM_{X_2})^+]^{-1}X_1'(M_{X_2}VM_{X_2})^+Y_j$$

which is applied in its rewritten form

$$v_i = \frac{1}{\sigma_{i0}^2}(M_{X_2}VM_{X_2})^+[Y_i - X_1\hat{\beta}_1(Y_1, \dots, Y_m)],\ i = 1, \dots, m,$$

(the way of rewriting is easy however tedious, therefore is omitted), we obtain

$$\widehat{g'\vartheta} = \sum_{i=1}^m \lambda_i \frac{1}{\sigma_{i0}^4}[Y_i - X_1\hat{\beta}_1(Y_1, \dots, Y_m)]'(M_{X_2}VM_{X_2})^+[Y_i - X_1\hat{\beta}_1(Y_1, \dots, Y_m)].$$

For determining the matrix $S_{(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+}$ the relationships

$$\operatorname{tr}\left[(M_{X_2}VM_{X_2})^+P_{X_1}^{(M_{X_2}VM_{X_2})^+}V(M_{X_2}VM_{X_2})^+V\right] = \operatorname{tr}\left(P_{X_1}^{(M_{X_2}VM_{X_2})^+}\right) = k_1$$

and

$$\operatorname{tr}[(M_{X_2}VM_{X_2})^+V(M_{X_2}VM_{X_2})^+V] = \operatorname{tr}\left(I - P_{X_2}^{V^{-1}}\right) = n - k_2,$$

where

$$P_{X_1}^{(M_{X_2}VM_{X_2})^+} = X_1[X_1'(M_{X_2}VM_{X_2})^+X_1]^{-1}X_1'(M_{X_2}VM_{X_2})^+,$$

and

$$P_{X_2}^{V^{-1}} = X_2(X_2'V^{-1}X_2)^{-1}X_2'V^{-1},$$

must be taken into account. After applying them we obtain

$$\left\{S_{(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+}\right\}_{i,j} = \begin{cases} \frac{n-k_2}{\sigma_{i0}^4} + k_1\frac{1/\sigma_{i0}^8}{\left(\sum_{s=1}^m 1/\sigma_{s0}^2\right)^2} - 2k_1\frac{1/\sigma_{i0}^6}{\sum_{s=1}^m 1/\sigma_{s0}^2}, & i = j \\ k_1\frac{[1/(\sigma_{i0}^2\sigma_{j0}^2)]^2}{\left(\sum_{s=1}^m 1/\sigma_{s0}^2\right)^2}, & i \neq j, \end{cases}$$

thus $S_{(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+}$ has the form given in (iii). □

Example 3.16. Let in Theorem 3.15 $m = 1$, i.e., the first epoch is considered only. Then MINQUE of σ_1^2 becomes the uniformly minimum variance quadratic unbiased invariant estimator of the well known form

$$\hat{\sigma}_1^2 = [1/(n-k_1-k_2)][Y_1 - X_1\hat{\beta}_1(Y_1)]'(M_{X_2}VM_{X_2})^+[Y_1 - X_1\hat{\beta}_1(Y_1)],$$

where

$$\hat{\beta}_1(Y_1) = [X_1'(M_{X_2}VM_{X_2})^+X_1]^{-1}X_1'(M_{X_2}VM_{X_2})^+Y_1.$$

It is to be remarked that after m (> 1) epochs the estimator of σ_1^2 given in Theorem 3.15 is better at least at ϑ_0, cf. A motivating example in [**Ká4**].

Remark 3.17. Up to now the investigated models are the simplest representants of the class of m-epoch models with stable and variable parameters. More complicated models arise when the design matrix (X_1, X_2) is not the same in each epoch, when nuisance parameters occur in separate epochs, when restrictions on parameters β_1 and $\beta_2^{(j)}$, respectively, must be respected, etc. Some investigation of other models from the mentioned class are in [**Ká2**] and [**K3**]; see also [**Sch1**].

4. Models with Variable Parameters

Sometimes it is possible to create such a design of experiment that it is not necessary to measure indirectly the stable parameter β_1 from the preceding section. Then the model

$$\left((Y_1, \dots, Y_m), X(\beta^{(1)}, \beta^{(2)}, \dots, \beta^{(m)}), I \otimes \sum_{i=1}^{p} \vartheta_i V_i\right),$$

can be considered; here X is an analogue of the matrix X_2 from Definition 3.1, $\beta^{(j)}$ is considered instead of $\beta_2^{(j)}$ and $I \otimes \vartheta_i V_i$ is the covariance matrix of the vector $(Y_1,' \dots, Y_m')'$.

Let $Y = (Y_1, \dots, Y_m)$, $B = (\beta^{(1)}, \dots, \beta^{(m)})$ and $\Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i$ — we have the well known model from multivariate statistics (cf. [**S**])

$$(Y, XB, I \otimes \Sigma(\vartheta)); \tag{4.1}$$

in contrast of the standard model here $I \otimes \Sigma(\vartheta)$ occurs instead of $\Sigma(\vartheta) \otimes I$; see also [**Ko1**, p. 301–313]. It is to be remarked that in the model $(Y, XB, \Sigma(\vartheta) \otimes I)$ there exists the uniformly (with respect to $\vartheta \in \underline{\vartheta}$) best linear unbiased estimator $\hat{B}$ of B in the form $\hat{B} = (X'X)^{-1}X'Y$. In our case only the ϑ_0-LBLUE of the form $\hat{B} = [X'\Sigma^{-1}(\vartheta_0)X]^{-1}X'\Sigma^{-1}(\vartheta_0)Y$ exists.

The ϑ_0-MINQUE of ϑ exists iff the matrix $C^{(I)}$,

$$\{C^{(I)}\}_{i,j} = \mathrm{tr}(M_X V_i M_X V_j), \; i, j = 1, \dots, p,$$

is regular. After m epochs the ϑ_0-MINQUE of ϑ is

$$\hat{\vartheta} = \frac{1}{m}\sum_{j=1}^{m} S^{-1}_{(M_X\Sigma_0 M_X)^+}\hat{\gamma}_j,$$

$\hat{\gamma}_j = (\hat{\gamma}_{j,1}, \dots, \hat{\gamma}_{j,p})'$, $\hat{\gamma}_{j,i} = Y_j'(M_X\Sigma_0 M_X)^+ V_i (M_X\Sigma_0 M_X)^+ Y_j$.

If the matrix Y is normally distributed, then

$$\operatorname{Var}[\hat{\vartheta}(Y)|\Sigma_0] = (2/m) S^{-1}_{(M_X\Sigma_0 M_X)^+}.$$

Thus it can be seen that from the viewpoint of estimation no new problems arise in this case. Some testing problems are investigated in [**K4**], which can be compared with the procedures in [**Ko1**]. However in deformation measurements with variable parameters the growth-curve model of R. F. Potthoff/S. N. Roy (1964) [**P**], cf. [**Ká3**], may be better suited than the model just mentioned.

The value of the parameter β in the j-th epoch, i.e., in the time t_j is supposed in the form (the i-th component)

$$\beta_i(t_j) = b_{i,1} + b_{i,2}\phi_1(t_j) + \dots + b_{i,s-1}\phi_{s-1}(t_j).$$

Here $\phi_1(\cdot), \dots, \phi_{s-1}(\cdot)$ are known linearly independent functions defined on R^1 having the property

$$\phi_r(t_1) = 0,\ r = 1, \dots, s-1,$$

e.g., $\phi_r(t) = (t - t_1)^r$, $t \in R^1$, $r = 1, \dots, s-1$. Thus the mean value of the observation matrix Y can be written in the form

$$E(Y|B) = X_{(n,k)} B_{(k,s)} Z_{(s,m)},$$

where

$$B = \begin{pmatrix} b_{1.1}, & b_{1,2}, & \dots, & b_{1,s} \\ \dots & \dots & \dots & \dots \\ b_{k,1}, & b_{k,2}, & \dots, & b_{k,s} \end{pmatrix}$$

and

$$Z = \begin{pmatrix} 1, & 1, & \dots, & 1 \\ \phi_1(t_1), & \phi_1(t_2), & \dots, & \phi_1(t_m) \\ \dots & \dots & \dots & \dots \\ \phi_{s-1}(t_1), & \phi_{s-1}(t_2), & ,\dots, & \phi_{s-1}(t_m) \end{pmatrix}.$$

Thus the deformation measurements with variable parameters result in the following growth-curve model with variance components

$$(Y,\ XBZ,\ \sum_{i=1}^{p}\vartheta_i(I\otimes V_i)). \tag{4.2}$$

Lemma 4.1. *Let $P_{(s,k)}$ be a given $s \times k$ matrix. The function $f(B) = \mathrm{tr}(PB)$, $B \in \mathcal{M}_{k,s}$, is unbiasedly estimable iff there exists an $m \times n$ matrix L such that $P = ZLX$.*

Proof. The proof is obvious: $f(B) = \mathrm{tr}(PB) = (\mathrm{vec}(P'))' \,\mathrm{vec}(B)$ estimable in the model $E(\mathrm{vec}(Y)) = (Z' \otimes X)\,\mathrm{vec}(B) \Leftrightarrow (\mathrm{vec}(P'))' = (\mathrm{vec}(L'))'(Z' \otimes X) = (\mathrm{vec}(X'L'Z'))' \Leftrightarrow P = (X'L'Z')' = ZLX$. □

Corollary 4.2. *The necessary and sufficient condition for the unbiased estimability of each linear function $f(B)$, $B \in \mathcal{M}_{k,s}$, in (4.1) is*

$$r(X) = k \quad \textit{and} \quad r(Z) = s,$$

which means that the number of the epochs must be equal or greater than s.

Lemma 4.3. *Let in (4.1) $r(X_{n,k)}) = k < n$ and $r(Z_{(s,m)}) = s < m$. Then the ϑ_0-LBLUE of the matrix B is*

$$\hat{B}(\underline{Y}) = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}\underline{Y}Z'(ZZ')^{-1}$$

and

$$\mathrm{Var}\left\{\mathrm{vec}\left[\hat{B}(\underline{Y})\right]|\Sigma_0\right\} = (ZZ')^{-1} \otimes (X'\Sigma_0^{-1}X)^{-1}.$$

Proof. The model (4.1) can be rewritten, according to [**Se1**], as

$$\left((Y_1', \dots, Y_m')', (Z' \otimes X)\,\mathrm{vec}(B), I \otimes \sum_{i=1}^{p} \vartheta_i V_i\right).$$

Now it is sufficient to use Lemma 2.3. □

Theorem 4.5. *Let in (4.1) $r(X_{n,k)}) = k < n$ and $r(Z_{(s,m)}) = s < m$. Then*
(i) *the ϑ_0-LBLUE of $B\phi(t_j)$ based on Y_j is*

$$\widehat{B\phi(t_j)}(Y_j) = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}Y_j$$

and

$$\mathrm{Var}\left[\widehat{B\phi(t_j)}(Y_j)|\Sigma_0\right] = (X'\Sigma_0^{-1}X)^{-1};$$

(ii) *the ϑ_0-LBLUE of $B\phi(t_j)$ based on the whole observation matrix $\underline{Y}$ is*

$$\widehat{B\phi(t_j)}(\underline{Y}) = \left(\widehat{B\phi(t_1)}(Y_1), \dots, \widehat{B\phi(t_m)}(Y_m)\right)\{P_{Z'}\}_{.j}\,,$$

where $P_{Z'} = Z'(ZZ')^{-1}Z$ and $\{P_{Z'}\}_{.j}$ is the j-th column of the matrix $P_{Z'}$;

$$\mathrm{Var}\left[\widehat{B\phi(t_j)}(\underline{Y})|\Sigma_0\right] = \{P_{Z'}\}_{j,j}\,(X'\Sigma_0^{-1}X)^{-1}.$$

Proof. (i) is a direct consequence of Lemma 2.3 if the relationships

$$E(Y_j|B) = XB\phi(t_j) \quad \text{and} \quad \operatorname{Var}(Y_j|\Sigma_0) = \Sigma_0$$

are taken into account.

(ii) Regarding Lemma 4.3. and (i)

$$\hat{B}(\underline{Y})Z = (\widehat{B\phi(t_1)}(Y_1), \dots, \widehat{B\phi(t_m)}(Y_m))P_{Z'}$$

which implies

$$\widehat{B\phi(t_j)}(\underline{Y}) = (\widehat{B\phi(t_1)}(Y_1), \dots, \widehat{B\phi(t_m)}(Y_m))\,\{P_{Z'}\}_{.j}\,.$$

Further, in consequence

$$\operatorname{Var}\left[\widehat{B\phi(t_j)}(Y_j)|\Sigma_0\right] = (X'\Sigma_0 X)^{-1},\ j = 1, \dots m,$$

(see (i)) and the fact that the estimators $\widehat{B\phi(t_1)}(Y_1), \dots, \widehat{B\phi(t_m)}(Y_m)$ are stochastically independent

$$\operatorname{Var}\left[\widehat{B\phi(t_j)}(\underline{Y})|\Sigma_0\right] = \left(\{P_{Z'}\}_{.j}\right)'\{P_{Z'}\}_{.j}\,(X'\Sigma_0^{-1}X)^{-1} = \{P_{Z'}\}_{j,j}\,(X'\Sigma_0^{-1}X)^{-1},$$

since the matrix $P_{Z'}$ is symmetric and idempotent. □

Corollary 4.6. *The estimator of B from Lemma 4.3 is identical with the ϑ_0-LBLUE of B based on $\hat{\beta}(t_1, Y_1), \dots, \hat{\beta}(t_m, Y_m)$ in the model*

$$\left[\begin{pmatrix} \hat{\beta}(t_1, Y_1) \\ \vdots \\ \hat{\beta}(t_m, Y_m) \end{pmatrix}, \begin{pmatrix} B\phi(t_1) \\ \vdots \\ B\phi(t_m) \end{pmatrix}, I \otimes (X'\Sigma_0^{-1}X)^{-1}\right],$$

where $\hat{\beta}(t_j, Y_j) = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}Y_j$ and B is the $k \times n$ matrix of unknown parameters.

Proof. As

$$\begin{pmatrix} B\phi(t_1) \\ \vdots \\ B\phi(t_m) \end{pmatrix} = (Z' \otimes I_{(k,k)})\operatorname{vec}(B),$$

Lemma 2.3 implies

$$\widehat{\operatorname{vec}(B)} =$$

$$= \left\{(Z \otimes I)[I \otimes (X'\Sigma_0^{-1}X](Z' \otimes I)\right\}^{-1}(Z \otimes I)[I \otimes (X'\Sigma_0^{-1}X)]\begin{pmatrix} \hat{\beta}(t_1, Y_1) \\ \vdots \\ \hat{\beta}(t_m, Y_m) \end{pmatrix}$$

$$= [(ZZ')^{-1}Z \otimes I]\begin{pmatrix} \hat{\beta}(t_1, Y_1) \\ \vdots \\ \hat{\beta}(t_m, Y_m) \end{pmatrix} = \operatorname{vec}\left\{[\hat{\beta}(t_1, Y_1), \dots, \hat{\beta}(t_m, Y_m)]Z'(ZZ')^{-1}\right\}$$

$$= \operatorname{vec}\left[(X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}\underline{Y}Z'(ZZ')^{-1}\right].$$

□

Remark 4.7. Corollary 4.6 is of great practical importance. During the deformation measurements we sequentially obtain the estimates

$$\hat{\beta}(t_1, y_1), \hat{\beta}(t_2, y_2), \dots .$$

After several epochs we are able to recognize the law of the deformations, over time, i.e., the proper combinations of functions $\phi_0(\cdot), \phi_1(\cdot), \dots$. In accordance with Corollary 4.6, the matrix B of the coefficients can be obtained directly from $\hat{\beta}(t_1, y_1), \dots, \hat{\beta}(t_m, y_m)$.

Remark 4.8. The matrix Z can be expressed as follows

$$Z = \begin{pmatrix} \phi_0(t_1), & \dots, & \phi_0(t_m) \\ \dots\dots & \dots\dots & \dots\dots \\ \phi_{s-1}(t_1), & \dots, & \phi_{s-1}(t_m) \end{pmatrix} = (\Phi(t_1), \dots, \Phi(t_m)) = \begin{pmatrix} \Phi_0' \\ \Phi_1' \\ \vdots \\ \Phi_{s-1}' \end{pmatrix}.$$

Thus

$$ZZ' = \begin{pmatrix} \Phi_0'\Phi_0, & \Phi_0'G \\ G'\Phi_0, & G'G \end{pmatrix}, \text{ where } G = (\Phi_1, \dots, \Phi_{s-1})$$

and

$$\begin{aligned} \left\{(ZZ')^{-1}\right\}_{1,1} &= \left[\Phi_0'\Phi_0 - \Phi_0'G(G'G)^{-1}G'\Phi_0\right]^{-1} \\ &= \frac{1}{\Phi_0'\Phi_0} + \frac{1}{\Phi_0'\Phi_0}\Phi_0'G(G'G - G'\Phi_0\frac{1}{\Phi_0'\Phi_0}\Phi_0'G)^{-1}G'\Phi_0\frac{1}{\Phi_0'\Phi_0}. \end{aligned}$$

Regarding (ii) from Theorem 4.5, we have

$$\operatorname{Var}\left[\widehat{B\Phi(t_1)}(\underline{Y})|\Sigma_0\right] = \Phi'(t_1)\begin{pmatrix} \Phi_0'\Phi_0, & \Phi_0'G \\ G'\Phi_0, & G'G \end{pmatrix}^{-1}\Phi(t_1)\operatorname{Var}\left[\widehat{B\Phi(t_1)}(Y_1)|\Sigma_0\right].$$

As

$$\Phi'(t_1)\begin{pmatrix} \Phi_0'\Phi_0, & \Phi_0'G \\ G'\Phi_0, & G'G \end{pmatrix}^{-1}\Phi(t_1) = \{P_{Z'}\}_{1,1}$$

and $P_{Z'}$ is the projection matrix, it is clear that $\{P_{Z'}\}_{1,1} \le 1$.

Special cases: If $\Phi_0, \Phi_1, \dots, \Phi_{s-1}$ is derived for the system of orthogonal Chebyshev polynomials on the set $\{t_1, \dots, t_m\}$, i.e.,

$$ZZ' = \begin{pmatrix} \Phi_0'\Phi_0, & 0, & \dots, & 0 \\ 0, & \Phi_1'\Phi_1, & ,\dots, & 0 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ 0, & 0, & ,\dots, & \Phi_{s-1}'\Phi_{s-1} \end{pmatrix},$$

then

$$\{P_{Z'}\}_{1,1} = \sum_{i=1}^{s-1} \frac{\phi_i^2(t_1)}{\Phi_i'\Phi_i}.$$

If $\phi_0(t_i) = 1$, $i = 1,\dots,n$, $\Phi(t_1) = e_1^{(s)} = (1,0,\dots,0)' \in R^s$ (in this case the functions $\phi_0(\cdot),\dots,\phi_{s-1}(\cdot)$ are not necesarily orthogonal on the set $\{t_1,\dots,t_m\}$),

$$\left\{Z'(ZZ')^{-1}Z\right\}_{1,1} = \frac{1}{m - \mathbf{1}'P_H\mathbf{1}},$$

where

$$\begin{pmatrix} (0_{s-1,1})' \\ H \end{pmatrix} = (\Phi_1,\dots,\Phi_{s-1}), \quad P_H = H(H'H)^{-1}H'.$$

The last statement is a consequence of the fact that

$$Z_{(s,m)} = \begin{pmatrix} 1, & \mathbf{1}' \\ 0_{(s-1,1)}, & H'_{(s-1,m-1)} \end{pmatrix},$$

which implies

$$\begin{aligned}\left\{Z'(ZZ')^{-1}Z\right\}_{1,1} &= \left\{\begin{pmatrix} 1, & 0' \\ \mathbf{1}, & H \end{pmatrix}\begin{pmatrix} m, & \mathbf{1}'H \\ H'\mathbf{1}, & H'H \end{pmatrix}^{-1}\begin{pmatrix} 1, & \mathbf{1}' \\ 0, & H' \end{pmatrix}\right\}_{1,1} \\ &= \left[m - \mathbf{1}'H(H'H)^{-1}H'\mathbf{1}\right]^{-1}.\end{aligned}$$

The number $\left\{Z'(ZZ')^{-1}Z\right\}_{1,1}$ is important, since we want to know how the accuracy in the estimation of the starting value $\beta(t_1)$ increases after m epochs.

If $\phi_0(\cdot) \equiv 1$, $\phi_1(t) = (t - t_1)$, $\phi_2(t) = (t - t_1)^2$, $t_i = t_1 + i$, $i = 0,\dots,4$, then $\left\{Z'(ZZ')^{-1}Z\right\}_{1,1} = 0.813$, what means a not negligible decrease of variances of the estimator.

Remark 4.9. The consideration in Remark 4.8 on the value $\{Z'(ZZ')^{-1}Z\}_{1,1}$ is justified and has a reasonable meaning if the functions $\phi_0(\cdot),\dots,\phi_{s-1}(\cdot)$, are chosen adequately. To verify the proper choice of these functions the following procedure can be used.

Let $\{T\beta(t_1),\dots,T\beta(t_m)\}$ be any trajectory important from the viewpoint of a static expert. Here T is a $q \times k$ matrix with rank $r(T) = q \leq k$. This trajectory expresses a law of the deformation over time for which the functions $\phi_0(\cdot),\dots,\phi_{s-1}(\cdot)$ must be chosen properly. If the choice is correct, then the trajectory is $\{TB\Phi(t_1),\dots,TB\Phi(t_m)\}$.

If vec($\underline{Y}$) is normally distributed and its actual covariance matrix is $I \otimes \Sigma_0$, then

$$v = \begin{pmatrix} \hat{\beta}(t_1, Y_1) \\ \vdots \\ \hat{\beta}(t_m, Y_m) \end{pmatrix} - \begin{pmatrix} \hat{B}(\underline{Y})\Phi(t_1) \\ \vdots \\ \hat{B}(\underline{Y})\Phi(t_m) \end{pmatrix} \sim N_{mk}(0, M_{Z'} \otimes (X'\Sigma_0^{-1}X)^{-1})$$

and $(I \otimes T)v \sim N_{mq}(0, M_{Z'} \otimes T(X'\Sigma_0^{-1}X)^{-1}T')$ where $M_{Z'} = I_{(m,m)} - P_{Z'}$ has rank $r(M_{Z'}) = m - s$.

Therefore the random variable

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\{M_{Z'}\}_{i,j}\left[\hat{\beta}(t_i, Y_i) - \hat{B}\Phi(t_i)\right]' T' \cdot \left[T(X'\Sigma_0^{-1}X)^{-1}T'\right]^{-1} T\left[\hat{\beta}(t_j, Y_j) - \hat{B}\Phi(t_j)\right]$$

has the central chi-square distribution with $(m-s)q$ degrees of freedom and it can be used as a test statistic for the null hypothesis that the trajectory is properly characterized by $\{TB\Phi(t_1), \dots, TB\Phi(t_m)\}$. This procedure is to be used for each trajectory important for the statics expert; simultaneously the Bonferroni correction (cf. e.g. [**Hu**, p. 492]) for the significance level must be taken into account (if f trajectories are taken into account simultaneously, then instead the value α of the significance level the value α/f must be used).

Theorem 4.10. *Let $\underline{Y}$ in the model (4.1) be normally distributed and the actual covariance matrix be Σ_0; let $L_{(q,k)}$ be any $q \times k$ matrix with the rank $r(L) = q \leq k$ and $R_{(s,r)}$ be any $s \times r$ matrix with the rank $r(R) = r \leq s$. If $LBR + H = 0$, then*

$$(L\hat{B}R + H)[R'(ZZ')^{-1}R]^{-1}(L\hat{B}R + H)' \sim W_q\left[r, L(X'\Sigma_0^{-1}X)^{-1}L'\right]$$

(the central Wishart distribution with r degrees of freedom and with the covariance matrix $L(X'\Sigma_0^{-1}X)^{-1}L'$).

Proof. It is necessary and sufficient to show that

$$\forall\{u \in R^q\} u'(L\hat{B}R + H)[R'(ZZ')^{-1}R]^{-1}(L\hat{B}R + H)'u \sim u'L(X'\Sigma_0^{-1}X)^{-1}L'u\chi_r^2,$$

where χ_r^2 is the central chi-square distributed random variable with r degrees of freedom.

As

$$R'\hat{B}'L'u \sim N_r[-H'u, (u'L(X'\Sigma_0^{-1}X)^{-1}L'u)R'(ZZ')^{-1}R]$$

and, with respect to our assumptions, $R'(ZZ')^{-1}R$ is regular and

$$u \neq 0 \Rightarrow u'L(X'\Sigma_0^{-1}X)^{-1}L'u > 0,$$

the statement follows from [**R1**, p. 535]; see also [**Ko2**]. □

Remark 4.11. The Wishart matrix from Theorem 4.10 can be used for testing a null-hypothesis in several ways; cf. [**R1**, p. 547]. One of them is given in the following corollary.

Corollary 4.12. *If $U \sim W_q(r, \Psi)$, where Ψ is p.d., then $\operatorname{tr}(\Psi^{-1}U) \sim \chi^2_{qr}$; the random variable*

$$\operatorname{tr}\left\{(L\hat{B}R + H)'[L(X'\Sigma_0^{-1}X)^{-1}L']^{-1}(L\hat{B}R + H)[R'(ZZ')^{-1}R]^{-1}\right\}$$

is distributed as χ^2_{qr}, which can be used for testing the above null-hypothesis.

This test is also a direct consequence of Lemmas 2.5 and 4.3.

If the matrix $\Sigma(\vartheta) = \sum_{i=1}^p \vartheta_i V_i$ in the model (4.1) is not known, the following theorem can be used (cf. also [**Z**]).

Theorem 4.13. *Let in* (4.1) $r(X) = k$ *and* $r(Z) = s$. *Then*

(i) *a function $g(\vartheta) = g'\vartheta$, $\vartheta \in \underline{\vartheta}$, can be estimated by the ϑ_0-MINQUE iff*

$$g \in \mathcal{M}(C^{(I)}),$$

where

$$\{C^{(I)}\}_{i,j} = (m-s)\operatorname{tr}(V_iV_j) + s\operatorname{tr}(M_XV_iM_XV_j),\ i,j = 1,\dots,p;$$

(ii) *if $g \in \mathcal{M}(C^{(I)})$, then the ϑ_0-MINQUE of the function $g(\cdot)$ is*

$$\begin{aligned}\widehat{g'\vartheta} = \sum_{i=1}^p \lambda_i \big\{&\operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'})\\ &+ \operatorname{tr}\left[\underline{Y}'(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}\right]\big\},\end{aligned}$$

where $\lambda = (\lambda_1', \dots, \lambda_p)'$ is a solution of the equation

$$\left[(m-s)S_{\Sigma_0^{-1}} + sS_{(M_X\Sigma_0M_X)^+}\right]\lambda = g.$$

Proof. With respect to Lemma 2.4,

$$\{C^{(I)}\}_{i,j} = \operatorname{tr}(M_{\underline{X}}\underline{V}_iM_{\underline{X}}\underline{V}_j),\ i,j = 1,\dots,p,$$

where

$$\underline{X} = Z' \otimes X \quad \text{and} \quad \underline{V}_i = I \otimes V_i.$$

Thus

$$\begin{aligned}M_{\underline{X}} &= I_{(m,m)} \otimes I_{(n,n)} - (Z' \otimes X)[(Z \otimes X')(Z' \otimes X)]^{-1}(Z \otimes X')\\ &= M_{Z'} \otimes I + P_{Z'} \otimes M_X\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(M_{\underline{X}}\underline{V}_iM_{\underline{X}}\underline{V}_j) &= \operatorname{tr}[M_{Z'} \otimes (V_iV_j)] + \operatorname{tr}[P_{Z'} \otimes (M_XV_iM_XV_j)]\\ &= (m-s)\operatorname{tr}(V_iV_j) + s\operatorname{tr}(M_XV_iM_XV_j).\end{aligned}$$

Further

$$\begin{aligned}(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+ &= I\otimes\Sigma_0^{-1} - (I\otimes\Sigma_0^{-1})(Z'\otimes X)\\ &\quad\cdot[(Z\otimes X')(I\otimes\Sigma_0^{-1})Z'\otimes X)]^{-1}(Z\otimes X')(I\otimes\Sigma_0^{-1})\\ &= M_{Z'}\otimes\Sigma_0^{-1} + P_{Z'}\otimes(M_X\Sigma_0 M_X)^+\end{aligned}$$

and

$$\begin{aligned}&\operatorname{tr}[(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+\underline{V}_i(M_{\underline{X}}\underline{\Sigma}_0 M_{\underline{X}})^+\underline{V}_j] =\\ &\quad= \operatorname{tr}\left\{M_{Z'}\otimes(\Sigma_0^{-1}V_i\Sigma_0^{-1}V_j) + P_{Z'}\otimes[(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+V_j]\right\}\\ &\quad= (m-s)\operatorname{tr}(\Sigma_0^{-1}V_i\Sigma_0^{-1}V_j) + s\operatorname{tr}[(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+V_j].\end{aligned}$$

Using the fact that

$$\begin{aligned}&[\operatorname{vec}(\underline{Y})]'(M_{\underline{X}}\underline{\Sigma}_0M_{\underline{X}})^+\underline{V}_i(M_{\underline{X}}\underline{\Sigma}_0M_{\underline{X}})^+\operatorname{vec}(\underline{Y}) =\\ &\qquad= [\operatorname{vec}(\underline{Y})]'\left\{M_{Z'}\otimes\Sigma_0^{-1}V_i\Sigma_0^{-1} + P_{Z'}\otimes[(M_X\Sigma_0M_X)^+\right.\\ &\qquad\quad\left.\cdot V_i(M_X\Sigma_0M_X)^+]\right\}\operatorname{vec}(\underline{Y})\\ &\qquad= \operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'}) + \operatorname{tr}[\underline{Y}'(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}],\end{aligned}$$

the proof can easily be finished. □

Remark 4.14. Let $s = m$ and $Z = I$, i.e., the model $(\underline{Y}, XB, I\otimes\Sigma_0)$ from the beginning of this section. In this case $C^{(I)} = sS_{M_X}$. If $V_1,\dots,V_p$ are linearly independent (i.e., the matrix Σ is properly parametrized) the matrix S_{M_X} need not be regular. However the matrix $(m-s)S_I + sS_{M_X}$ is regular, since S_I is p.d. This fact must be respected in preparing the design of the deformation measurements.

Remark 4.15. Analogously as in Section 3 the estimator of ϑ can be based also either on the terms

$$\operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'}),\ i=1,\dots,p,$$

or on the terms

$$\operatorname{tr}(\underline{Y}'(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}),\ i=1,\dots,p.$$

Let us compare the efficiency of such estimators for m increasing at least in the case of normality. For the sake of simplicity let $S_{(M_x\Sigma_0M_x)^+}$ be p.d. The estimator of ϑ based on $\operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'})$ is denoted as $\hat{\vartheta}_1$, the other as $\hat{\vartheta}_2$.

Since

$$E\left[\operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'})|\Sigma = \sum_{i=1}^p\vartheta_iV_i\right] = (m-s)\{S_{\Sigma_0^{-1}}\}_{i.}\vartheta,\ \vartheta\in\underline{\vartheta},$$

the former is

$$\hat{\vartheta}_1 = \frac{1}{m-s}\, S^{-1}_{\Sigma_0^{-1}} \begin{pmatrix} \operatorname{tr}[\underline{Y}'\Sigma_0^{-1}V_1\Sigma_0^{-1}\underline{Y}M_{Z'}) \\ \vdots \\ \operatorname{tr}[\underline{Y}'\Sigma_0^{-1}V_p\Sigma_0^{-1}\underline{Y}M_{Z'}) \end{pmatrix}$$

and in the case of normality of the observation matrix $\underline{Y}$

$$\begin{aligned} &\operatorname{cov}\left[\operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'}), \operatorname{tr}[\underline{Y}'\Sigma_0^{-1}V_j\Sigma_0^{-1}\underline{Y}M_{Z'})|\Sigma_0\right] = \\ &\quad = 2\operatorname{tr}\left\{(I\otimes\Sigma_0)[M_{Z'}\otimes(\Sigma_0^{-1}V_i\Sigma_0^{-1})](I\otimes\Sigma_0)[M_{Z'}\otimes(\Sigma_0^{-1}V_j\Sigma_0^{-1})]\right\} \\ &\quad = 2(m-s)\left\{S_{\Sigma_0^{-1}}\right\}_{i,j}. \end{aligned}$$

Thus

$$\operatorname{Var}(\hat{\vartheta}_1|\Sigma_0) = \frac{2}{m-s} S^{-1}_{\Sigma_0^{-1}}.$$

Analogously we obtain

$$\hat{\vartheta}_2 = \frac{1}{s} S^{-1}_{(M_X\Sigma_0M_X)^+} \begin{pmatrix} \operatorname{tr}[\underline{Y}'(M_X\Sigma_0M_X)^+V_1(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}] \\ \vdots \\ \operatorname{tr}[\underline{Y}'(M_X\Sigma_0M_X)^+V_p(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}] \end{pmatrix},$$

$$\operatorname{Var}(\hat{\vartheta}_2|\Sigma_0) = \frac{2}{s}\, S^{-1}_{(M_X\Sigma_0M_X)^+}$$

(realize that it is independent of m).

The estimator $\hat{\vartheta}$ from Theorem 4.13 is

$$\hat{\vartheta} = [(m-s)S_{\Sigma_0^{-1}} + sS_{(M_X\Sigma_0M_X)^+}]^{-1}\hat{\gamma},$$

where $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_p)'$,

$$\hat{\gamma}_i = \operatorname{tr}(\underline{Y}'\Sigma_0^{-1}V_i\Sigma_0^{-1}\underline{Y}M_{Z'}) + \operatorname{tr}[\underline{Y}'(M_X\Sigma_0M_X)^+V_i(M_X\Sigma_0M_X)^+\underline{Y}P_{Z'}],$$

$i = 1, \dots, p$, and

$$\operatorname{Var}(\hat{\vartheta}|\Sigma_0) = 2[(m-s)S_{\Sigma_0^{-1}} + sS_{(M_X\Sigma_0M_X)^+}]^{-1}.$$

Therefore

$$\lim_{m\to\infty} \operatorname{Var}(\hat{\vartheta}_1|\Sigma_0)[\operatorname{Var}(\hat{\vartheta}|\Sigma_0)]^{-1} = I_{(p,p)}$$

and

$$\lim_{m\to\infty} \operatorname{Var}(\hat{\vartheta}_1|\Sigma_0)[\operatorname{Var}(\hat{\vartheta}_2|\Sigma_0)]^{-1} = 0.$$

Obviously

$$\operatorname{Var}(\hat{\vartheta}|\Sigma_0) \le \operatorname{Var}(\hat{\vartheta}_1|\Sigma_0)$$

and

$$\mathrm{Var}(\hat{\vartheta}|\Sigma_0) \leq \mathrm{Var}(\hat{\vartheta}_2|\Sigma_0).$$

Concluding remark. It is to be said that there are numerous models of deformation measurements which are either not yet investigated or investigations of them are in "statu nascendi" only. Because of their practical importance it is desirable to attract the interest of mathematicians to this class of problems.

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L. Kubáček, Mathematical Institute, Slovak Academy of Sciences, 814 73 Bratislava, Slovak Republic