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OPTIMAL CONTROL OF A VARIATIONAL INEQUALITY WITH POSSIBLY NONSYMMETRIC LINEAR OPERATOR. APPLICATION TO THE OBSTACLE PROBLEMS IN MATHEMATICAL PHYSICS.

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Abstract. This paper is concerned with an optimal control problem for variational inequalities, where the linear not necessary symmetric operators as well as the convex sets of possible states depend on the control parameter. Existence of an optimal control problem is proven on the abstract level. An abstract framework for the theoretical study of obstacle problems in mathematical physics in the variational inequality context is presented. Moreover, some sufficient conditions for the existence of an optimal control are given.

Introduction

In this paper we deal with the question of the existence of an optimal control function for a stationary variational inequalities, where the linear not necessary symmetric operators as well as the convex sets of possible states depend on the control parameter. The optimal control problem for a system governed by an elliptic variational inequality is proposed by Lions [**10**] and discussed in Mignot [**12**], Barbu [**2**], Sokolowski and Zolesio [**19**], Haslinger and Neittaanmäki [**18**], Murat [**16**]. In these papers authors concentrate on the case of a symmetric operator. The most characteristic property of variational inequalities is the fact that their solution does not depend smoothly, in general, on the control. A special type of the convergence of sequences of sets and functionals introduced by Mosco plays an important role in our considerations. We introduce an abstract framework for the theoretical study of the optimal control problem in the variational inequality context. In Section 1 we present a general theorem, yielding the existence of at least one optimal control. We formulate the optimal control problem of an obstacle problem in mathematical physics in Sections 2, 3 and apply the general existence theorem. Such problems play a very important role in various branches of physics and mechanics. The latter includes an obstacle Fourier problem occuring in the modeling of several heat-transfer phenomena. In Section 4 we formulate the optimal control problem of some unilateral problems, which describe the stationary

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equilibrium of a liquid in a region Ω surrounded by a membrane $\partial\Omega$ that allows the liquid to enter the region Ω, whereas it prevents the liquid to flow out.

1. Problem Statement and Main Result

1.1. On the convergence of sets and of functions.

Let $V(\Omega)$ be as normed linear space. Following Mosco [**14**] we introduce a convergence of sequences of subsets of $V(\Omega)$.

Definition 1. A sequence $\{K_n(\Omega)\}_n$ of subsets of $V(\Omega)$ converges to a set $K(\Omega) \subset V(\Omega)$ if

1. $K(\Omega)$ contains all weak limits of sequences $\{v_{n_k}\}_{n_k}$ $(v_{n_k} \in K_{n_k}(\Omega))$, where $\{K_{n_k}(\Omega)\}_{n_k}$ are arbitrary subsequences of $\{K_n(\Omega)\}_n$;
2. Every element $v \in K(\Omega)$ is the strong limit of some sequence $\{v_n\}_n$, $v_n \in K_n(\Omega)$.

Notation. $K(\Omega) = \operatorname{Lim}_{n\to\infty} K_n(\Omega)$.

Let $\mathcal{W}\colon V(\Omega) \to (-\infty, \infty]$ be a functional. The set

$$\text{epi } \mathcal{W} := \{(v, \beta) \in V(\Omega) \times \mathbb{R} : \mathcal{W}(v) \leq \beta\}$$

is called the epigraf of $\mathcal{W}$, and the effective domain of $\mathcal{W}$ is a subset of $V(\Omega)$,

$$D\mathcal{W} \ \ (\text{or dom } \mathcal{W}) = \{v : \mathcal{W}(v) < +\infty, \ v \in V(\Omega)\}.$$

Moreover, the subdifferential $\partial\mathcal{W}$ is an operator from $V(\Omega)$ to 2^{V^*} given by $\partial\mathcal{W}(z) = \{z^* \in V^*(\Omega),\ \langle z^*, v - z\rangle_{V(\Omega)} \leq \mathcal{W}(v) - \mathcal{W}(z)$ for all $v \in V(\Omega)$, for $z \in V(\Omega)$ with $\mathcal{W}(z) < \infty$ and by $\partial\mathcal{W}(z) = \emptyset$ for $z \in V(\Omega)$ with $\mathcal{W}(z) = \infty\}$.

Definition 2. A sequence $\{\mathcal{W}_n\}$ of functionals from $V(\Omega)$ into $(-\infty, \infty]$ converges to $\mathcal{W}\colon V(\Omega) \to (-\infty, \infty]$ in $V(\Omega)$, if

$$\text{epi } \mathcal{W} = \operatorname*{Lim}_{n\to\infty} \text{epi } \mathcal{W}_n.$$

We use the notation $\mathcal{W} = \operatorname{Lim}_{n\to\infty} \mathcal{W}_n$.

Let us recall the following lemma of Mosco on the convergence of functionals in $V(\Omega)$.

Lemma 1. *Let* $\mathcal{W}_n\colon V(\Omega) \to (-\infty, \infty]$, $n = 1, 2, \dots$. *Then* $\mathcal{W} = \operatorname{Lim}_{n\to\infty} \mathcal{W}_n$ *in* $V(\Omega)$ *if and only if* (1.) *and* (2.) *below hold:*

1. *For every* $v \in V(\Omega)$ *there exists a sequence* $\{v_n\}_n$ *in* $V(\Omega)$ *such that* $\lim_{n\to\infty} v_n = v$ *(strongly) in* $V(\Omega)$ *and* $\limsup_{n\to\infty} \mathcal{W}_n(v_n) \leq \mathcal{W}(v)$.

2. *For every subsequence $\{\mathcal{W}_{n_k}\}_{n_k}$ of $\{\mathcal{W}_n\}_n$ and every sequence $\{v_k\}_k$ in $V(\Omega)$ weakly convergent to $v \in V(\Omega)$ the inequality*

$$\mathcal{W}(v) \leq \liminf_{n\to\infty} \mathcal{W}_{n_k}(v_k)$$

holds.

We shall denote by $\mathcal{E}_0(V(\Omega))$ the family of all lower semicontinuous convex functionals $\mathcal{W}\colon V(\Omega) \to (-\infty, \infty]$, not identically equal to $+\infty$.

Remark 1. Due to the previous lemma the condition $\mathcal{W} = \mathrm{Lim}_{\,n\to\infty} \mathcal{W}_n$ implies that for every $v \in V(\Omega)$ there exists a sequence $\{v_n\}_n \subset V(\Omega)$ such that $\lim_{n\to\infty} v_n = v$ (strongly) in $V(\Omega)$ and $\lim_{n\to\infty} \mathcal{W}_n(v_n) = \mathcal{W}(v)$.

1.2. Problem statement.

Let $U(\Omega)$ be a reflexive Banach space of controls with a orm $\|\cdot\|_{U(\Omega)}$. Let $U_{ad}(\Omega) \subset U(\Omega)$ be a set of admissible controls. We assume that $U_{ad}(\Omega)$ is compact in $U(\Omega)$. Further, denote by $V(\Omega)$ a real Hilbert space with an inner product $(\cdot,\cdot)_{V(\Omega)}$ and a norm $\|\cdot\|_{V(\Omega)}$, by $V^*(\Omega)$ its dual space with a norm $\|\cdot\|_{V^*(\Omega)}$ and with the duality pairing $\langle\cdot,\cdot\rangle_{V(\Omega)}$.

Let constants M_0, M_1 $(0 < M_0 < M_1)$ be given. We denote by $E(M_0, M_1)$ the class of the linear, continuous (not necessary symmetric) operators $\mathcal{A}\colon V(\Omega) \to V^*(\Omega)$ such that

$$M_0\|v\|^2_{V(\Omega)} \leq \langle \mathcal{A}v, v\rangle_{V(\Omega)} \leq M_1\|v\|^2_{V(\Omega)}, \text{ for all } v \in V(\Omega).$$

We introduce the systems $\{\mathcal{K}(e,\Omega)\}$, $\{A(e)\}$ of convex closed subsets $\mathcal{K}(e,\Omega) \subset V(\Omega)$ and linear bounded operators $A(e) \in L(V(\Omega, V^*(\Omega))$, $e \in U_{ad}(\Omega)$, satysfying the following assumptions:

$$\text{(H1)}\quad \left\{\begin{array}{ll} 1^\circ. & \bigcap\limits_{e\in U_{ad}(\Omega)} \mathcal{K}(e,\Omega) \neq \emptyset; \\ 2^\circ. & e_n \to e \text{ (strongly) in } U(\Omega) \Rightarrow \mathcal{K}(e,\Omega) = \mathop{\mathrm{Lim}}\limits_{n\to\infty} \mathcal{K}(e_n,\Omega); \\ 3^\circ. & \|A(e)\|_{L(V(\Omega),V^*(\Omega))} \leq M \text{ for all } e \in U_{ad}(\Omega); \\ 4^\circ. & \langle A(e)v, v\rangle_{V(\Omega)} \geq \alpha\|v\|^2_{V(\Omega)},\ \alpha > 0, \text{ for all } e \in U_{ad}(\Omega) \text{ and} \\ & v \in V(\Omega) \text{ (a real number } \alpha \text{ not depending on } e \text{ and } v, \\ & \text{further the operator } A(e) \text{ is said to be uniformly coercive} \\ & \text{with respect to } U_{ad}(\Omega)); \\ 5^\circ. & e_n \to e \text{ (strongly) in } U(\Omega) \Rightarrow A(e_n) \to A(e) \text{ in} \\ & L(V(\Omega), V^*(\Omega)),\ e,\ e_n \in U_{ad}(\Omega). \end{array}\right.$$

Thus, by virtue of ((H1), $3^\circ, 4^\circ$), $A(e_n)$, $n = 1, 2, \ldots$ and $A(e)$ are elements of the class $E(\alpha, M)$ for each sequence $\{e_n\}_n$, where $e_n \to e$ (strongly) in $U(\Omega)$.

Moreover, we suppose:

(E1)
1°. There is a system of functionals $\{\Phi(e_n,\cdot)\}_n$ on $V(\Omega)$ with values in $(-\infty,\infty]$ (not identically equal to $+\infty$) semi-continuous and convex on $V(\Omega)$, $\{v \in V(\Omega) : \Phi(e_n,v) < \infty\} \subset \mathcal{K}(e_n,\Omega)$, $\{v \in V(\Omega) : \Phi(e,v) < \infty\} \subset \mathcal{K}(e,\Omega)$, $\Phi(e,\cdot) = \operatorname{Lim}_{n\to\infty} \Phi(e_n,\cdot)$ as $e_n \to e$ (strongly) in $U(\Omega)$, $e_n, e \in U_{ad}(\Omega)$.
2°. $\{L(e_n)\}_n$ is a sequence in $V^*(\Omega)$ such that $L(e_n) \to L(e)$ (strongly) in $V^*(\Omega)$ as $e_n \to e$ (strongly) in $U(\Omega)$, e_n, $e \in U_{ad}(\Omega)$.

Further we assume that for each sequence $\{e_n\}$, $e_n \to e$ (strongly) in $U(\Omega)$ there is a bounded sequence $\{a_n\}_n$ with $a_n \in \mathcal{K}(e_n,\Omega)$ and $\Phi(e_n,a_n) < \infty$ for all n, e, $e_n \in U_{ad}(\Omega)$ such that

$$\limsup_{n\to\infty} \Phi(e_n,a_n) < \infty. \tag{1.1}$$

There exist two positive constants c_1, c_2 such that for each sequence $\{e_n\}$, $e_n \to e$ (strongly) in $U(\Omega)$, $e_n, e \in U_{ad}(\Omega)$,

$$\begin{aligned} \Phi(e_n,v_n) &\geq -c_1\|v_n\|_{V(\Omega)} - c_2 \quad \text{for } n = 1,2,\dots \text{ (see [\textbf{15}])} \\ \Phi(e,v) &\geq -c_1\|v\|_{V(\Omega)} - c_2. \end{aligned} \tag{1.2}$$

Then since $A(e_n) \in E(\alpha,M)$ for any sequence of pairs $\{[e_n,v_n]\}_n$, $e, e_n \in U_{ad}(\Omega)$ $n = 1,2,\dots$ with $\|v_n\|_{V(\Omega)} \to \infty$ and $e_n \to e$ (strongly) in $U(\Omega)$ we have

$$\frac{[\langle A(e_n)v_n, v_n - a_n\rangle_{V(\Omega)} + \Phi(e_n,v_n)]}{\|v_n\|_{V(\Omega)}} \to \infty. \tag{1.3}$$

Moreover, for each n

$$\frac{[\langle A(e_n)v, v - a_n\rangle_{V(\Omega)} + \Phi(e_n,v)]}{\|v\|_{V(\Omega)}} \to \infty. \tag{1.4}$$

as $\|v\|_{V(\Omega)} \to \infty$, $v \in \mathcal{K}(e_n\Omega)$ where $e_n \in U_{ad}(\Omega)$, $n = 1,2,\dots,n$ is arbitrary but fixed in $U_{ad}(\Omega)$, and $A(e_n) \in E(\alpha,M)$.

Remark 2. By virtue of ((H1), 3°, 4°) and (1.1) we can write

$$[\langle A(e_n)v_n, v_n - a_n\rangle_{V(\Omega)} + \Phi(e_n,v_n)] \geq \alpha\|v_n - a_n\|^2_{V(\Omega)} - c_3\|v_n - a_n\|_{V(\Omega)} - c_4,$$

where a_n is bounded in $\mathcal{K}(e_n,\Omega)$ $(n = 1,2,\dots)$ and when $\|v_n\|_{V(\Omega)} \to \infty$ then also $\|v_n - a_n\|_{V(\Omega)} \to \infty$. In a similar way (for each fixed n) we obtain relation (1.4).

We set $L(e_n) = f + Be_n$, where $B \in L(U(\Omega), V^*(\Omega))$. For fixed $n = 1, 2, \dots$ and any given $L(e_n) \in V^*(\Omega)$ (if we suppose (1.4) and (1.1)) there exists a unique solution $u(e_n) \in \mathcal{K}(e_n, \Omega)$ of the state inequality

$$\begin{aligned} \langle A(e_n)u(e_n),\, v - u(e_n)\rangle_{V(\Omega)} + \Phi(e_n, v) - \Phi(e, u(e_n)) \\ \geq \langle L(e_n), v - u(e_n)\rangle_{V(\Omega)} \quad \text{for all } v \in \mathcal{K}(e_n, \Omega). \end{aligned} \tag{1.5}$$

This follows from the Kenmochi Theorem ([**8**]).

Further, consider a functional $\mathcal{L}\colon U(\Omega) \times V(\Omega) \to \mathbb{R}$ for which the following condition holds:

$$\text{(E2)} \quad \begin{cases} e_n \to e \text{ (strongly) in } U(\Omega),\ v_n \rightharpoonup \text{ in } V(\Omega) \text{ (weakly)} \implies \\ \implies \mathcal{L}(e, v) \leq \liminf_{n\to\infty} \mathcal{L}(e_n, v_n). \end{cases}$$

We shall formulate the optimal control in the following way:

Problem $(\mathcal{P}_*)$**.** Find a control $e_* \in U_{ad}(\Omega)$ such that

$$\begin{aligned} \langle A(e_*)u(e_*),\, v - u(e_*)\rangle_{V(\Omega)} + \Phi(e_*, v) - \Phi(e_*, u(e_*)) \\ \leq \langle L(e_*), v - u(e_*)\rangle_{V(\Omega)} \quad \text{for all } v \in \mathcal{K}(e_*, \Omega) \end{aligned} \tag{1.6}$$

$$\mathcal{L}(e_*, u(e_*)) = \inf_{e \in U_{ad(\Omega)}} \mathcal{L}(e, u(e)). \tag{1.7}$$

1.3. Main result.

Theorem 1. *Let the assumption* (H1), (E1), (E2), (1.1), (1.2), (1.3) *be satisfied. Then there exist at least one solution e_* of the optimal control problem* $(\mathcal{P}_*)$.

Proof. As the solution $u(e)$ of the variational inequality (1.5) is uniquely determined for every $e \in U_{ad}(\Omega)$, we can introduce the functional $J(e)$ as

$$J(e) = \mathcal{L}(e, u(e)), \quad e \in U_{ad}(\Omega). \tag{1.8}$$

Due to the compactness of $U_{ad}(\Omega)$ in $U(\Omega)$, there exists a sequence $\{e_n\} \subset U_{ad}(\Omega)$ such that

$$\lim J(e_n) = \inf_{e \in U_{ad}(\Omega)} J(e) \tag{1.9}$$

$$\lim_{n\to\infty} e_n = e_* \text{ in } U(\Omega),\ e_* \in U_{ad}(\Omega). \tag{1.10}$$

Denoting $u(e_n) := u_n \in \mathcal{K}(e_n, \Omega)$ we obtain the inequality

$$\begin{aligned} \langle A(e_n)u_n, u_n - v\rangle_{V(\Omega)} - \langle L(e_n), u_n - v\rangle_{V(\Omega)} \\ \leq \Phi(e_n, v) - \Phi(e_n, u_n) \quad \text{for all } v \in \mathcal{K}(e_n, \Omega). \end{aligned} \tag{1.11}$$

In particular, taking a_n for v in (1.11) we obtain

$$\langle A(e_n)u(e_n) - L(e_n), u_n - a_n\rangle_{V(\Omega)} + \phi(e_n, u_n) \leq \Phi(e_n, a_n) \quad \text{for every } n. \tag{1.12}$$

Hence the relations (1.1), (1.3) and ((E1), 2°) imply that $\{u_n\}_n$ is a bounded sequence. This implies the existence of a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ and an element $u_* \in V(\Omega)$ such that (For simplicity we write u_k, $L(e_k)$, $\Phi(e_k, \cdot)$, a_k and $A(e_k)$ for u_{n_k}, $L(e_{n_k})$, $\Phi(e_{n_k}, \cdot)$, a_{n_k} and $A(e_{n_k})$, respectively.)

$$u_k \rightharpoonup u_* \quad \text{(weakly) in } V(\Omega). \tag{1.13}$$

As $u_k \in \mathcal{K}(e_k, \Omega)$, the assumption ((H1), 2°) implies

$$u_* \in \mathcal{K}(e_*, \Omega). \tag{1.14}$$

Then we observe from Lemma 1, ((E1), 1°), (1.1), (1.2) that

$$\begin{aligned}\Phi(e_*, u_*) &\leq \liminf_{k\to\infty} \Phi(e_k, u_k)\\ &\leq \limsup_{k\to\infty} \{\Phi(e_k, a_k) - \langle A(e_k)u_k - L(e_k), u_k - a_k\rangle_{V(\Omega)}\} < \infty\end{aligned} \tag{1.15}$$

since by virtue of the monotonicity of $A(e_k)$ one has

$$\begin{aligned}|\langle A(e_k)u_k, a_k - u_k\rangle_{V(\Omega)}| &\leq \langle A(e_k)a_k, a_k\rangle_{V(\Omega)} + |\langle A(e_k)a_k, u_k\rangle_{V(\Omega)}|\\ &\leq 2Mc^2, \quad \text{where } \|u_k\|_{V(\Omega)}, \|a_k\|_{V(\Omega)} \leq c.\end{aligned}$$

On the other hand, by virtue of Lemma 1, ((E1), 1°) and Remark 1 there exists a sequence $\{h_k\}_k \subset V(\Omega)$ such that

$$\begin{gathered}\lim_{k\to\infty} h_k = u_* \quad \text{in } V(\Omega)\\ \lim_{k\to\infty} \Phi(e_k, h_k) = \Phi(e_*, u_*).\end{gathered} \tag{1.16}$$

Here, note that $h_k \in \mathcal{K}(e_k, \Omega)$ for all k, which follows from the assumption ((E1), 1°) and (1.15), (1.16), so that (inserting $v := h_k$ into (1.11))

$$\langle A(e_k)u_k - L(e_k), u_k - h_k\rangle_{V(\Omega)} \leq \Phi(e_k, h_k) - \Phi(e_k, u_k), \quad \text{for all } k. \tag{1.17}$$

Moreover, from ((H1), 3°) and (1.13) we obtain

$$\|A(e_k)u_k\|_{V^*(\Omega)} \leq c \quad \text{for } k = 1, 2, \ldots. \tag{1.18}$$

Then there exists an element $\chi \in V^*(\Omega)$ and a subsequence $\{A(e_j)u_j\}_j$ of $\{A(e_k)u_k\}_k$ such that

$$A(e_j)u_j \rightharpoonup \chi \quad \text{(weakly) in } V^*(\Omega). \tag{1.19}$$

Combining (1.13) and (1.16, 1) we have

$$(u_j - h_j) \rightharpoonup 0 \ \text{ (weakly) in } V(\Omega).$$

Thus, by passing to the limit in (1.17) we have

$$\begin{aligned}
(1.20) \qquad & \limsup_{j\to\infty} \langle A(e_j)u_j, u_j - u_*\rangle_{V(\Omega)} \\
& \le \limsup_{j\to\infty} \langle A(e_j)u_j - L(e_j), u_j - h_j\rangle_{V(\Omega)} \\
& \le \limsup_{j\to\infty} \Phi(e_j, h_j) - \liminf_{j\to\infty} \Phi(e_j, u_j) \le 0 \ \text{ for all } j.
\end{aligned}$$

However, combining the relation (1.19) with the inequality (1.20) we arrive at

$$(1.21) \qquad \limsup_{j\to\infty} \langle A(e_j)u_j, u_j\rangle_{V(\Omega)} \le \langle \chi, u_*\rangle_{V(\Omega)} \,.$$

Moreover, the monotonicity of $A(e_j)$ on $V(\Omega)$ ($A(e_j) \in E(\alpha, M)$, $j = 1, 2, \dots$) implies (in view of (1.21))

$$(1.22) \quad \langle \chi, u_*\rangle \ge \limsup_{j\to\infty} [\langle A(e_j)v, u_j - v\rangle_{V(\Omega)} + \langle A(e_j)u_j, v\rangle_{V(\Omega)}], \ j = 1, 2, \dots.$$

Relations (1.10), (1.13), (1.19) and ((H1), 5°), (1.22) enable us to write

$$\langle \chi - A(e_*)v, u_* - v\rangle_{V(\Omega)} \ge 0 \ \text{ for all } v \in V(\Omega).$$

Let $v = u_* + t(w - u_*)$, $t \in \mathbb{R}^+$ and $w \in V(\Omega)$. Then we get

$$(1.23) \qquad \langle \chi - A(e_*)[u_* + t(w - u_*)], u_* - w\rangle_{V(\Omega)} \ge 0 \ \text{ for any } w \in V(\Omega).$$

For $v = u_* - t(w - u_*)$ we can analogously write

$$(1.24) \qquad \langle \chi - A(e_*)[u_* - t(w - u_*)], w - u_*\rangle_{V(\Omega)} \ge 0.$$

Then combining (1.23) with (1.24) for $t \to 0$ we see that

$$\langle \chi - A(e_*)u_*, u_* - w\rangle_{V(\Omega)} = 0 \quad \text{for any } w \in V(\Omega).$$

This means that

$$(1.25) \qquad \chi = A(e_*)u_*$$

$$(1.26) \qquad A(e_j)u_j \rightharpoonup A(e_*)u_* \ \text{ (weakly) in } V^*(\Omega).$$

Using again the monotonicity of $A(e_j)$ we have

$$\langle A(e_j)u_j, u_j - u_* \rangle_{V(\Omega)} \geq \langle A(e_j)u_*, u_j - u_* \rangle_{V(\Omega)}, \quad j = 1, 2, \dots .$$

Next, by the convergences (1.10) and (1.13), by assumption ((H1), 5°) and by the last inequality we obtain

$$\liminf_{j\to\infty} \langle A(e_j)u_j, u_j - u_* \rangle_{V(\Omega)} \geq 0$$

which compared with (1.20) leads to

$$\lim_{j\to\infty} \langle A(e_j)u_j, u_j - u_* \rangle_{V(\Omega)} = 0 \,. \tag{1.27}$$

Clearly (by virtue of (1.26) and (1.27))

$$\lim_{j\to\infty} \langle A(e_j)u_j, u_j \rangle_{V(\Omega)} = \langle A(e_*)u_*, u_* \rangle_{V(\Omega)} \,. \tag{1.28}$$

We shall show that

$$\langle A(e_*)u_* - L(e_*)u_* - v \rangle_{V(\Omega)} \leq \Phi(e_*, v) - \Phi(e_*, u_*) \quad \text{for all } v \in \mathcal{K}(e_*, \Omega). \tag{1.29}$$

Let v be any element of $\mathcal{K}(e_*, \Omega)$. If $\Phi(e_*, v) = +\infty$, then (1.29) is trivial. Thus, assume $\Phi(e_*, v) < \infty$. According to Lemma 1 and ((E1), 1°) again, there is a sequence $\{\omega_j\}_j$ with $\omega_j \in \mathcal{K}(e_j, \Omega)$ for all j strongly convergent to v such that

$$\lim_{j\to\infty} \Phi(e_j, \omega_j) = \Phi(e_*, v). \tag{1.30}$$

Since $L(e_j) \to L(e_*)$ (strongly) in $V^*(\Omega)$ as $j \to \infty$ and

$$\langle A(e_j)u_j - L(e_j), u_j - \omega_j \rangle_{V(\Omega)} \leq \Phi(e_j, \omega_j) - L(e_j, u_j)$$

for all j, we obtain (1.29) by letting $j \to \infty$ and using (1.15), (1.28) and (1.30). As the element $v \in \mathcal{K}(e, \Omega)$ is chosen arbitrary we get $u_0 \equiv u(e_0)$ and

$$u(e_n) : (= u_n) \rightharpoonup u(e_*) : (= u_*) \ \ \text{(weakly) in } V(\Omega).$$

Then (E2), (1.9) yield

$$\mathcal{L}(e_*(u, (e_*)) \leq \liminf_{j\to\infty} \mathcal{L}(e_n, u(e_n)) = \inf_{e \in U_{ad}(\Omega)} \mathcal{L}(e, u(e)),$$

hence

$$\mathcal{L}(e_*, u(e_*)) = \inf_{e \in U_{ad}(\Omega)} \mathcal{L}(e, u(e))$$

which completes the proof.

Due to $A(e_*)$, $A(e_n) \in E(\alpha, M)$ for $n = 1, 2, \dots$, the strong convergence will follow from relation

$$\begin{aligned}
\alpha \limsup_{n\to\infty} \|u(e_n) - u(e_*)\|^2_{V(\Omega)} &\leq \lim_{n\to\infty} \langle A(e_n)(u(e_n) - u(e_*)), u(e_n) - u(e_*) \rangle_{V(\Omega)} \\
&= \lim_{n\to\infty} \big\{ \langle A(e_n)u(e_n), u(e_n) \rangle_{V(\Omega)} + \langle A(e_n)u(e_*), u(e_*) \rangle_{V(\Omega)} \\
&\quad - \langle A(e_n)u(e_*), u(e_n) \rangle_{V(\Omega)} - \langle A(e_n)u(e_n), u(e_*) \rangle_{V(\Omega)} \big\} = 0
\end{aligned}$$

(by virtue of ((H1), 4°, 5°) and (1.13), (1.28)). □

2. Optimal Control of an Obstacle Fourier Problem (Fixed Convex Set)

In order to motivate the study of the abstract control problem ($\mathcal{P}$) we consider the so-called obstacle Fourier problem for linear second-order elliptic partial differential operators. The obstacle Fourier problem occurs in the modelling of several heat-transfer phenomena. On the other hand, problems of this genre can be found in diverse fields of applications. One might as well be interested in finding the profile which gives the minimum drag as in constructing a magnet which has a constant magnetic field in a prescribed region. It is as easy (or as difficult) to minimize the weight of a huge cooling tower of a modern nuclear power station as to maximize the performance of the tiniest of transistors of a complicted silicon chip.

2.1. Problem statement.

We start with notation. Let Ω denote an open bounded connected subset belonging to the 3 dimensional real space $\mathbb{R}^3$ (we denote by x $(=\{x_i\}_{i=1}^3)$ the generic point from $\mathbb{R}^3$), with the boundary $\partial\Omega$. Let $a \cdot b$ denote the usual scalar product in $\mathbb{R}^3$, i.e. $a \cdot b = \sum_{i=1}^3 a_i b_i$, for any $a, b \in \mathbb{R}^3$. $L_p(\Omega)$, $1 \leq p \leq \infty$, and $H^1(\Omega)$, $H_0^1(\Omega)$ are the usual Banach spaces of real-valued functions defined on Ω, their norms are denoted by $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)} \cdot H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ the generalized scalar product between these spaces. Next we suppose that $\partial\Omega$ is sufficiently smooth (Lipschitz continuous, for example). We have $\mathcal{D}(\overline{\Omega}) = C^\infty(\overline{\Omega})$ and $\mathcal{M}_0$ is a linear continuous operator from $H^1(\Omega)$ to $L_2(\partial\Omega)$ (the trace operator).

Now we describe the optimal control problem considered here. First, let $U(\Omega)$ (control space) $= C^0(\overline{\Omega})$ and the set of admissible control functions, given by

$$U_{ad}(\Omega) = \{e \in W_\infty^1(\Omega) : 0, e_{\min} \leq e \leq e_{\max},\ |\partial e/\partial x_i| \leq c_i,\ i = 1, 2, 3\}$$

where c_i are given constants.

We note that $U_{ad}(\Omega)$ is clearly compact in the topology of $C^0(\overline{\Omega})$. For an arbitrary fixed $e \in U_{ad}(\Omega)$ let the state of the control system (Fourier obstacle problem) be given by solutions of the nonlinear elliptic boundary value problem

$$(2.1)\qquad \begin{cases} \mathcal{R}(e)u \geq L(e) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_U \\ ([\mathcal{A}(e)]\,\mathrm{grad}\ u) \cdot n + ku = H & \text{on } \partial\Omega_H \\ ([\mathcal{A}(e)]\,\mathrm{grad}\ u) \cdot n + ku \in \beta(u) & \text{on } \partial\Omega_S \\ u \geq \psi & \text{a.e. on } \Omega \\ (\mathcal{R}(e)u - L(e))(u - \psi) = 0 & \text{a.e. on } \Omega \\ \mathcal{R}(e)u = -\mathrm{div}\,\{([\mathcal{A}(e)])\,\mathrm{grad}\ u + ([Q(e)])u\} & \\ \qquad\qquad + ([\mathcal{B}(e)]) \cdot \mathrm{grad}\ u + a_0(e)u & \end{cases}$$

where $\beta(\cdot)$ is a maximal monotone operator, $\partial\Omega = \partial\Omega_U \cup \partial\Omega_H \cup \partial\Omega_S$ (such that $\partial\Omega_H \cap \partial\Omega_S = 0$, $\partial\Omega_U \cap \partial\Omega_H = 0$, $\partial\Omega_U \cap \partial\Omega_S = 0$), H and k are given functions defined over $\partial\Omega$, such that $H \in L_2(\partial\Omega_H)$, $k \in C^0(\overline{\partial\Omega_H \cup \partial\Omega_S})$, the function ψ represents the obstacle and $\psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$, $\psi(\partial\Omega) < 0$, n is the outward unit vector normal at $\partial\Omega$.
$[\mathcal{A}(e)] = [\mathcal{A}(\cdot, e(\cdot))] = [a_{ij}(\cdot, e(\cdot)]$ denotes a $(3,3)$-matrix (the system of linear operators from $\mathbb{R}^3$ to $\mathbb{R}^3$ depending upon x over Ω)

$$[\mathcal{Q}(e)] = [\mathcal{Q}(\cdot, e(\cdot))] = [a_i(\cdot, e(\cdot)],$$
$$[\mathcal{B}(e)] = [\mathcal{B}(\cdot, e(\cdot))] = [b_i(\cdot, e(\cdot)] \ \text{ denotes a (1,3)-matrix}$$

$a_0(e) = a_0(\cdot, e(\cdot))$ — a scalar function, $e \in U_{ad}(\Omega)$.

We assume that $\{[\mathcal{A}(x,e)]\}$, $\{[\mathcal{Q}(x,e)]\}$, $\{[\mathcal{B}(x,e)]\}$, $a_0(x,e)$ are defined on $\Omega \times [e_{\min}, e_{max}]$ and satisfies the following conditions:

(F1)
- 1°. $a_{ij}(\cdot,t)$, $a_i(\cdot,t)$, $b_i(\cdot,t)$, $a_0(\cdot,t)$ $(i,j = 1,2,3)$ are continuous function on $\overline{\Omega}$ for every $t \in [e_{\min}, e_{\max}]$ and $a_{ij}(x,\cdot)$, $a_i(x,\cdot)$, $b_i(x,\cdot)$, $a_0(x,\cdot)$ are a continuous on $[e_{\min}, e_{\max}]$ for every $x \in \Omega$.
  Moreover $a_0(x,t) \geq c_0 > 0$ a.e. on Ω.
- 2°. Of course, we have also to assume the ellipticity condition: $[\mathcal{A}(x,e)]\xi \cdot \xi \geq \alpha_* |\xi|^2_{\mathbb{R}^3}$ for any $\xi \in \mathbb{R}^3$, for any $e \in U_{ad}(\Omega)$ a.e. $x \in \Omega$, $\alpha_* =$ const. > 0 to be fullfilled.

In the right-hand side of (2.1), $L(e)$ (for any $e \in U_{ad}(\Omega)$) denotes a fixed functional belonging to $L_2(\Omega)$.

Finally, we choose the functional

$$\mathcal{L}(e,v) = \|v - z_d\|^2_{L_2(\Omega)} + N\|e\|^2_{U(\Omega)}, \quad N > 0 \tag{2.2}$$

as the cost functional of our control problem.

In this notation the optimal control problem in distributed parameters considered here reads $\min \mathcal{L}(e,u)$ subject (2.1).

We shall briefly denote it by problem $(\mathcal{P})$. Preparing our treatment we deal with the state inequation (2.1) for an arbitrary fixed $e \in U_{ad}(\Omega)$.

Because of above assumptions we have to work in the framework of the Sobolev space $V(\Omega)$ $(\subset H^1(\Omega))$. This means that $u \in V(\Omega)$ is a solution of (2.1) if and only if u is a solution of the nonsymmetric operator inequation

$$\begin{cases} A(e)u \geq L(e) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_U \\ ([\mathcal{A}(e)]\,\mathrm{grad}\ u)\cdot n + ku = H & \text{on } \partial\Omega_H \\ ([\mathcal{A}(e)]\,\mathrm{grad}\ u)\cdot n + ku \in \beta(u) & \text{on } \partial\Omega_S \\ u \geq \psi & \text{a.e. on } \Omega \end{cases} \tag{2.3}$$

where $A(e)$ (the nonsymmetric operator of Fourier obstacle problem) is a linear bounded operator acting from $V(\Omega)$ into $V^*(\Omega)$. It is defined by

$$\langle A(e)v, z\rangle_{V(\Omega)} = a(e, v, z) \text{ for any } v, z \in V(\Omega) \text{ and } e \in U_{ad}(\Omega) \tag{2.4}$$

where on the open Ω we define the bilinear (nonsymmetric) form $a(e, \cdot, \cdot)\colon V(\Omega) \times V(\Omega) \to \mathbb{R}$ for all $e \in U_{ad}(\Omega)$ by:

$$\begin{aligned} a(e, v, z) = \int_\Omega \{[\mathcal{A}(e)] \operatorname{grad} v \cdot \operatorname{grad} z + ([\mathcal{Q}(e)] \cdot \operatorname{grad} z)v \\ + ([\mathcal{B}(e)] \cdot \operatorname{grad} v)z + a_0 vz\}\, d\Omega + \int_{\partial\Omega_H} k\mathcal{M}_0 v\mathcal{M}_0 z\, ds \end{aligned} \tag{2.5}$$

and the right-hand side is an element belonging to $L_2(\Omega)$ given by: $(\langle L(e), \cdot\rangle_{V(\Omega)} : V(\Omega) \to \mathbb{R})$

$$\langle L(e), v\rangle_{V(\Omega)} = \int_\Omega (f + Be)v\, d\Omega + \int_{\partial\Omega_H} H\mathcal{M}_0 v\, ds \text{ for any } v \in V(\Omega), \tag{2.6}$$

$e \in U_{ad}(\Omega)$ and $B \in L(U(\Omega), L_2(\Omega))$.

Remark 3. Let $\mathcal{K}(\Omega) = V(\Omega) = H_0^1(\Omega)$. Then, it is not hard to see that $A(e)$ is a Fredholm operator ([**4**]). Therefore the state equation $A(e)u = L(e)$ is solvable if and only if the fixed $e \in U_{ad}(\Omega)$ belongs to the set $\mathcal{H}(\Omega) = \{e \in U_{ad}(\Omega) : \langle L(e), v\rangle_{V(\Omega)} = 0 \text{ for any } v \in \operatorname{Ker}(A^*(e))\}$ where $\operatorname{Ker}(A^*(e))$ is the kernel of the adjoint operator $A^*(e)$ to $A(e)$, $\mathcal{H}(\Omega)$ is called set of admissible controls and $\mathcal{Z}(\Omega) = \{[e, u] \in L_2(\Omega) \times H_0^1(\Omega) : A(e)u = L(e),\ e \in \mathcal{H}(\Omega)\}$ is said to be set of admissible pairs.

2.2. Existence Theorem.

The bilinear form (2.5) can be shown to be coercive under less restrictive assumption than: $[\mathcal{Q}(e)]$ and $[\mathcal{B}(e)]$ vanish identically in Ω. However, we have always to require, that the lower-order coefficients be conveniently small, in some sense to be specified, with respect to various parameters such as meas Ω and the constant α_* of uniform ellipticity. The case of a noncoercive bilinear form:
We now define

$$\begin{aligned} |[\mathcal{A}(e)](x)| &= \sup_{\xi\in\mathbb{R}^3-\{0\}} |[\mathcal{A}(e)](x)\xi|/|\xi|_{\mathbb{R}^3} \\ |[\mathcal{Q}(e)](x)| &= \sup_{\xi\in\mathbb{R}^3-\{0\}} |[\mathcal{Q}(e)](x)\xi|/|\xi|_{\mathbb{R}^3} \\ |[\mathcal{B}(e)](x)| &= \sup_{\xi\in\mathbb{R}^3-\{0\}} |[\mathcal{B}(e)](x)\xi|/|\xi|_{\mathbb{R}^3} \end{aligned} \tag{2.7}$$

Then by ((F1), 1°, 2°) we clearly find that the functions $x \to |[\mathcal{A}(e)](x)|$, $x \to |[\mathcal{Q}(e)](x)|$ and $x \to |[\mathcal{B}(e)](x)|$ belong to $L_\infty(\Omega)$. We consider bilinear form

$\langle A(e)v, z\rangle_{V(\Omega)}$ on $V(\Omega)$, given by (2.4) where the $a_{ij}(e)$, $a_i(e)$, $b_i(e)$, $a_0(e)$ verify ((F1), 1°, 2°) and

$$\text{(F2)} \qquad \int_\Omega \{[\mathcal{B}(e)] \cdot \operatorname{grad} v + a_0 v\}\, d\Omega \; + \int_{\partial\Omega_H} k\mathcal{M}_0 v\, ds \geq 0$$

for any $v \in V(\Omega)$, $v \geq 0$ and for any $e \in U_{ad}(\Omega)$. (We note that the assumption (F2) corresponds, for differentiable $[\mathcal{B}(e)]$ to the conditions: $a_0(e) - [\mathcal{B}(e)] \cdot \operatorname{grad} v \geq 0$ in Ω and $k + [\mathcal{B}(e)] \cdot n \geq 0$ on $\partial\Omega_H$).

The lower order terms imply, in general, that the bilinear form (2.5) is not coercive, (this means that the Lions-Stampacchia theorem ([**9**]) is not directly applicable to the state obstacle problem).

With respect to the Poincare inequality and the Sobolev embeddings one has, for some $C_r = C_r(\Omega, \partial\Omega_u, r) > 0$

$$\text{(2.8)} \qquad \|v\|_{L_r(\Omega)} \leq c_r \|\operatorname{grad} v\|_{L_2(\Omega)} = c_r \|v\|_{V(\Omega)} \quad \text{for any } v \in V(\Omega)$$

where $r = 2n/(n-2)$ if $n \geq 3$ and r is any number $1 < r < \infty$ if $n = 2$. Moreover, for any $\varepsilon > 0$ and $f \in L_p(\Omega)$, $1 \leq p < \infty$ we can write, ([**17**])

$$\text{(2.9)} \qquad f = f_* + f_{**} \quad \text{with } \|f_{**}\|_{L_p(\Omega)} \leq \varepsilon \quad \text{and } \|f_*\|_{L_\infty(\Omega)} \leq M(\varepsilon).$$

We have the following inequalities:

$$\text{(2.10)} \qquad \left| \int_\Omega ([\mathcal{Q}(e)] \cdot \operatorname{grad} v) v \, d\Omega \right| \leq (\alpha_*/4) \|\operatorname{grad} v\|^2_{L_2(\Omega)} + \alpha_*^{-2} \|[\mathcal{Q}(e)]\|^2_{L_\infty(\Omega)} \|v\|^2_{L_2(\Omega)}$$

$$\text{(2.11)} \qquad \left| \int_\Omega ([\mathcal{B}(e)] \cdot \operatorname{grad} v) v \, d\Omega \right| \leq (\alpha_*/4) \|\operatorname{grad} v\|^2_{L_2(\Omega)} + \alpha_*^{-2} \|[\mathcal{B}(e)]\|^2_{L_\infty(\Omega)} \|v\|^2_{L_2(\Omega)}.$$

Thus (by ((H1), 2°), (2.10), (2.11), (F1)) the bilinear form (2.5) is coercive on $V(\Omega)$ relative to $L_2(\Omega)$:

$$\text{(2.12)} \qquad \langle A(e)v, v\rangle_{V(\Omega)} + \lambda \|v\|^2_{L_2(\Omega)} \geq \hat{\alpha}_* \|v\|^2_{V(\Omega)} \quad \text{for } v \in V(\Omega),\ e \in U_{ad}(\Omega),\ \alpha_* > 0$$

with

$$\lambda = \alpha_*^{-1} (\|[\mathcal{Q}(e)]\|^2_{L_\infty(\Omega)} + \|[\mathcal{B}(e)]\|^2_{L_\infty(\Omega)}) + \|a_0(e)\|^2_{L_\infty(\Omega)} + (\alpha_*/2)$$
$$\hat{\alpha}_* = \alpha_*/2.$$

Lemma 3. *Assume* (F1), (F2) *and* (2.12). *If in addition, we assume the following estimate*

$$\sum_{i=1}^{3} \|a_i(x,t) - b_i(x,t)\|_{L_p(\Omega)} \le \alpha_*/(2C_r) \tag{2.13}$$

with $r = 2p/(p-2)$, $t \in [e_{\min}, e_{\max}]$, *then* (2.12) *holds with* $\lambda = 0$, *i.e.* $\langle A(e)\cdot,\cdot\rangle_{V(\Omega)}$ *is coercive on* $V(\Omega)$.

Proof. Parallel to that of Proposition 7.2 ([**17**]). □

Remark 4. For the general case we can apply (2.9) to $([\mathcal{Q}(e)] - [\mathcal{B}(e)])$ to obtain

$$\sum_{i=1}^{3} \|a_{i**}(x,t) - b_{i**}(x,t)\|_{L_p(\Omega)} \le \alpha_*/(4C_r)$$

and

$$\sum_{i=1}^{3} \|a_{i*}(x,t) - b_{i*}(x,t))\|_{L_\infty(\Omega)} \le M(\varepsilon).$$

We get

$$\begin{aligned}
&\left| \int_\Omega (([\mathcal{Q}(e)] - [\mathcal{B}(e)]) \cdot \operatorname{grad} v) v \, d\Omega \right| \\
&\qquad \le \int_\Omega |(([\mathcal{Q}_*(e)] - [\mathcal{B}_*(e)]) \cdot \operatorname{grad} v) v| \; d\Omega \\
&\qquad\quad + \int_\Omega |(([\mathcal{Q}_{**}(e)] - [\mathcal{B}_{**}(e)]) \cdot \operatorname{grad} v) v| \, d\Omega \\
&\qquad \le M(\varepsilon)\|v\|_{L_2(\Omega)} \|\operatorname{grad} v\|_{L_2(\Omega)} + (\alpha_*/4)\|\operatorname{grad} v\|^2_{L_2(\Omega)} \\
&\qquad \le (4M^2(\varepsilon)/\alpha_*)\|v\|^2_{L_2(\Omega)} + (\alpha_*/2)\|\operatorname{grad} v\|^2_{L_2(\Omega)}
\end{aligned}$$

and (2.12) follows for any $\lambda \ge \lambda_* = 4M^2(\varepsilon)/\alpha_*$.

Lemma 4. *The family* $\{A(e)\}$, $e \in U_{ad}(\Omega)$ *of operators, defined by* (2.4), (2.5) *satisfies the assumptions* ((H1), 3°, 4°, 5°).

Proof. We have

$$\begin{aligned}
|\langle A(e)v, z\rangle_{V(\Omega)}| \le \operatorname{Max}\Big[&\|[\mathcal{A}(e)]\|_{L_\infty(\Omega)},\ \|[\mathcal{Q}(e)]\|_{L_\infty(\Omega)},\ \|[\mathcal{B}(e)]\|_{L_\infty(\Omega)}, \\
&\|a_0(e)\|_{L_\infty(\Omega)},\ c(\Omega)\|k\|_{L_\infty(\partial\Omega_H)}\Big] \|v\|_{V(\Omega)} \|z\|_{V(\Omega)}
\end{aligned}$$

for any $v, z \in V(\Omega)$, $e \in U_{ad}(\Omega)$. (It is a simple application of Schwarz inequality by using (2.8) since $|[\mathcal{A}(e)]|$, $|[\mathcal{Q}(e)]|$, $|[\mathcal{B}(e)]| \in L_\infty(\Omega)$ and $k \in L_\infty(\partial\Omega_H \cup \partial\Omega_S)$.)

Then from the above relation we get ((H1), 3°). On the other hand ((H1), 4°) is immediate consequence of Lemma 3.

Let $e_n \to e$ (strongly) in $C^0(\overline{\Omega})$ for $n \to \infty$, ($e, e_n \in U_{ad}(\Omega)$). Then (for $v, w \in V(\Omega)$) one has

$$
\begin{aligned}
(2.14)\qquad & |\langle A(e_n)v, w\rangle_{V(\Omega)}| - \langle A(e)v, w\rangle_{V(\Omega)}| \\
&\le \int_\Omega |([\mathcal{A}(e_n)] - [\mathcal{A}(e)]) \operatorname{grad} v \cdot \operatorname{grad} w|\, d\Omega \\
&\quad + \int_\Omega |([\mathcal{Q}(e_n)] - [\mathcal{Q}(e)]) \cdot \operatorname{grad} wv|\, d\Omega \\
&\quad + |([\mathcal{B}(e_n)] - [\mathcal{B}(e)]) \cdot \operatorname{grad} vw|\, d\Omega + |([a_0(e_n)] - [a_0(e)])vw|\, d\Omega \\
&\le \max_{i,j} \max_{x\in\overline{\Omega}} |a_{ij}(e_n) - a_{ij}(e)| \int_\Omega |\operatorname{grad} v \cdot \operatorname{grad} w|\, d\Omega \\
&\quad + \max_{i} \max_{x\in\overline{\Omega}} |a_{i}(e_n) - a_{i}(e)| \int_\Omega |\operatorname{grad} w|\,|v|\, d\Omega \\
&\quad + \max_{i} \max_{x\in\overline{\Omega}} |b_{i}(e_n) - b_{i}(e)| \int_\Omega |\operatorname{grad} v|\,|w|\, d\Omega \\
&\quad + \max_{x\in\overline{\Omega}} |a_{0}(e_n) - a_{0}(e)| \int_\Omega |vw|\, d\Omega.
\end{aligned}
$$

On the other hand by virtue of (F1) and (2.14) if we apply Theorem 3.10 ([**11**]), we can write

$$
\begin{aligned}
\lim_{n\to\infty} \max_{i,j} \max_{x\in\overline{\Omega}} |a_{ij}(e_n) - a_{ij}(e)| &= 0,\\
\lim_{n\to\infty} \max_{i} \max_{x\in\overline{\Omega}} |a_{i}(e_n) - a_{i}(e)| &= 0,\\
\lim_{n\to\infty} \max_{i} \max_{x\in\overline{\Omega}} |b_{i}(e_n) - b_{i}(e)| &= 0,\\
\lim_{n\to\infty} \max_{x\in\overline{\Omega}} |a_{0}(e_n) - a_{0}(e)| &= 0.
\end{aligned}
$$

Inserting this into (2.14) we obtain ((H1), 5°). □

Since $\beta(\cdot)$ in (2.3) is a maximal monotone operator on $\mathbb{R}$, a convex lower continuous proper functional $j(\cdot)$ can be determined up to an additive constant (see [**2**]) such that $\beta(\cdot) = \partial j(\cdot)$. A convex lower semicontinuous proper functional $\Phi^*(\cdot)$ is then defined on $L_2(\partial\Omega_S)$ by the relation

$$
\Phi^*(\cdot) = \int_{\partial\Omega_S} j(\cdot)\, ds \quad \text{if } j(\cdot) \in L_2(\partial\Omega_S)
$$

and $\Phi^*(\cdot) = \infty$ otherwise.
The restriction of $\Phi^*(\cdot)$ to $H^{1/2}(\partial\Omega_S)$ is denote by $\Phi(\cdot)$. Next since $\Phi(\cdot)$ is independent on e, verification of ((E1), 1°) is trivial. It remains to chose $\mathcal{K}(\Omega)$

$(\subset V(\Omega))$. We set

$$\mathcal{K}(\Omega) = \{v \in V(\Omega) : v(x) \geq \psi(x) \text{ a.e. on } \Omega\}$$

where $\psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ and $\psi(\partial\Omega) < 0$.

Lemma 5. *The set $\mathcal{K}(\Omega)$ is nonempty and closed in $V(\Omega)$.*

Proof. See Glowinski ([**7**]). □

Thus, verification of (1.3) and (1.4) is trivial (clear as $\mathcal{K}(\Omega)$ does not depend on e). Next from (2.6) and since $f \in L_2(\Omega)$, $H \in L_2(\partial\Omega_H)$, $B \in L(U(\Omega), L_2(\Omega))$ and from the Schwarz inequality in $L_2(\Omega)$ and $L_2(\partial\Omega)$, we have

$$\begin{aligned} |\langle L(e), v\rangle_{V(\Omega)}| &\leq (\|f\|_{L_2(\Omega)} + \|Be\|_{L_2(\Omega)})\|v\|_{L_2(\Omega)} + \|H\|_{L_2(\partial\Omega_H)}\|\mathcal{M}_0 v\|_{L_2(\partial\Omega_H)} \\ &\leq (\|f\|_{L_2(\Omega)} + \|Be\|_{L_2(\Omega)} + C(\Omega)\|H\|_{L_2(\partial\Omega_H)})\|v\|_{V(\Omega)} \end{aligned}$$

for $v \in V(\Omega)$ and for any $e \in U_{ad}(\Omega)$.
This means, that the continuity of $L(e)$ in e is assured. The condition ((E1), $2°$) is evident. Finally, to each control e we associate a cost functional (2.2). The functional $\mathcal{L}(e,v)\colon U_{ad}(\Omega) \times V(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous, consequently, we may write immediately,

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{L}(e_n, v_n) &= \liminf_{n\to\infty}(\|v_n - z_d\|^2_{L_2(\Omega)} + N\|e_n\|^2_{U(\Omega)}) \\ &\geq \|v - z_d\|^2_{L_2(\Omega)} + N\|e\|^2_{U(\Omega)} = \mathcal{L}(e,v). \end{aligned}$$

The problem of optimal control is now to find, $\inf\{J(e) : e \in U_{ad}(\Omega)\}$. Thence Theorem 1 shows that a solution e_* exists.

3. Optimal Control of an Obstacle Problem (The convex sets depend on the control parameter)

Let Ω be a bounded open set in $\mathbb{R}^2$ and let a bilinear form $a(e,v,z)$ be given on $V(\Omega)$ $(= H^1_0(\Omega))$ by (2.5) (where $k = 0$). The set of admissible functions (controls) has the form

$$\begin{aligned} U_{ad}(\Omega) = \Big\{e \in H^2(\overline{\Omega}) : 0 < e_{\min} \leq e(x) \leq e_{\max} \text{ on } \Omega,\ \|e\|_{H^2(\Omega)} \leq c_1, \\ \int_\Omega e(x)\, d\Omega = c_2,\ e(s) = \nu(s), \text{ on } \partial\Omega,\ \nu \in C(\overline{\partial\Omega})\Big\}. \end{aligned}$$

We suppose that c_1, c_2, c_3, $e_{\min}$, $e_{\max}$ are given constants in such a way that $U_{ad}(\Omega) \neq \emptyset$. $U(\Omega) = H^1(\Omega) \cap C(\overline{\Omega})$ (the set of controls).
Now we introduce the system of sets $\{\mathcal{K}(e,\Omega)\}$, $e \in U_{ad}(\Omega)$, $\mathcal{K}(e,\Omega) = \{v \in V(\Omega) :$

$v(x) \geq \mathcal{S}(x) + e(x)$ a.e. on $\Omega\}$ where the obstacle is analytically described by a function: $\mathcal{S}(x) \in H^2(\Omega)$ fulfilling the condition

$$\mathcal{S}(s) + \nu(s) \leq 0 \quad \text{for all } s \in \partial\Omega. \tag{3.1}$$

Thus, $\mathcal{K}(e,\Omega)$ is nonempty for every $e \in U_{ad}(\Omega)$ due to assumption (3.1). Indeed, we have $z \in \mathcal{K}(e,\Omega)$, $z = \max\{0, \mathcal{S}+e\}$ (see [**9**]). It can be easily seen that $\mathcal{K}(e,\Omega)$ is convex and closed in $V(\Omega)$. The system $\{\mathcal{K}(e,\Omega)\}$ fulfils the condition ((H1), $2°$). Indeed, if $\lim_{n\to\infty} e_n = e$ in $U(\Omega)$ $(= H^1(\Omega) \cap C(\overline{\Omega}))$, $e_n \in U_{ad}(\Omega)$, then there exists a subsequence $\{e_n\}$ weakly convergent in $H^2(\Omega)$ to the element $e \in U_{ad}(\Omega)$. Let $z_k \rightharpoonup z$ (weakly) in $V(\Omega)$, $z_k \in \mathcal{K}(e_k,\Omega)$, $k = 1, 2, \ldots$, $z \in V(\Omega)$. We then have $z_k(x) \geq S(x) + e_k(x)$ for a.e. $x \in \Omega$ which implies, with respect to the compact embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ $z(x) \geq \mathcal{S}(x) + e(x)$ for a.e. $x \in \Omega$ and hence $z \in \mathcal{K}(e,\Omega)$. If $v \in \mathcal{K}(e,\Omega)$, then we put $v_m = v + (e_m - e)$. The elements $\{v_m\}$ satisfy the conditions $v_m \in \mathcal{K}(e_m,\Omega)$, $\lim_{m\to\infty} v_m = v$ (strongly) in $V(\Omega)$. Hence the condition ((H1), $2°$) holds. We set

$$\langle A(e)v, z\rangle_{V(\Omega)} = a(e,v,z) \quad \text{for any } v, z \in V(\Omega) \text{ and } e \in U_{ad}(\Omega) \tag{3.2}$$

where $A(e) \in L(V(\Omega), V^*(\Omega))$.
The state function $u(e) \in \mathcal{K}(e,\Omega)$ is a solution of the state variational inequality

$$\langle A(e)u(e), v - u(e)\rangle_{V(\Omega)} \geq \langle L(e), v - u(e)\rangle_{V(\Omega)}, \quad \text{for all } v \in \mathcal{K}(e,\Omega) \tag{3.3}$$

where the operators $A(e)$ are given by (2.5), for ($k = 0$ and $u = 0$ on $\partial\Omega$) and the function, $L(e) = f + Be$. Let us consider the cost functional of the form

$$\mathcal{I}(e) = \mathcal{L}(e,v) = \mu_A(e,\Omega) + \|e\|^2_{L_2(\Omega)} \tag{3.4}$$

where $\mu_A(e,\Omega)$ is Radon measure, $e \in U_{ad}(\Omega)$. From the regularity results obtained by Brezis, Kinderlehrer ([**2**], [**9**]) it follows that $u(e) \in H^2(\Omega) \cap H^1_0(\Omega)$. It is evident that $\theta + u(e) \in \mathcal{K}(e,\Omega)$ for any $\theta \in V(\Omega)$, $\theta \geq 0$ therefore

$$\langle A(e)u(e), \theta\rangle_{V(\Omega)} - \langle L(e), \theta\rangle_{V(\Omega)} \geq 0 \quad \text{for any } \theta \geq 0,\ e \in U_{ad}(\Omega). \tag{3.5}$$

Hence there exists a non-negative Radon-measure $\mu_A(e,\Omega)$ given by:

$$\begin{aligned} \int_\Omega \theta\, d\mu_A(e,\Omega) &= (\mathcal{R}(e)u(e) - L(e), \theta)_{L_2(\Omega)} \\ &= \langle A(e)u(e), \theta\rangle_{V(\Omega)} - \langle L(e), \theta\rangle_{V(\Omega)} \quad \text{for any } \theta \in C_0^\infty(\Omega) \end{aligned} \tag{3.6}$$

with the property that for $\mathcal{Z}$ (coincidence set, $x \in \Omega : v(x) = \mathcal{S}(x) + e(x)$) compact, $\mu_A(e, \Omega\backslash\mathcal{Z}) = 0$. (It shoud be noted that in general the set $\mathcal{Z}$ is not closed.)

In order to estabilish the existence of an optimal control function we have to verify

$$\begin{aligned} e_n \to e_* \quad &\text{(strongly) in } U(\Omega), \\ &(e_*, e_n \in U_{ad}(\Omega)) \Rightarrow u(e_n) \rightharpoonup u(e_*) \quad \text{(weakly) in } V(\Omega). \end{aligned} \tag{3.7}$$

We shall proceed in a similar way as in ([**3**]). First we recall an important result of F. Murat in ([**16**]).

Lemma 6. *If $\{\mathcal{F}_n\} \subset V^*(\Omega)$ is a sequence such that: $\mathcal{F}_n \geq 0$ (in the distributional sense) and $\mathcal{F}_n \rightharpoonup \mathcal{F}$ (weakly) in $V^*(\Omega)$ then: $\mathcal{F}_n \to \mathcal{F}$ (strongly) in $W_q^{-1}(\Omega) = (W_p^1(\Omega))^*$ for all $q < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. The inequality $\mathcal{F}_n \geq 0$ in $V^*(\Omega)$ means*

$$\langle \mathcal{F}_n, \nu \rangle_{V(\Omega)} \geq 0 \quad \textit{for all} \quad \nu \in C_0^\infty(\Omega), \ \nu \geq 0 \ \textit{on} \ \Omega. \tag{3.8}$$

Lemma 7. *Let $u(e) \in \mathcal{K}(e, \Omega)$ be a (unique) solution of the inequality* (3.3), *$e \in U_{ad}(\Omega)$. Then the relation* (3.7) *holds.*

Proof. Let $\lim_{n\to\infty} e_n = e_*$ (strongly) in $U(\Omega)$, $(e_*, e_n \in U_{ad}(\Omega))$. It results from the form of the set $U_{ad}(\Omega)$ that

$$e_n \rightharpoonup e_* \ \ \text{(weakly) in } W_p^1(\Omega) \text{ for every } p \geq 1. \tag{3.9}$$

Let us denote $u_n : = u(e_n)$, $n = 1, 2, \ldots$. We recall that the elements u_n are the solutions of the variational inequalities.

$$\langle A_n u_n, v - u_n \rangle_{V(\Omega)} \geq \langle L_n, v - u_n \rangle_{V(\Omega)} \quad \text{for all } v \in \mathcal{K}(e_n, \Omega), \tag{3.10}$$

where we have denoted $A_n = A(e_n)$, $L_n = L(e_n)$, $n = 1, 2, \ldots$.
In the same way as in Section 2, we can prove boundedness

$$\|u_n\|_{V(\Omega)} \leq c, \quad n = 1, 2, \ldots. \tag{3.11}$$

Hence there exists a subsequence of $\{[u_{n_k}, e_{n_k}]\}_k$ (still denoted by $\{[u_n, e_n]\}_n$) such that

$$u_n \rightharpoonup u_* \quad \text{(weakly) in } V(\Omega) \tag{3.12}$$

$$u_n \to u_* \quad \text{(strongly) in } L_2(\Omega) \tag{3.13}$$

$$e_n \rightrightarrows e_* \quad \text{(uniformly) in } U(\Omega) \tag{3.14}$$

$$e_n \rightharpoonup e_* \quad \text{(weakly) in } W_p^1(\Omega) \text{ for all } p \geq 1 \tag{3.15}$$

$$A_n \to A_* \quad \text{in } L(V(\Omega), V^*(\Omega)). \tag{3.16}$$

As $u_n \in \mathcal{K}(e_n, \Omega)$, we have the inequalities $u_n \geq \mathcal{S} + e_n$ a.e. on Ω, $n = 1, 2, \ldots$ and the relations (3.13), (3.14) imply: $u \geq \mathcal{S} + e_*$ a.e. on Ω and hence $u_* \in \mathcal{K}(e_*, \Omega)$. Let us rewrite the inequality (3.10) in the form

$$\langle A_n u_n - L_n, v - u_n \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}(e_n, \Omega). \tag{3.17}$$

Taking $v = u_n + \theta$, $\theta \geq 0$, $\theta \in C_0^\infty(\Omega)$ we obtain

$$\langle A_n u_n, -L_n, \theta \rangle \geq 0 \ \text{ in } V^*(\Omega), \ n = 1, 2, \ldots. \tag{3.18}$$

Using the form (3.2), the conditions (F1), the limits (3.12), (3.13) and (3.14), (3.16) we arrive at

$$A_n u_n - L_n \rightharpoonup A_* u_* - L_* \quad \text{(weakly) in } V^*(\Omega) \tag{3.19}$$

where $A_* = A(e_*)$, $L_* = L(e_*)$.
Applying now Lemma 6 we obtain

$$A_n u_n - L_n \to A_* u_* - L_* \quad \text{(strongly) in } W_q^{-1}(\Omega),\ q < 2. \tag{3.20}$$

Setting $v = z + (e_n - e_*)$ in (3.17) for any $z \in \mathcal{K}(e_*, \Omega)$ we have the relations

$$\begin{aligned}\langle A_* u_n, z\rangle_{V(\Omega)} &= \langle (A_* - A_n)u_n, z\rangle_{V(\Omega)} + \langle A_n u_n, z\rangle_{V(\Omega)} \\ &\geq \langle (A_* - A_n)u_n, z\rangle_{V(\Omega)} + \langle A_n u_n, u_n\rangle_{V(\Omega)} + \langle L_n, z - u_n\rangle_{V(\Omega)} \\ &\quad - \langle A_n u_n - L_n, e_n - e_*\rangle_{V(\Omega)}\,.\end{aligned}$$

Using the relations (3.12), (3.15) and (3.16), (3.20) and the weak lower semicontinuity

$$\langle A_* u_*, u_*\rangle_{V(\Omega)} \leq \liminf_{n\to\infty} \langle A_* u_n, u_n\rangle_{V(\Omega)}$$

we arrive at the inequality

$$\langle A_* u_*, z\rangle_{V(\Omega)} \geq \langle A_* u_*, u_*\rangle_{V(\Omega)} + \langle L_*, z - u_*\rangle_{V(\Omega)} \quad \text{for all } z \in \mathcal{K}(e_*, \Omega)$$

and hence $u_* = u(e_*)$ and the relation (3.7) is verified. □

For any fixed compact $\mathcal{O} \subset \Omega$, the following estimate holds

$$\mu_A(e_n, \mathcal{O}) \leq c_A \quad \text{(uniformly bounded with respect to } n). \tag{3.21}$$

Indeed, let $\theta \in C_0^\infty(\Omega)$, $\theta \equiv 1$ on $\mathcal{O}$ and $\theta \geq 0$ on Ω. Then we may write

$$\begin{aligned}\mu_A(e_n, \mathcal{O}) &\leq \int_\Omega \theta\, d\mu_A(e_n, \Omega) = \langle A(e_n)u(e_n) - L(e_n), \theta\rangle_{V(\Omega)} \\ &\leq |\,\langle A(e_n)u(e_n) - L(e_n), \theta\rangle_{V(\Omega)}\,| \\ &\leq \|A(e_n)u(e_n) - L(e_n)\|_{V^*(\Omega)} \|\theta\|_{V(\Omega)} \leq c_A.\end{aligned}$$

This means, that there exist a subsequence $\{\mu_A(e_{n_k}, \mathcal{O}_n)\}_k$ (still denoted by $\{\mu_A(e_n, \mathcal{O})\}_n$) such that

$$\mu_A(e_n, \mathcal{O}) \rightharpoonup \mu_A(e_*, \mathcal{O}) \tag{3.22}$$

(weakly) as $n \to \infty$ in $V^*(\Omega)$.
Hence, for any finite continuous function θ on Ω the following convergence

$$\int_\Omega \theta\, d\mu_A(e_n, \Omega) \to \int_\Omega \theta\, d\mu_A(e_*, \Omega) \tag{3.23}$$

hold. Moreover, the family of Radon measures $\{\mu_A(e_n,\Omega)\}$ is denoted correct (since the pairing $[e_n, u(e_n)]$ is the solution of the variational inequality (3.3)). Then, by virtue of weakly converges of measures $\{\mu_A(e_n,\Omega)\}_n$ in particular, follows

$$\liminf_{n\to\infty} \mu_A(e_n,\Omega) \geq \mu_A(e_*,\Omega). \tag{3.24}$$

Let $\{e_n\}$ be a minimizing sequence for the functional $\mathcal{I}(e)$:

$$\lim_{n\to\infty} \mathcal{I}(e_n) = \inf_{e\in U_{ad}(\Omega)} \mathcal{I}(e)$$

(we put $\inf_{e\in U_{ad}(\Omega)} \mathcal{I}(e) = -\infty$, if the set $\{\mathcal{I}(e)\}$ is not lower bounded). Since the set $U_{ad}(\Omega)$ is compact in $U(\Omega)$, there exist $e_0 \in U_{ad}(\Omega)$ and a subsequence $\{e_k\}$ such that $\lim_{k\to\infty} e_k = e_0$ in $U(\Omega)$.

Finally we get (using (3.24) and the semicontinuity of norm $\|\cdot\|_{L_2(\Omega)}$)

$$\inf_{e\in U_{ad}(\Omega)} \mathcal{I}(e) = \lim_{k\to\infty} \mathcal{I}(e_k) = \liminf_{k\to\infty} \mathcal{L}(e_k, u_k) \geq \mathcal{L}(e_0, u(e_0)) = \mathcal{I}(e_0).$$

This means that $e_0 \in U_{ad}(\Omega)$ is an optimal control function of the optimal control problem $(\mathcal{P})$.

4. Control of Some Unilateral Problems

Let Ω be a bounded open domain annulus of $\mathbb{R}^2$ with smooth boundary $\partial\Omega = \partial\Omega_* \cup \partial\Omega_{**}$. Let $\mathfrak{A}(e)$ be a second order elliptic operator in Ω, given by

$$\mathfrak{A}(e)v = -\nabla\cdot(\mathcal{F}[\mathcal{Q}(x)])\nabla v + \mathcal{F}a_0(x)v,\ \ v \in V(\Omega) \tag{4.1}$$

where $V(\Omega) = H^1(\Omega)$ and $\mathcal{Q}(x)$ is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ depending upon x over Ω, ∇ denotes the vector $\{\partial/\partial x_i\}_{i=1}^2$ and $e = [\mathcal{F}, \mathcal{Z}]$ is the control variable.

We suppose that the following hypotheses concerning $[\mathcal{Q}]$, a_0, L hold:

$$\text{(L1)}\quad \begin{cases} L \in L_2(\Omega),\ a_0 \in L_\infty(\Omega),\ a_0(x) \geq \alpha_0\ (>0) \text{ a.e. on } \Omega,\ a_{ij} \in L_\infty(\Omega) \\ \text{and } \partial a_{ij}/\partial x_k \in L_\infty(\Omega) \text{ for any } 1 \leq i,j \leq 2,\ k = 1,2. \\ \text{There exists } \alpha > 0 \text{ such that } [\mathcal{Q}(x)]\xi\cdot\xi \geq \alpha|\xi|^2_{\mathbb{R}^2} \text{ a.e. on } \Omega \text{ for any} \\ \xi = \{\xi_i\}_{i=1}^2 \in \mathbb{R}^2. \end{cases}$$

The state of the system is given by $u(e)$, solution of the unilateral problem:

$$\begin{cases} 1^\circ.\ \mathfrak{A}(e)u(e) = L \\ 2^\circ.\ u(e) - \mathcal{Z} = 0 \quad \text{on } \partial\Omega_* \\ 3^\circ.\ (u(e) - \mathcal{Z}) \geq 0,\ \ \partial u(e)/\partial n_{\mathfrak{A}} \geq 0, \\ \qquad (u(e) - \mathcal{Z})(\partial u(e)/\partial n_{\mathfrak{A}}) = 0 \quad \text{on } \partial\Omega_{**}. \end{cases} \tag{4.2}$$

In (4.2) $\mathcal{Z}$ is a function on $\partial\Omega$ which is the control variable and $\partial/\partial n_{\mathfrak{A}}$ denotes the conormal derivate associated to $\mathfrak{A}(e)$: $\partial v/\partial n_{\mathfrak{A}} = (\mathcal{Z}[\mathcal{Q}(x)]\nabla v)\cdot n$ and $n = \{n_i\}$ is a normal to $\partial\Omega_{**}$ directed toward the exterior of Ω. The cost functional is defined by

$$\mathcal{I}(e) = \mathcal{L}(e, u(e)) = \int_{\partial\Omega_{**}} (\partial u(e)/\partial n_{\mathfrak{A}} - z_d)^2 \, ds + N \int_{\partial\Omega} \mathcal{Z}^2 \, ds \tag{4.3}$$

where z_d is given in $L_2(\partial\Omega_{**})$ and where N is given > 0.

Let $\mathcal{U}$ be the set of control functions, given by:

$$\mathcal{U} = U(\Omega) \times U(\partial\Omega) \ \text{ where } U(\Omega) = L_2(\Omega),\ U(\partial\Omega) = H^1(\partial\Omega).$$

Let us introduce the set of admissible control functions:

$$\mathcal{U}_{ad} = U_{ad}(\Omega) \times U_{ad}(\partial\Omega)$$

where

$$\begin{aligned} U_{ad}(\Omega) = \big\{\mathcal{F} \in H^1(\Omega) \cap C(\overline{\Omega}) \ &: 0 < \mathcal{F}_{\min} \le \mathcal{F}(x) \le \mathcal{F}_{\max} \\ &\quad \text{for all } x \in \Omega,\ \|\mathcal{F}\|_{H^1(\Omega)} \le c_1\big\} \\ U_{ad}(\partial\Omega) = \big\{\mathcal{Z} \in H^2(\partial\Omega) \ &: 0 < \mathcal{Z}_{\min} \le \mathcal{Z}(x) \le \mathcal{Z}_{\max} \\ &\quad \text{for all } x \in \partial\Omega,\ \|\mathcal{Z}\|_{H^2(\partial\Omega)} \le c_2\big\} \end{aligned}$$

where positive constants $\mathcal{F}_{\min}$, $\mathcal{F}_{\max}$, $\mathcal{Z}_{\min}$, $Z_{\max}$, c_1, c_2, are chosen in such a way that $\mathcal{U}_{ad} \ne \emptyset$. The set of admissible states of the system (4.2) is defined as

$$\mathcal{K}(e,\Omega) = \big\{v \in V(\Omega) : \mathcal{M}_0 v = \mathcal{Z} \text{ a.e. on } \partial\Omega_*,\ \mathcal{M}_0 v \ge \mathcal{Z} \text{ a.e. on } \partial\Omega_{**}\big\}. \tag{4.4}$$

Due to the structure of (4.3) it is natural to take $\mathcal{Z} \in L_2(\partial\Omega)$, and in order for $u(e)$ to make sense to add condition $\mathcal{K}(e,\Omega) \ne \emptyset$ for each $e \in \mathcal{U}_{ad}$. However, this is not sufficient in general for (4.3) to make sense, since in general $\partial u(e)/\partial n_{\mathfrak{A}}$ does not belong to $L_2(\partial\Omega_{**})$. Moreover, this is not sufficient in general for ((H1), 5°) to make sense (in order to characterize the dependence $e \to \mathcal{K}(e,\Omega)$ we need the special type of convergence of sequences of sets introduced by Mosco).

In order to make (4.2) precise we define the following bilinear form on $V(\Omega)$

$$a(e,v,z) = \int_\Omega (\mathcal{F}[\mathcal{Q}]\nabla v)\cdot\nabla z \, d\Omega + \int_\Omega \mathcal{F} a_0 v z \, d\Omega \tag{4.5}$$
$$\text{for any } v,\ z \in V(\Omega),\ e \in \mathcal{U}_{ad}.$$

Lemma 8. *The set $\mathcal{K}(e,\Omega)$ is nonempty, convex and closed.*

Proof. Suppose that a function $\mathcal{H} \in V(\Omega)$ determines the control function on $\partial\Omega$ by its traces. Since $\mathcal{H} \in \mathcal{K}(e,\Omega)$, $\mathcal{K}(e,\Omega)$ is nonempty. The convexity of $\mathcal{K}(e,\Omega)$ is obvious. If $\{v_n\}_n \subset \mathcal{K}(e,\Omega)$ and $v_n \to v$ (strongly) in $V(\Omega)$, then $\mathcal{M}_0 v_n \to \mathcal{M}_0 v$ (strongly) in $L_2(\partial\Omega)$, since $\mathcal{M}_0$ (the operator of trace): $H^1(\Omega) \to L_2(\partial\Omega)$ is continuous. Since $v_n \in \mathcal{K}(e,\Omega)$, $\mathcal{M}_0 v_n = \mathcal{Z}$ a.e. on $\partial\Omega_*$ and $\mathcal{M}_0 v_n \geq \mathcal{Z}$ a.e. on $\partial\Omega_{**}$ for each $e \in U_{ad}$. Therefore $\mathcal{M}_0 v = \mathcal{Z}$ a.e. on $\partial\Omega_*$ and $\mathcal{M}_0 v \geq \mathcal{Z}$ a.e. on $\partial\Omega_{**}$. Hence $v \in \mathcal{K}(e,\Omega)$ which shows that $\mathcal{K}(e,\Omega)$ is closed.

Let us observe that if $u(e) \in V(\Omega)$ and satisfies ((4.2), 1°) then $\partial u(e)/\partial n_{\mathfrak{A}}$ is defined and belongs to $H^{-1/2}(\partial\Omega_{**})$. This allows us to introduce the set

$$\mathcal{N}(\partial\Omega) = \big\{\mathcal{Z} : \mathcal{Z} \in L_2(\partial\Omega) : \mathcal{K}(e,\Omega) \neq \emptyset,\ \partial u(e)/\partial n_{\mathfrak{A}} \in L_2(\partial\Omega_{**})\big\}.$$

Let us remark that

$$U_{ad}(\partial\Omega)\ \ (= H^2(\partial\Omega)) \subset N(\partial\Omega). \tag{4.6}$$

Indeed if $\mathcal{Z} \in H^2(\partial\Omega)$ then $u(e) \in H^2(\Omega)$ so that $\partial u(e)/\partial n_{\mathfrak{A}} \in H^{1/2}(\partial\Omega_{**}) \subset L_2(\partial\Omega_{**})$, hence (4.6) follows. □

Lemma 9. *The system of convex closed sets $\mathcal{K}(e,\Omega)$ defined by* (4.4) *fulfils the condition* ((H1), 2°).

Proof. Let $e_n \to e$ (strongly) in $\mathcal{U}$ (or $\mathcal{Z}_n \to \mathcal{Z}$ strongly in $U(\partial\Omega)$), $e_n \in \mathcal{U}_{ad}$. Then there exists a subsequence $\{\mathcal{Z}_k\}_k$ of $\{\mathcal{Z}_n\}_n$ weakly convergent in $H^2(\partial\Omega)$ to the element $\mathcal{Z} \in U_{ad}(\partial\Omega)$. Let $v_n \rightharpoonup v$ weakly convergent in $V(\Omega)$, ($v_n \in \mathcal{K}(e_n,\Omega)$). Then we have: $\mathcal{M}_0 v_n - \mathcal{Z}_n = 0$ a.e. on $\partial\Omega_*$, $\mathcal{M}_0 v_n - \mathcal{Z}_n \geq 0$ a.e. on $\partial\Omega_{**}$, which by virtue of the compact embedding $H^{1/2}(\partial\Omega) \hookrightarrow L_2(\partial\Omega)$ implies that, $\mathcal{M}_0 v - \mathcal{Z} = 0$ a.e. on $\partial\Omega_*$ and $\mathcal{M}_0 v - \mathcal{Z} \geq 0$ a.e. on $\partial\Omega_{**}$. Hence $v \in \mathcal{K}(e,\Omega)$. Next, let $v \in \mathcal{K}(e,\Omega)$, then we put $v_n = v + (\mathcal{H}_n - \mathcal{H})$. The elements $\{v_n\}$ satisfy the conditions: $v_n \in \mathcal{K}(e_n,\Omega)$ and $\lim_{n\to\infty} v_n = v$ (strongly) in $V(\Omega)$. Hence the condition ((H1), 2°) holds. □

The optimal control problem in distributed parameters considered here reads:

$$\min \mathcal{L}(e,u)\ \ \text{subject to (4.2)}. \tag{4.7}$$

This means that $u(e) \in \mathcal{K}(e,\Omega)$ is a solution of (4.2) if and only if $u(e)$ is a solution of the operator equation

$$A(e)u(e) = L \tag{4.8}$$

where $A(e)$ is a linear bounded operator acting from $V(\Omega)$ into $V^*(\Omega)$. It is defined by

$$\langle A(e)v, z\rangle_{V(\Omega)} = a(e,v,z)\ \ \text{for any } v,z \in V(\Omega),\ e \in \mathcal{U}_{ad}. \tag{4.9}$$

Lemma 10. *The family* $\{A(e)\}$, $e \in \mathcal{U}_{ad}$ *of operators defined by* (4.5) *and* (4.9) *satisfies the assumptions* ((H1), 3°, 4°, 5°).

Proof. We now define

$$|[\mathcal{Q}(x)]| = \sup_{\xi \in R^2 - \{0\}} |[\mathcal{Q}(x)]\xi| / |\xi|_{\mathbb{R}^2}\,,$$

from (L1) we clearly find that the function $x \to |[\mathcal{Q}(x)]|$ belongs to $L_\infty(\Omega)$, we denote by $\|[\mathcal{Q}]\|_{L_\infty(\Omega)}$ the $L_\infty(\Omega)$-norm of the above function. Then, by virtue of (4.5) and from the Schwarz inequality, we can write

$$\begin{aligned}
(4.10) \qquad |a(e,v,z)| &\leq \mathcal{F}_{\max}(\|[\mathcal{Q}]\|_{L_\infty(\Omega)} \left(\int_\Omega |\nabla v^2|\, d\Omega\right)^{1/2} \left(\int_\Omega |\nabla z|^2\, d\Omega\right)^{1/2} \\
&\quad + \|a_0\|_{L_\infty(\Omega)} \|v\|_{L_\infty(\Omega)} \|z\|_{L_\infty(\Omega)} \\
&\leq \mathcal{F}_{\max} \mathrm{Max}\,(\|[\mathcal{Q}]\|_{L_\infty(\Omega)}, \|a_0\|_{L_\infty(\Omega)}) \|v\|_{V(\Omega)} \|z\|_{V(\Omega)}
\end{aligned}$$

for any $v, z \in V(\Omega)$ and $e \in \mathcal{U}_{ad}$. Relation (4.10) implies the continuity of $a(e,\cdot,\cdot)$ for each $e \in \mathcal{U}_{ad}$. On the other hand by (L1) we obtain

$$(4.11) \qquad a(e,v,v) \geq \mathcal{F}_{\min} \mathrm{Min}\,(\alpha, \alpha_0) \|v\|^2_{V(\Omega)} \quad \text{for any } v \in V(\Omega)$$

and for each $e \in \mathcal{U}_{ad}$ which shows the uniformly coercivity of $a(e,\cdot,\cdot)$ (with respect to $U_{ad}(\Omega)$). Now, ((H1), 3°, 4°) is an immediate consequence of (4.10) and (4.11). Let $e, e_n \in \mathcal{U}_{ad}$ be such that $e_n \to e$ (strongly) in $\mathcal{U}$. Then

$$\begin{aligned}
(4.12) \qquad & |a(e_n,v,z) - a(e,v,z)| \\
&\leq \|e_n - e\|_{C(\overline{\Omega})} \mathrm{Max}\,(\|[\mathcal{Q}]\|_{L_\infty(\Omega)}, \|a_0\|_{L_\infty(\Omega)}) \|v\|_{V(\Omega)} \|z\|_{V(\Omega)} \to 0
\end{aligned}$$

for every $v, z \in V(\Omega)$ which shows the condition ((H1), 5°). ☐

Finally, from Theorem 1 and Lemmas 8, 9 we conclude that:

$$(4.13) \qquad u(e_n) \to u(e) \ \text{(strongly) in } V(\Omega) \text{ if } e_n \to e \ \text{(strongly) in } \mathcal{U}.$$

Moreover, due to (4.6) and (4.13) there exists a subsequeance $\{u(e_k)\}$ weakly convergent in $H^2(\Omega)$ to the element $u(e)$. We then have

$$(4.14) \qquad \partial u(e_n)/\partial n_{\mathfrak{A}} \rightharpoonup \partial u(e)/\partial n_{\mathfrak{A}} \ \text{(weakly) in } H^{1/2}(\partial\Omega_{**}).$$

Let $\{e_n\}_n$ be a minimizing sequence

$$(4.15) \qquad \mathcal{I}(e_n) \to \inf \mathcal{I}(e), \quad e \in \mathcal{U}_{ad}.$$

Since the set $\mathcal{U}_{ad}$ is compact in $\mathcal{U}$, there exists $e_* \in \mathcal{U}_{ad}$ and a subsequence $\{e_{k_*}\}$ of $\{e_k\}$ such that

$$(4.16) \qquad \lim e_{k_*} = e_*\,.$$

It follows from (E2) that according to (4.14) and (4.16)

$$\mathcal{I}(e_*) = \mathcal{L}(e_*, u(e_*)) \leq \liminf_{k_* \to \infty} \mathcal{L}(e_{k_*}, u(e_{k_*})) = \liminf_{k_* \to \infty} \mathcal{I}(e_{k_*}) = \inf_{e \in \mathcal{U}_{ad}} \mathcal{I}(e).$$

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