Acta Math. Univ. Comenianae
Vol. LXIII, 1(1994), pp. 39–53

CONTINUITY OF THE HAUSDORFF DIMENSION FOR INVARIANT SUBSETS OF INTERVAL MAPS

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Abstract. Let $T\colon [0,1] \to [0,1]$ be an expanding piecewise monotonic map, and consider the set R of all points, whose orbits omit a certain finite union of open intervals. It is shown that the Hausdorff dimension $\mathrm{HD}\,(R)$ depends continuously on small perturbations of the endpoints of these open intervals. A similar result for the topological pressure is also obtained. Furthermore it is shown that for every $t \in [0,1]$ there exists a closed, T-invariant $R_t \subseteq [0,1]$ with $\mathrm{HD}\,(R_t) = t$. Finally it is proved that the Hausdorff dimension of the set of all points, whose orbit is not dense, is 1.

Introduction

Let $T\colon [0,1] \to [0,1]$ be an expanding piecewise monotonic map, that means there exists a finite partition $\mathcal{Z}$ of $[0,1]$ into pairwise disjoint open intervals with $\cup_{Z\in\mathcal{Z}}\overline{Z} = [0,1]$, $T|Z$ is continuous and strictly monotone for all $Z \in \mathcal{Z}$, $T'|Z$ can be extended to a continuous function on $\overline{Z}$ for all $Z \in \mathcal{Z}$, and there exists an $n \in \mathbb{N}$ with $\inf_{x\in[0,1]} |(T^n)'(x)| > 1$. Fix a $K \in \mathbb{N}$. Let $(a_1,a_2) \cup (a_3,a_4) \cup \cdots \cup (a_{2K-1},a_{2K})$ be a finite union of open subintervals of $[0,1]$, and set $R(a_1,a_2,\dots,a_{2K}) := \cap_{n=0}^{\infty}\overline{[0,1] \setminus T^{-n}\left(\cup_{k=1}^{K}(a_{2k-1},a_{2k})\right)}$. We investigate the influence of small perturbations of the endpoints of $\cup_{k=1}^{K}(a_{2k-1},a_{2k})$ on the set $R(a_1,a_2,\dots,a_{2K})$.

In [**4**] there are considered piecewise monotonic maps $T\colon X \to \mathbb{R}$, where X is a finite union of intervals, and the set $R(T) := \cap_{n=0}^{\infty}\overline{T^{-n}X}$ is investigated. Hence we have $R(a_1,a_2,\dots,a_{2K}) = R\left(T|[0,1] \setminus \cup_{k=1}^{K}(a_{2k-1},a_{2k})\right)$. But the results of [**4**] need not be applicable in our case, since $R\left(T|[0,1] \setminus \cup_{k=1}^{K}(\tilde a_{2k-1},\tilde a_{2k})\right)$ need not be close to $R\left(T|[0,1] \setminus \cup_{k=1}^{K}(a_{2k-1},a_{2k})\right)$ in the sense defined in [**4**], if $|\tilde a_j - a_j| < \varepsilon$ for all $j \in \{1,2,\dots,2K\}$. For example, if $K = 1$, $a_1 = \inf Z$ for a $Z \in \mathcal{Z}$ and $a_1 \neq 0$, $a_1 < a_2$, $\varepsilon > 0$ satisfies $a_1 + \varepsilon < a_2$, then $T|[0,1] \setminus (a_1+\varepsilon, a_2)$ has more intervals of monotonicity than $T|[0,1] \setminus (a_1,a_2)$ (namely the interval $(a_1, a_1+\varepsilon)$), but closeness in the sense of [**4**] implies that the maps have the same number of intervals of monotonicity.

Received August 30, 1993.

1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 58F03, 54H20, 58F15.

Research was supported by Projekt Nr. P8193–PHY of the Austrian Fond zur Förderung der wissenschaftlichen Forschung.

In Theorem 1 of this paper it is shown, that the function $(a_1, a_2, \dots, a_{2K}) \mapsto p(R(a_1, a_2, \dots, a_{2K}), T, f)$, where $p(.,.,.)$ denotes the topological pressure, is upper semi-continuous. Furthermore it says, that this function is continuous at $(a_1, a_2, \dots, a_{2K})$, if a certain condition generalizing $p(R(a_1, a_2, \dots, a_{2K}), T, f) > \sup_{x \in R(a_1, a_2, \dots, a_{2K})} f(x)$ is satisfied. This implies the continuity of the topological entropy. Theorem 2 says that the map $(a_1, a_2, \dots, a_{2K}) \mapsto \mathrm{HD}\left(R(a_1, a_2, \dots, a_{2K})\right)$ is continuous. Such results are obtained by Mariusz Urbański in the case of an expanding C^2-diffeomorphism T of the circle ([**5**], [**6**]). In [**5**] he showed that the topological entropy and the Hausdorff dimension are continuous, if $K = 1$ and $(a_1, a_2) = (0 - \varepsilon, 0 + \varepsilon)$, where 0 is a fixed point of T (intervals on the circle are defined in the usual way). He generalized this result to $K \geq 1$ and arbitrary $(a_1, a_2, \dots, a_{2K})$ in [**6**] (Theorem 4 in [**6**]). We show in Theorem 3 that for every $t \in [0, 1]$ there exists a closed, T-invariant $R_t \subseteq [0, 1]$ with $\mathrm{HD}\,(R_t) = t$. The results in [**1**] give that we can choose R_t, such that R_t is topologically transitive, and $R_t = \cap_{n=0}^{\infty} \overline{F_t \setminus T^{-n} G_t}$, where F_t and G_t are finite unions of intervals. Finally Theorem 4 says that the Hausdorff dimension of the set of all points, whose orbit is not dense, is 1. In the case of an expanding C^2-diffeomorphism of a circle, the results of Theorem 3 and Theorem 4 can be easily deduced from [**5**] (Corollary 4 of [**6**] is the analogon of Theorem 3 of this paper).

The proof uses a graph $(\mathcal{D}, \to)$, called Markov diagram, associated to $(R(a_1, a_2, \dots, a_{2K}), T)$. Lemma 2 says that the initial part of the Markov diagram of $(R(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{2K}), T)$ is similar to that of $(R(a_1, a_2, \dots, a_{2K}), T)$, if $|\tilde{a}_j - a_j| < \delta$ for all $j \in \{1, 2, \dots, 2K\}$ and a sufficiently small δ. Although the result and the proof of this lemma look similar to those of Lemma 6 in [**4**], the details are different. As in [**4**] this lemma, an approximation of f by piecewise constant functions, and Lemma 6 of [**3**] imply Theorem 1. Using Theorem 2 of [**3**] this implies Theorem 2. Theorem 3 and Theorem 4 are easy consequences of Theorem 2.

1. Definitions and Notations

A map $T\colon [0, 1] \to [0, 1]$ is called **piecewise monotone**, if there exists a finite partition $\mathcal{Z}$ of $[0, 1]$, such that $T|Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. We call $\mathcal{Z}$ a **finite partition** of $[0, 1]$, if $\mathcal{Z}$ consists of pairwise disjoint open intervals with $\cup_{Z \in \mathcal{Z}} \overline{Z} = [0, 1]$. A function $f\colon [0, 1] \to \mathbb{R}$ is called **piecewise continuous** with respect to the finite partition $\mathcal{Z}(f)$ of $[0, 1]$, if $f|Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. We say that $f\colon [0, 1] \to \mathbb{R}$ is **piecewise constant** with respect to the finite partition $\mathcal{Z}(f)$ of $[0, 1]$, if $f|Z$ is constant for all $Z \in \mathcal{Z}(f)$. A piecewise monotonic map $T\colon [0, 1] \to [0, 1]$ is called **expanding**, if there exists a $j \in \mathbb{N}$, such that $(T^j)'$ is a piecewise continuous function and $\inf_{x \in [0,1]} |(T^j)'(x)| > 1$. At this point we want to remark, that all results of this paper hold also for the situation considered in

[**4**], that means $T\colon X \to \mathbb{R}$ is piecewise monotone, where X is a finite union of closed intervals.

Let $K \in \mathbb{N}$ and suppose that $0 \le a_1 \le a_2 \le \cdots \le a_{2K-1} \le a_{2K} \le 1$ with $a_j < a_{j+2}$ for $j \in \{1, 2, \ldots, 2K-2\}$. Set $Q := (a_1, a_2, \ldots, a_{2K-1}, a_{2K})$. Let $\mathcal{Q}_K$ be the set of all such Q's. Now define for $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$

$$X(Q) := [0,1] \setminus \left(\bigcup_{k=1}^{K} (a_{2k-1}, a_{2k})\right) \tag{1.1}$$

and

$$R(Q) := \bigcap_{j=0}^{\infty} \overline{[0,1] \setminus T^{-j}\left(\bigcup_{k=1}^{K} (a_{2k-1}, a_{2k})\right)}\ . \tag{1.2}$$

Let $\mathcal{Z}(Q)$ be the set of all maximal open subintervals of $X(Q) \cap (\cup_{Z\in\mathcal{Z}} Z)$ and set $X_1(Q) := X(Q) \setminus \{x : x \text{ is isolated in } X(Q)\}$. We say that $\mathcal{Y}$ is a **finite partition of** $X(Q)$, if $\mathcal{Y}$ consists of pairwise disjoint open intervals with $\cup_{Y\in\mathcal{Y}} \overline{Y} = X_1(Q)$. Observing that the results of [**4**] remain true, if we allow X to be a finite union of closed intervals and isolated points, we have that $\big(T|X(Q), \mathcal{Z}(Q)\big)$ is a piecewise monotonic map of class R^0 in the sense defined in [**4**]. Furthermore we have $R\big(T|X(Q)\big) = R(Q)$.

Now we want to define a topology on $\mathcal{Q}_K$. Let $\varepsilon > 0$. Then $Q := (a_1, a_2, \ldots, a_{2K-1}, a_{2K})$ and $\tilde{Q} := (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{2K-1}, \tilde{a}_{2K})$ are said to be ε-close, if $|a_k - \tilde{a}_k| < \varepsilon$ for all $k \in \{1, 2, \ldots, 2K\}$. Observe that $\big(T|X(Q), \mathcal{Z}(Q)\big)$ and $\big(T|X(\tilde{Q}), \mathcal{Z}(\tilde{Q})\big)$ need not be ε-close with respect to the R^0-topology defined in [**4**].

Next we modify $([0,1], T)$ in order to get a topological dynamical system. This will be done in a similar way as in [**4**].

Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, let $Q \in \mathcal{Q}_K$, and let $\mathcal{Y}$ be a finite partition of $X(Q)$, which refines $\mathcal{Z}(Q)$. We assume throughout this paper, that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. Set $E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \cup \{\inf Z, \sup Z : Z \in \mathcal{Z}\}$. Now define $W := \big(\cup_{j=0}^{\infty} T^{-j}(E \setminus \{0,1\})\big) \setminus \{0,1\}$, set $\mathbb{R}_{\mathcal{Y}} := \mathbb{R} \setminus W \cup \{x^-, x^+ : x \in W\}$, and define $y < x^- < x^+ < z$, if $y < x < z$ holds in $\mathbb{R}$. This means, that we have doubled all endpoints of elements of $\mathcal{Y}$ or $\mathcal{Z}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi(x) < \pi(y)$ implies $x < y$. As in [**4**] we can introduce a metric d on $\mathbb{R}_{\mathcal{Y}}$, which generates the order topology.

Let $X_{\mathcal{Y}}$ be the closure of $[0,1] \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$ and define $X_{\mathcal{Y}}(Q) := \{x \in \mathbb{R}_{\mathcal{Y}} : \pi(x) \in X(Q)\}$. Observe that $X_{\mathcal{Y}}$ and $X_{\mathcal{Y}}(Q)$ are compact. For a perfect subset A of $\mathbb{R}$ let $\hat{A}$ be the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Now set $\hat{\mathcal{Y}} := \{\hat{Y} : Y \in \mathcal{Y}\}$, $\hat{\mathcal{Z}} := \{\hat{Z} : Z \in \mathcal{Z}\}$ and $\hat{\mathcal{Z}}(Q) := \{\hat{Z} : Z \in \mathcal{Z}(Q)\}$. The map $T|[0,1] \setminus (W \cup E)$

can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}}\colon X_{\mathcal{Y}} \to X_{\mathcal{Y}}$. Then $(T_{\mathcal{Y}}, \hat{\mathcal{Z}})$ is a continuous piecewise monotonic map of class R^0 on $X_{\mathcal{Y}}$ in the sense defined in [**4**]. If there is no confusion we shall use the notation $\mathcal{Y}$ instead of $\hat{\mathcal{Y}}$, $\mathcal{Z}$ instead of $\hat{\mathcal{Z}}$, and $\mathcal{Z}(Q)$ instead of $\hat{\mathcal{Z}}(Q)$. The set $R_{\mathcal{Y}} := \bigcap_{j=0}^{\infty} T_{\mathcal{Y}}{}^{-j} X_{\mathcal{Y}}(Q)$ satisfies $R_{\mathcal{Y}} = \bigcap_{j=0}^{\infty} \overline{T_{\mathcal{Y}}{}^{-j} X_{\mathcal{Y}}(Q)} = \{x \in \mathbb{R}_{\mathcal{Y}} : \pi(x) \in R(Q)\}$. We call $T_{\mathcal{Y}}$ the completion of T with respect to $\mathcal{Y}$. If $f\colon [0,1] \to \mathbb{R}$ is piecewise continuous with respect to $\mathcal{Z}$, then there exists a unique continuous function $f_{\mathcal{Y}}\colon X_{\mathcal{Y}} \to \mathbb{R}$ with $f_{\mathcal{Y}}(x) = f(x)$ for all $x \in [0,1] \setminus (W \cup E)$. Then $f_{\mathcal{Y}}$ is called the completion of f with respect to $\mathcal{Y}$.

A **topological dynamical system** (X,T) is a continuous map T of a compact metric space X into itself. Hence $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system.

If (X,T) is a topological dynamical system, and $f\colon X \to \mathbb{R}$ is a continuous function, then the **topological pressure** $p(X,T,f)$ is defined by

$$p(X,T,f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{E} \sum_{x\in E} \exp\left(\sum_{j=0}^{n-1} f(T^j x)\right), \tag{1.3}$$

where the supremum is taken over all (n,ε)-separated subsets E of X. A set $E \subseteq X$ is called (n,ε)-separated, if for every $x \neq y \in E$ there exists a $j \in \{0,1,\ldots,n-1\}$ with $d(T^j x, T^j y) > \varepsilon$.

Now we define

$$p(R(Q),T,f) := p(R_{\mathcal{Y}}, T_{\mathcal{Y}}, f_{\mathcal{Y}}). \tag{1.4}$$

Lemma 2 of [**3**] says, that this definition does not depend on the partition $\mathcal{Y}$. Furthermore we define for $n \in \mathbb{N}$

$$S_n(R(Q),f) := \sup_{x\in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}{}^j x). \tag{1.5}$$

Observe that this definition does not depend on the partition $\mathcal{Y}$. Note that these definitions are a bit different from those in [**4**].

Now we define the Hausdorff dimension. For an $A \subseteq \mathbb{R}$, $A \neq \emptyset$ define $\operatorname{diam} A := \sup_{x,y\in A} |x-y|$. Let $Y \subseteq \mathbb{R}$. For $t \geq 0$ and $\varepsilon > 0$ set

$$m(Y,t,\varepsilon) := \inf\Big\{ \sum_{A\in\mathcal{A}} (\operatorname{diam} A)^t \;:\; \mathcal{A} \text{ is an at most countable cover of } Y \text{ with } \operatorname{diam} A < \varepsilon \text{ for all } A \in \mathcal{A} \Big\}.$$

Then define the **Hausdorff dimension** $\operatorname{HD}(Y)$ of Y by

$$\operatorname{HD}(Y) := \inf\left\{ t \geq 0 : \lim_{\varepsilon\to 0} m(Y,t,\varepsilon) = 0 \right\}. \tag{1.6}$$

In [**3**] this definition is slightly modified, which allows to define the Hausdorff dimension also on $X_{\mathcal{Y}}$ — the space, where the completion $T_{\mathcal{Y}}$ of a piecewise monotonic map T acts — in a way, such that $\mathrm{HD}\,(R_{\mathcal{Y}}) = \mathrm{HD}\,\big(R(Q)\big)$. At this point we remark, that all results of this paper hold also in the situation considered in [**3**], where a bit more general situation is treated.

Now we shall define an at most countable oriented graph $(\mathcal{D}, \to)$, called Markov diagram, which describes the orbit structure of $(R(Q), T)$ (cf. [**1**], [**2**]). As we shall need also a description of the Markov diagram in a different way, we shall introduce also versions $(\mathcal{A}, \to)$ of the Markov diagram, which are similar to the variants of the Markov diagram introduced in [**4**].

Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, let $Q \in \mathcal{Q}_K$, and let $\mathcal{Y}$ be a finite partition of $X(Q)$, which refines $\mathcal{Z}(Q)$. Let $T_{\mathcal{Y}}$ be the completion of T with respect to $\mathcal{Y}$. Let I be the set of all isolated points of $X(Q)$, and $I_{\mathcal{Y}}$ be the set of all isolated points of $X_{\mathcal{Y}}(Q)$. Note that $I \subseteq \pi(I_{\mathcal{Y}})$, but equality is not true in general. Then there exist $b_1, b_2, \dots, b_{2N} \in X_{\mathcal{Y}}(Q)$ with $b_1 < b_2 < \cdots < b_{2N}$, such that $\hat{\mathcal{Y}} = \big\{[b_{2j-1}, b_{2j}] : j \in \{1, 2, \dots, N\}\big\}$. Furthermore there exist a $J \in \mathbb{N}$ and $b_{2N+1}, b_{2N+2}, \dots, b_J \in X_{\mathcal{Y}}(Q)$ with $b_{2N+1} < b_{2N+2} < \cdots < b_J$, such that $I_{\mathcal{Y}} = \{b_{2N+1}, b_{2N+2}, \dots, b_J\}$. Set $I_0 := \{b_1, b_2, \dots, b_J\}$. Let $Y_0 \in \hat{\mathcal{Y}}$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}(Q)$ is called **successor** of D, if there exists a $Y \in \hat{\mathcal{Y}}$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. We get that every successor C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the **Markov diagram** of T with respect to $\mathcal{Y}$. The set $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{D \in \mathcal{D} : \exists C \in \mathcal{D}_{r-1} \text{ with } C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \cup_{r=0}^{\infty} \mathcal{D}_r$.

If $C \in \mathcal{D}$ and $x \in I_0$, then we introduce an arrow $C \to \{x\}$, if and only if $x \in T_{\mathcal{Y}} C$. Let $x \in I_0$. Then we set $j(x) := \min\{j \in \mathbb{N} : T_{\mathcal{Y}}{}^j x \notin X_{\mathcal{Y}}(Q)\}$, where we set $j(x) := \infty$, if $T_{\mathcal{Y}}{}^j x \in X_{\mathcal{Y}}(Q)$ for all $j \in \mathbb{N}$. Now define $\mathcal{D}(x) := \big\{\{T_{\mathcal{Y}}{}^j x\} : j \in \mathbb{N}_0, j < j(x)\big\}$, define $\mathcal{D}_r(x) := \big\{\{T_{\mathcal{Y}}{}^j x\} : j \in \mathbb{N}_0, j < \min\{j(x), r+1\}\big\}$ for $r \in \mathbb{N}_0$, and introduce the arrow $\{T_{\mathcal{Y}}{}^{j-1} x\} \to \{T_{\mathcal{Y}}{}^j x\}$, if $\{T_{\mathcal{Y}}{}^j x\} \in \mathcal{D}(x)$ and $j \in \mathbb{N}$ (there are no other arrows beginning in $\{T_{\mathcal{Y}}{}^{j-1} x\}$). If $B \subseteq I_0$, then define $\mathcal{D}(B) := \mathcal{D} \cup \bigcup_{x \in B} \mathcal{D}(x)$, and $\mathcal{D}_r(B) := \mathcal{D}_r \cup \bigcup_{x \in B} \mathcal{D}_r(x)$ for $r \in \mathbb{N}_0$.

Now define

$$b_{i,j} := T_{\mathcal{Y}}{}^j b_i \quad \text{for } i \in \{1, 2, \dots, J\} \text{ and } j \in \mathbb{N}_0,\ 0 \le j < j(b_i). \tag{1.7}$$

For $i \in \{1, 2, \dots, 2N\}$ set $j(i) := j(b_i)$ and for $i \in \{2N+1, 2N+2, \dots, J+2N\}$ set $j(i) := j(b_{i-2N})$. Now set $\mathcal{M}^* := \{(i,j) : i \in \{1, 2, \dots, J+2N\}, j \in \mathbb{N}_0, 0 \le j < j(i)\}$, and for $r \in \mathbb{N}_0$ define $\mathcal{M}^*{}_r := \{(i,j) \in \mathcal{M}^* : j \le r\}$. Now we define a map $A^*\colon \mathcal{M}^* \to \mathcal{D}(I_0)$ with $A^*(\mathcal{M}^*) = \mathcal{D}(I_0)$ and $A^*(\mathcal{M}^*{}_r) = \mathcal{D}_r(I_0)$ for all

$r \in \mathbb{N}_0$, such that $b_{i,j}$ is an endpoint of $A^*(i,j)$ for all $(i,j) \in \mathcal{M}^*$. This map will be surjective, but need not be injective, that means a $C \in \mathcal{D}$ can be represented by different elements of $\mathcal{M}^*$. Furthermore we define arrows between elements of $\mathcal{M}^*$, such that $c \to d$ in $\mathcal{M}^*$ implies $A^*(c) \to A^*(d)$ in $\mathcal{D}(I_0)$, and for every $c \in \mathcal{M}^*$ the map A^* is bijective from $\{d \in \mathcal{M}^* : c \to d\}$ to $\{D \in \mathcal{D}(I_0) : A^*(c) \to D\}$. Furthermore we shall have, that $c \in \mathcal{M}^*{}_r$ implies the existence of a $d \in \mathcal{M}^*{}_r$ with $A^*(c) \subseteq A^*(d)$ and either $A^*(c) = [b_d, b_c]$ or $A^*(c) = [b_c, b_d]$.

If $(i,j) \in \mathcal{M}^*$ and $i > 2N$, then we define $A^*(i,j) := \{b_{i-2N,j}\}$. For $j \in \{1,2,\dots,N\}$ set $A^*(2j-1,0) := A^*(2j,0) := [b_{2j-1}, b_{2j}]$. Hence we have that $b_{i,0}$ is an endpoint of $A^*(i,0)$ for all $i \in \{1,2,\dots,J+2N\}$. Now suppose that $A^*|\mathcal{M}^*{}_r$ is constructed, and all arrows beginning in $\mathcal{M}^*{}_{r-1}$ are described for an $r \in \mathbb{N}_0$. Let $i \in \{1,2,\dots,J+2N\}$, and suppose that $j(i) \geq r+1$. Then $(i,r) \in \mathcal{M}^*{}_r$ and $A^*(i,r) \in \mathcal{D}_r(I_0)$. We have that $A^*(i,r) \subseteq A^*(u,v)$ and either $A^*(i,r) = [b_{u,v}, b_{i,r}]$ or $A^*(i,r) = [b_{i,r}, b_{u,v}]$ for a $(u,v) \in \mathcal{M}^*{}_r$. First we suppose, that there exists an $s \in \{0,1,\dots,r-1\}$ with $A^*(i,r) = A^*(i,s)$. In this case we introduce an arrow $(i,r) \to d$ if and only if either $d = (i,r+1)$ or $d \neq (i,s+1)$ and $(i,s) \to d$. Furthermore we set $A^*(i,r+1) = A^*(i,s+1)$. Now we consider the case $A^*(i,r) \neq A^*(i,s)$ for all $s \in \{0,1,\dots,r-1\}$. Set $\mathcal{C} := \{C \in \mathcal{D}(I_0)\colon A^*(i,r) \to C, T_{\mathcal{Y}} b_{i,r} \notin C, T_{\mathcal{Y}} b_{u,v} \notin C\}$. For every $C \in \mathcal{C}$ there exists an $i(C) \in \{1,2,\dots,J+2N\}$ with $A^*(i(C),0) = C$. We introduce an arrow $(i,r) \to (i(C),0)$. If $A^*(i,r)$ has a successor C with $T_{\mathcal{Y}} b_{u,v} \in C$ and $T_{\mathcal{Y}} b_{i,r} \notin C$, then we introduce an arrow $(i,r) \to (u,v+1)$, and if there are two successors with this property, then then we introduce also an arrow $(i,r) \to (u+2N, v+1)$. If $j(i) > r+1$, then there exists a successor D of $A^*(i,r)$ with $b_{i,r+1} = T_{\mathcal{Y}} b_{i,r} \in D$ (suppose $\operatorname{card} D > 1$, if there are two successors with this property). We introduce an arrow $(i,r) \to (i,r+1)$ and define $A^*(i,r+1) := D$, and if there are two successors with this property, then we introduce also an arrow $(i,r) \to (i+2N, r+1)$. We have that $b_{i,r+1}$ is an endpoint of $A^*(i,r+1)$. If $T_{\mathcal{Y}} b_{u,v} \in A^*(i,r+1)$, then $A^*(i,r+1) \subseteq A^*(u,v+1)$ and we have either $A^*(i,r+1) = [b_{u,v+1}, b_{i,r+1}]$ or $A^*(i,r+1) = [b_{i,r+1}, b_{u,v+1}]$. Otherwise there exists a $w \in \{1,2,\dots,J+2N\}$ with $A^*(i,r+1) \subseteq A^*(w,0)$, such that either $A^*(i,r+1) = [b_{w,0}, b_{i,r+1}]$ or $A^*(i,r+1) = [b_{i,r+1}, b_{w,0}]$. This finishes the construction of the oriented graph $(\mathcal{M}^*, \to)$ and the function A^*.

Instead of $\mathcal{M}^*$ we consider also sets $\mathcal{M}$ defined as follows. Let $\chi\colon \{1,2,\dots,J+2N\} \to \{1,2,\dots,J+2N\}$ be bijective. Set $\mathcal{M} := \{(i,j) : (\chi(i),j) \in \mathcal{M}^*\}$ and for $(i,j) \in \mathcal{M}$ define $A(i,j) := A^*(\chi(i),j)$. If $(i,j),(u,v) \in \mathcal{M}$, then we introduce an arrow $(i,j) \to (u,v)$ in $\mathcal{M}$, if and only if $(\chi(i),j) \to (\chi(u),v)$ in $\mathcal{M}^*$. For $r \in \mathbb{N}_0$ define $\mathcal{M}_r := \{(i,j) : (\chi(i),j) \in \mathcal{M}^*{}_r\}$.

We call $(\mathcal{A}, \to)$ a **version of the Markov diagram** of T with respect to $\mathcal{Y}$, if there exists a $B \subseteq I_0$, such that $\mathcal{A} \subseteq \mathcal{M}$ satisfies the following properties.

(1) If $i \in \{1,2,\dots,J+2N\}$ and $j \in \mathbb{N}_0$, then $(i,j) \in \mathcal{A}$ implies $(i,l) \in \mathcal{A}$ for $l \in \{0,1,\dots,j\}$.

(2) $c, d \in \mathcal{A}$ and $c \to d$ in $\mathcal{M}$ imply $c \to d$ in $\mathcal{A}$.
(3) $c, d \in \mathcal{A}$ and $c \to d$ in $\mathcal{A}$ imply either $c \to d$ in $\mathcal{M}$ or there exists a $d_0 \in \mathcal{M} \setminus \mathcal{A}$ with $c \to d_0$ in $\mathcal{M}$ and $A(d) = A(d_0)$.
(4) For $c \in \mathcal{A}$ the map $A : \{d \in \mathcal{A} : c \to d\} \to \{D \in \mathcal{D}(B) : A(c) \to D\}$ is bijective.
(5) $A(\mathcal{A} \cap \mathcal{M}_r) = \mathcal{D}_r(B)$ for all $r \in \mathbb{N}_0$.

For $r \in \mathbb{N}_0$ set $\mathcal{A}_r := \mathcal{A} \cap \mathcal{M}_r$. If $I_\mathcal{Y} \subseteq B$, then $(\mathcal{A}, \to)$ is called a **full version of the Markov diagram** of T with respect to $\mathcal{Y}$.

The main difference between these versions of the Markov diagram introduced above and the variants of the Markov diagram introduced in [**4**] is, that the orbits of elements of I_0 can be included in a version (but they cannot be included in a variant). Besides we allow a permutation of the set $\{1, 2, \ldots, J + 2N\}$, which will be useful in the proof of Lemma 2.

Now suppose, that $T : [0,1] \to [0,1]$ is a piecewise monotonic map with respect to $\mathcal{Z}$, that $f : [0,1] \to \mathbb{R}$ is piecewise constant with respect to $\mathcal{Z}$, that $K \in \mathbb{N}$, that $Q \in \mathcal{Q}_K$, and that $\mathcal{Y}$ is a finite partition of $X(Q)$, which refines $\mathcal{Z}(Q)$. Let $(\mathcal{A}, \to)$ be a version of the Markov diagram of T with respect to $\mathcal{Y}$. For $c \in \mathcal{A}$ let f_c be the unique real number with $f_\mathcal{Y}(x) = f_c$ for all $x \in A(c)$. Then we define for $c, d \in \mathcal{A}$

$$(1.8) \qquad F_{c,d}(f) := \begin{cases} e^{f_c} & \text{if } c \to d, \\ 0 & \text{otherwise.} \end{cases}$$

Set $F(f) := \big(F_{c,d}(f)\big)_{c,d \in \mathcal{A}}$, and for $\mathcal{C} \subseteq \mathcal{A}$ set $F_\mathcal{C}(f) := \big(F_{c,d}(f)\big)_{c,d \in \mathcal{C}}$. It is shown in [**3**], that $u \mapsto uF_\mathcal{C}(f)$ is an $\ell^1(\mathcal{C})$-operator and $v \mapsto F_\mathcal{C}(f)v$ is an $\ell^\infty(\mathcal{C})$-operator, where both operators have the same norm $\|F_\mathcal{C}(f)\|$ and the same spectral radius $r\big(F_\mathcal{C}(f)\big)$. Observing that (2.7)–(2.11), Lemma 4 and the remark after Lemma 3 of [**4**] remain true in our situation we get

$$(1.9) \qquad \|F_\mathcal{C}(f)^n\| = \sup_{c \in \mathcal{C}} \sum_{c_0 = c \to c_1 \to \cdots \to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}}$$

for every $n \in \mathbb{N}$ and every $\mathcal{C} \subseteq \mathcal{A}$, where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $c_0 = c$,

$$(1.10) \qquad r\big(F_\mathcal{C}(f)\big) = \lim_{n \to \infty} \|F_\mathcal{C}(f)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|F_\mathcal{C}(f)^n\|^{\frac{1}{n}}$$

for every $\mathcal{C} \subseteq \mathcal{A}$,

$$(1.11) \qquad \|F(f)^n\| = \|F_{\mathcal{A}_n}(f)^n\| = \sup_{c \in \mathcal{A}_0} \sum_{c_0 = c \to c_1 \to \cdots \to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}}$$

for every $n \in \mathbb{N}$, where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{A}$ with $c_0 = c$, and

$$r\big(F(f)\big) = \lim_{n\to\infty} \|F_{\mathcal{A}_n}(f)^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|F_{\mathcal{A}_n}(f)^n\|^{\frac{1}{n}} . \tag{1.12}$$

Furthermore we get using the proof of Lemma 6 in [**3**] that

$$\log r\big(F(f)\big) \le p(R(Q), T, f) . \tag{1.13}$$

If $(\mathcal{A}, \to)$ is a full version of the Markov diagram of T with respect to $\mathcal{Y}$, or if $p(R(Q), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$, then we have

$$\log r\big(F(f)\big) = p(R(Q), T, f). \tag{1.14}$$

Lemma 1. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$, let $f\colon [0,1] \to \mathbb{R}$ be a piecewise constant function with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, and let $Q \in \mathcal{Q}_K$. Suppose that $p(R(Q), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$. Then for every $\varepsilon > 0$ there exists an $r \in \mathbb{N}$, such that for every version $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Z}(Q)$ there exists an irreducible $\mathcal{C} \subseteq \mathcal{A}_r$ with $A(c) \in \mathcal{D}$ for all $c \in \mathcal{C}$, such that $\log r\big(F_{\mathcal{C}}(f)\big) > p(R(Q), T, f) - \varepsilon$.*

Proof. We can suppose that ε is small enough to ensure $p(R(Q), T, f) - \varepsilon > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$. As Lemma 5 of [**4**] remains true in our situation, it implies our result excluding the property $A(c) \in \mathcal{D}$ for all $c \in \mathcal{C}$. Suppose that there exists a $c \in \mathcal{C}$ with $A(c) \notin \mathcal{D}$. Then $A(c) = \{x\}$ for an $x \in X_{\mathcal{Z}(Q)}(Q)$. As every $c \in \mathcal{A}$ with $\operatorname{card} A(c) = 1$ has at most one successor, and as $\mathcal{C}$ is irreducible, we have that every $d \in \mathcal{C}$ has at most one successor and $\operatorname{card} A(d) = 1$. This implies by (1.5), (1.9) and (1.10) that $\log r\big(F_{\mathcal{C}}(f)\big) \le \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$. As $\log r\big(F_{\mathcal{C}}(f)\big) > p(R(Q), T, f) - \varepsilon$, this contradicts $p(R(Q), T, f) - \varepsilon > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$, which finishes the proof. □

2. Continuity of the Markov Diagram

In this section let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to $\mathcal{Z}$. Let $K \in \mathbb{N}$, and let $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$. Suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$, which refines $\mathcal{Z}$, such that $a_j \in \{\inf Y, \sup Y : Y \in \mathcal{Y}\}$ for every $j \in \{1, 2, \ldots, 2K-1, 2K\}$. Let $\mathcal{Y}(Q)$ be the set of all maximal open subintervals of $X(Q) \cup \big(\bigcup_{Y\in\mathcal{Y}} Y\big)$, and let $T_{\mathcal{Y}(Q)}$ be the completion of T with respect to $\mathcal{Y}(Q)$. Throughout this section we shall use the notations T_Q, X_Q, ... instead of $T_{\mathcal{Y}(Q)}$, $X_{\mathcal{Y}(Q)}$, ... As in Section 1 let I be the set of all isolated points in $X(Q)$, I_Q the set of all isolated points in $X_Q(Q)$, and $I_0 := I_Q \cup \{\inf Y, \sup Y : Y \in \hat{\mathcal{Y}}(Q)\}$.

If $\tilde{Q} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{2K-1}, \tilde{a}_{2K}) \in \mathcal{Q}_K$, then denote the completion of T with respect to $\mathcal{Y}(\tilde{Q})$ by $T_{\tilde{Q}}$ ($\mathcal{Y}(\tilde{Q})$ is the set of all maximal open subintervals of $X(\tilde{Q}) \cup (\cup_{Y \in \mathcal{Y}} Y)$). Again we shall use throughout this section the notations $T_{\tilde{Q}}$, $X_{\tilde{Q}}$, ... instead of $T_{\mathcal{Y}(\tilde{Q})}$, $X_{\mathcal{Y}(\tilde{Q})}$, ... Now we shall define a map $Y \colon X_{\tilde{Q}} \to \mathcal{Y}$. To this end we set first $E_1 := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \setminus \{0,1\}$. Let $x \in X_{\tilde{Q}}$. If $\tilde{\pi}(x) \notin E_1$, then there exists a unique $Y \in \mathcal{Y}$ with $\tilde{\pi}(x) \in \overline{Y}$. Set $Y(x) := Y$ in this case. Otherwise we have either $x = \tilde{\pi}(x)^-$ or $x = \tilde{\pi}(x)^+$, and there exist exactly two $Y^-, Y^+ \in \mathcal{Y}$ with $Y^- < Y^+$, such that $\tilde{\pi}(x) \in \overline{Y^-} \cap \overline{Y^+}$. Now set $Y(x) := Y^-$, if $x = \tilde{\pi}(x)^-$, and $Y(x) := Y^+$, if $x = \tilde{\pi}(x)^+$. Observe that this definition contains the definition of $Y \colon X_Q \to \mathcal{Y}$.

The aim of this section is to show, that the Markov diagrams of T with respect to $\mathcal{Y}(Q)$, resp. $\mathcal{Y}(\tilde{Q})$ have similar initial parts, if Q and $\tilde{Q}$ are sufficiently close. The method of the proof of this result is the same as in the proof of Lemma 6 in [**4**], but the details are different. As the proof is very technical we omit it.

Lemma 2. *Let $T \colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$, let $K \in \mathbb{N}$, and let $Q = (a_1, a_2, \dots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$. Suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$, which refines $\mathcal{Z}$, such that $a_j \in \{\inf Y, \sup Y : Y \in \mathcal{Y}\}$ for every $j \in \{1, 2, \dots, 2K-1, 2K\}$. Then for every $r \in \mathbb{N}$ there exists a $\delta > 0$, such that for every $\tilde{Q} \in \mathcal{Q}_K$, which is δ-close to Q, there exists a version $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}(Q)$, and a full version $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}(\tilde{Q})$ with the following properties.*

1. *There exists a function $\varphi \colon \tilde{\mathcal{A}}_r \to \mathcal{A}_r$, such that $\varphi(\tilde{\mathcal{A}}_0) = \mathcal{A}_0$, and $\operatorname{card} \varphi^{-1}(c) \le 2$ for every $c \in \mathcal{A}_r$. If $c \in \mathcal{A}_r$ and either $\operatorname{card} \varphi^{-1}(c) > 1$ or $c \notin \varphi(\tilde{\mathcal{A}}_r)$, then $A(c) = \{x\}$ for an $x \in X_Q(Q)$.*
2. *For $c, d \in \tilde{\mathcal{A}}_r$ with $A(\varphi(c)) \in \mathcal{D}$ the property $c \to d$ in $\tilde{\mathcal{A}}$ implies $\varphi(c) \to \varphi(d)$ in $\mathcal{A}$. Furthermore $c, d \in \tilde{\mathcal{A}}_r$, $\varphi(c) \to \varphi(d)$ in $\mathcal{A}$ and d is not a successor of c in $\tilde{\mathcal{A}}$ imply that $A(\varphi(d)) = \{x\}$, where x is contained in $\{T_Q \inf A(\varphi(c)), T_Q \sup A(\varphi(c))\}$. If $c, d \in \tilde{\mathcal{A}}_r$, $c \to d$ in $\tilde{\mathcal{A}}$, and $\varphi(d)$ is not a successor of $\varphi(c)$ in $\mathcal{A}$, then there exist $c_1, d_1 \in \tilde{\mathcal{A}}_r$ with $c_1 \to d_1$ in $\tilde{\mathcal{A}}$, $\varphi(c_1) = \varphi(c)$, $\varphi(c) \to \varphi(d_1)$ in $\mathcal{A}$, and $A(\varphi(d_1)) = A(\varphi(d))$.*
3. *If $c \in \tilde{\mathcal{A}}_r$ and $Y \in \mathcal{Y}$ satisfy $Y(x) = Y$ for all $x \in A(\varphi(c))$, then $Y(x) = Y$ for all $x \in \tilde{A}(c)$.*
4. *If $c \in \tilde{\mathcal{A}}_0$, and $d_0 = \varphi(c) \to d_1 \to \cdots \to d_r$ is a path of length r in $\mathcal{A}$, then there exist at most $r+1$ different paths $c_0 = c \to c_1 \to \cdots \to c_r$ in $\tilde{\mathcal{A}}$ with $A(\varphi(c_j)) = A(d_j)$ for $j \in \{1, 2, \dots, r\}$.*

3. Continuity of the Pressure and the Hausdorff Dimension

In this section we shall use the results of Section 2 to prove continuity results about the pressure and the Hausdorff dimension.

Theorem 1. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$, let $f\colon [0,1] \to \mathbb{R}$ be piecewise continuous with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, and let $Q \in \mathcal{Q}_K$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $\tilde{Q} \in \mathcal{Q}_K$ is δ-close to Q implies*

$$p(R(\tilde{Q}), T, f) < p(R(Q), T, f) + \varepsilon\ .$$

Furthermore, if $p(R(Q), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$, then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $\tilde{Q} \in \mathcal{Q}_K$ is δ-close to Q implies

$$|p(R(\tilde{Q}), T, f) - p(R(Q), T, f)| < \varepsilon\ .$$

Proof. Suppose that $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$. Let $\varepsilon > 0$. By the piecewise continuity of f there exists a finite partition $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$ of $[0,1]$ refining $\mathcal{Z}$, such that $a_j \in \{\inf Y, \sup Y : Y \in \mathcal{Y}\}$ for every $j \in \{1, 2, \ldots, 2K\}$ and $\sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f(x) - f(y)| < \frac{\varepsilon}{2}$. If $x \in Y$ for a $Y \in \mathcal{Y}$, then define $f_1(x) := \inf_{y\in Y} f(y)$. Then $f_1\colon [0,1] \to \mathbb{R}$ is a piecewise constant function with respect to $\mathcal{Y}$, and we have for every $x \in [0,1]$

$$f(x) - \frac{\varepsilon}{2} \le f_1(x) \le f(x)\ . \tag{3.1}$$

This implies for every $\tilde{Q} \in \mathcal{Q}_K$

$$p(R(\tilde{Q}), T, f) - \frac{\varepsilon}{2} \le p(R(\tilde{Q}), T, f_1) \le p(R(\tilde{Q}), T, f). \tag{3.2}$$

We show at first, that there exists a $\delta > 0$, such that $p(R(\tilde{Q}), T, f) < p(R(Q), T, f) + \varepsilon$, if $\tilde{Q} \in \mathcal{Q}_K$ is δ-close to Q.

Set $R := \exp\left(p(R(Q), T, f) + \frac{\varepsilon}{2}\right)$. By (1.13) and (3.2) we get $r\big(F_{\mathcal{A}}(f_1)\big) < R$ for every version $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}(Q)$. As $\lim_{r\to\infty} \sqrt[r]{r+1} = 1$ we get using (1.10) that there exists an $r \in \mathbb{N}$ with

$$(r+1)\|F_{\mathcal{A}}(f_1)^r\| < R^r \tag{3.3}$$

for every version $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}(Q)$. We fix this r for the rest of this part of this proof.

By Lemma 2 there exists a $\delta > 0$, such that the conclusions of Lemma 2 are true for every $\tilde{Q} \in \mathcal{Q}_K$, which is δ-close to Q.

Let $\tilde{Q} \in \mathcal{Q}_K$ be δ-close to Q, and suppose that $(\mathcal{A}, \to)$, resp. $(\tilde{\mathcal{A}}, \to)$ are the versions of the Markov diagrams of T with respect to $\mathcal{Y}(Q)$, resp. $\mathcal{Y}(\tilde{Q})$ occuring in the conclusion of Lemma 2. For $c \in \mathcal{A}$ let f_c be the unique real number with $f_1(x) = f_c$ for all $x \in A(c)$, and for $c \in \tilde{\mathcal{A}}$ let $\tilde{f}_c$ be the unique real number with $f_1(x) = \tilde{f}_c$ for all $x \in \tilde{A}(c)$. Set $F(f_1) := \big(F_{c,d}(f_1)\big)_{c,d\in\mathcal{A}}$ and $\tilde{F}(f_1) :=$

$\big(\tilde{F}_{c,d}(f_1)\big)_{c,d\in\tilde{\mathcal{A}}}$. Let $\varphi\colon \tilde{\mathcal{A}}_r \to \mathcal{A}_r$ be the function occurring in the conclusion of Lemma 2. By (1.11) and (1.12) we get

$$r\big(\tilde{F}(f_1)\big)^r \le \sup_{c\in\tilde{\mathcal{A}}_0} \sum_{c_0=c\to c_1\to\cdots\to c_r} \prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}} , \tag{3.4}$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\tilde{\mathcal{A}}$ with $c_0 = c$. As $c \in \tilde{\mathcal{A}}_0$ we have $c_0, c_1, \dots, c_r \in \tilde{\mathcal{A}}_r$.

Fix $c \in \tilde{\mathcal{A}}_0$. If $c_0 \to c_1 \to \cdots \to c_r$ is a path of length r in $\tilde{\mathcal{A}}$ with $c_0 = c$, then (1) and (2) of Lemma 2 gives that there exists a path $d_0 \to d_1 \to \cdots \to d_r$ of length r in $\mathcal{A}_r$ with $d_0 = \varphi(c_0) \in \mathcal{A}_0$ and $A(d_j) = A\big(\varphi(c_j)\big)$ for all $j \in \{0, 1, \dots, r\}$. Set $\chi(c_0 \to c_1 \to \cdots \to c_r) := d_0 \to d_1 \to \cdots \to d_r$. By (3) of Lemma 2 we get $\prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}} = \prod_{j=0}^{r-1} e^{f_{d_j}}$. Furthermore (4) of Lemma 2 gives that for a fixed $d_0 \to d_1 \to \cdots \to d_r$ there are at most $r+1$ different paths $c_0 \to c_1 \to \cdots \to c_r$ in $\tilde{\mathcal{A}}$ with $c_0 = c$ and $\chi(c_0 \to c_1 \to \cdots \to c_r) := d_0 \to d_1 \to \cdots \to d_r$. This implies

$$\sum_{c_0=c\to c_1\to\cdots\to c_r} \prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}} \le (r+1) \sum_{d_0=\varphi(c)\to d_1\to\cdots\to d_r} \prod_{j=0}^{r-1} e^{f_{d_j}} , \tag{3.5}$$

where the first sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\tilde{\mathcal{A}}$ with $c_0 = c$, and the second over all paths $d_0 \to d_1 \to \cdots \to d_r$ of length r in $\mathcal{A}$ with $d_0 = \varphi(c)$.

Now (1.11), (3.3), (3.4) and (3.5) imply $r\big(\tilde{F}(f_1)\big)^r \le (r+1)\|F(f_1)^r\| < R^r$. Hence $r\big(\tilde{F}(f_1)\big) < R$. As $(\tilde{\mathcal{A}}, \to)$ is a full version of the Markov diagram of T with respect to $\mathcal{Y}(\tilde{Q})$, (1.14) gives $p(R(\tilde{Q}), T, f_1) = \log r\big(\tilde{F}(f_1)\big)$. Now (3.2) and the definition of R imply $p(R(\tilde{Q}), T, f) \le p(R(\tilde{Q}), T, f_1) + \frac{\varepsilon}{2} = \log r\big(\tilde{F}(f_1)\big) + \frac{\varepsilon}{2} < \log R + \frac{\varepsilon}{2} = p(R(Q), T, f) + \varepsilon$, which shows the first part of this theorem.

It remains to show that there exists a $\delta > 0$, such that $\tilde{Q} \in \mathcal{Q}_K$ is δ-close to Q implies $p(R(\tilde{Q}), T, f) > p(R(Q), T, f) - \varepsilon$, if $p(R(Q), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$. We can assume that ε is small enough to ensure $p(R(Q), T, f) > \varepsilon + \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$. By (3.1) and (3.2) this implies

$$p(R(Q), T, f_1) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f_1) + \frac{\varepsilon}{2} > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f_1) . \tag{3.6}$$

Using (3.2) we get by Lemma 1 that there exists an $r \in \mathbb{N}$, such that for every version $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}(Q)$ there exists an irreducible $\mathcal{C} \subseteq \mathcal{A}_r$ with $A(c) \in \mathcal{D}$ for all $c \in \mathcal{C}$ and

$$\log r\big(F_{\mathcal{C}}(f_1)\big) > p(R(Q), T, f_1) - \frac{\varepsilon}{2} \ge p(R(Q), T, f) - \varepsilon . \tag{3.7}$$

Fix this r for the rest of this proof.

By Lemma 2 there exists a $\delta > 0$, such that the conclusions of Lemma 2 are true for every $\tilde{Q} \in \mathcal{Q}_K$, which is δ-close to Q.

Let $\tilde{Q} \in \mathcal{Q}_K$ be δ-close to Q, and suppose that $(\mathcal{A}, \to)$, resp. $(\tilde{\mathcal{A}}, \to)$ are the versions of the Markov diagrams of T with respect to $\mathcal{Y}(Q)$, resp. $\mathcal{Y}(\tilde{Q})$ occurring in the conclusion of Lemma 2. Define f_c, $\tilde{f}_c$, $\tilde{F}(f_1)$ and φ analoguous as in the first part of this proof. Let $\mathcal{C} \subseteq \mathcal{A}_r$ be irreducible with $A(c) \in \mathcal{D}$ for all $c \in \mathcal{C}$, such that (3.7) is satisfied. Now (1) and (2) of Lemma 2 imply that $\varphi\colon \tilde{\mathcal{C}} \to \mathcal{C}$ is bijective and satisfies for $c, d \in \tilde{\mathcal{C}}$ that $c \to d$ in $\tilde{\mathcal{A}}$ is equivalent to $\varphi(c) \to \varphi(d)$ in $\mathcal{A}$, where $\tilde{\mathcal{C}} := \varphi^{-1}(\mathcal{C}) \subseteq \tilde{\mathcal{A}}_r$. Using (3) of Lemma 2 we get $\sum_{c_0 = c \to c_1 \to \cdots \to c_n} \prod_{j=0}^{n-1} e^{\tilde{f}_{c_j}} = \sum_{d_0 = \varphi(c) \to d_1 \to \cdots \to d_n} \prod_{j=0}^{n-1} e^{f_{d_j}}$ for every $c \in \tilde{\mathcal{C}}$ and every $n \in \mathbb{N}$, where the first sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\tilde{\mathcal{C}}$ with $c_0 = c$, and the second over all paths $d_0 \to d_1 \to \cdots \to d_n$ of length n in $\mathcal{C}$ with $d_0 = \varphi(c)$. By (1.9) and (1.10) this implies $r\big(F_{\mathcal{C}}(f_1)\big) = r\big(\tilde{F}_{\tilde{\mathcal{C}}}(f_1)\big) \le r\big(\tilde{F}(f_1)\big)$. Hence (1.13), (3.2) and (3.7) give $p(R(\tilde{Q}), T, f) \ge p(R(\tilde{Q}), T, f_1) \ge \log r\big(\tilde{F}(f_1)\big) \ge \log r\big(F_{\mathcal{C}}(f_1)\big) > p(R(Q), T, f) - \varepsilon$, which finishes the proof. $\square$

We give an example, where the pressure is not lower semi-continuous. Let $T\colon [0,1] \to [0,1]$, $\mathcal{Z}$ and $f\colon [0,1] \to \mathbb{R}$ be defined as in (4.4) and (4.5) of [**4**], that means $\mathcal{Z} := \{(0, \frac{1}{6}), (\frac{1}{6}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)\}$,

$$Tx = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{6}], \\ \frac{2}{3} - 2x & \text{for } x \in [\frac{1}{6}, \frac{1}{3}], \\ 2x - \frac{2}{3} & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2 - 2x & \text{for } x \in [\frac{2}{3}, 1], \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{3}], \\ 30x - 10 & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 30 - 30x & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

Set $K := 1$ and set $Q := (\frac{2}{3}, 1) \in \mathcal{Q}_1$ (note that elements of $\mathcal{Q}_1$ are not intervals!). Then we have $R(Q) = [0, \frac{2}{3}] \cup \{1\}$, the nonwandering set of $(R(Q), T)$ is $[0, \frac{1}{3}] \cup \{\frac{2}{3}\}$ and $p(R(Q), T, f) = 10$. The function f is so large at the isolated fixed point $\frac{2}{3}$, such that it dominates the pressure on the rest of the nonwandering set. As we shall see below this fixed point can be destroyed by an arbitrarily small perturbation. The condition $p(R(Q), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f)$ excludes such a phenomenon. For $\varepsilon \in (0, \frac{1}{3})$ define $Q_\varepsilon := (\frac{2}{3} - \varepsilon, 1) \in \mathcal{Q}_1$. Then Q_ε is ε-close to Q. We have $R(Q_\varepsilon) = [0, \frac{2}{3} - \varepsilon] \cup \{1\}$, the nonwandering set of $(R(Q_\varepsilon), T)$ is $[0, \frac{1}{3}]$, and $p(R(Q_\varepsilon), T, f) = \log 2$, which shows that the pressure is not lower semi-continuous in this case.

Now we shall show that the topological entropy is continuous. If we set $f = 0$ in Theorem 1, we get that $|h_{\text{top}}(R(\tilde{Q}), T) - h_{\text{top}}(R(Q), T)| < \varepsilon$ for every $\tilde{Q} \in \mathcal{Q}_K$, which is sufficiently close to Q, if $h_{\text{top}}(R(Q), T) > 0$. If otherwise

$h_{\mathrm{top}}(R(Q),T)=0$, then Theorem 1 gives also $|h_{\mathrm{top}}(R(\tilde{Q}),T)-h_{\mathrm{top}}(R(Q),T)|<\varepsilon$ for every $\tilde{Q}\in\mathcal{Q}_K$, which is sufficiently close to Q, since $h_{\mathrm{top}}(R(\tilde{Q}),T)\geq 0$. Hence we have proved the following result.

Corollary 1.1. *Let $T\colon[0,1]\to[0,1]$ be a piecewise monotonic map, let $K\in\mathbb{N}$, and let $Q\in\mathcal{Q}_K$. Then for every $\varepsilon>0$ there exists a $\delta>0$, such that $\tilde{Q}\in\mathcal{Q}_K$ is δ-close to Q implies*

$$|h_{\mathrm{top}}(R(\tilde{Q}),T)-h_{\mathrm{top}}(R(Q),T)|<\varepsilon\ .$$

Now we shall show that $Q\mapsto\mathrm{HD}\,\big(R(Q)\big)$ is continuous, if T is expanding. To this end we need the following result, which is proved in [**3**] (see also Lemma 7 of [**4**]).

Lemma 3. *Let $T\colon[0,1]\to[0,1]$ be an expanding piecewise monotonic map, let $K\in\mathbb{N}$, and let $Q\in\mathcal{Q}_K$. Then the map $t\mapsto p(R(Q),T,-t\log|T'|)$ defined on $\mathbb{R}$ is continuous and strictly decreasing, has a unique zero t_R, and $\mathrm{HD}\,\big(R(Q)\big)=t_R$.*

Using Lemma 3 and Theorem 1 a proof analoguous to the proof of Theorem 3 in [**4**] shows the continuity of the Hausdorff dimension.

Theorem 2. *Let $T\colon[0,1]\to[0,1]$ be an expanding piecewise monotonic map, let $K\in\mathbb{N}$, and let $Q\in\mathcal{Q}_K$. Then for every $\varepsilon>0$ there exists a $\delta>0$, such that $\tilde{Q}\in\mathcal{Q}_K$ is δ-close to Q implies*

$$\big|\mathrm{HD}\,\big(R(\tilde{Q})\big)-\mathrm{HD}\,\big(R(Q)\big)\big|<\varepsilon\ .$$

Theorem 2 and Corollary 1.1 are generalizations of Theorem 4 in [**6**], where continuity of the topological entropy and the Hausdorff dimension is shown, if T is an expanding C^2-diffeomorphism of the circle. In [**5**] it is shown that $t\mapsto h_{\mathrm{top}}(R_t)$ and $t\mapsto\mathrm{HD}\,(R_t)$ are continuous for expanding C^2-diffeomorphisms of the circle, where $R_t:=\cap_{j=0}^{\infty}\mathbb{T}^1\setminus T^{-j}(0-t,0+t)$ (we assume that 0 is a fixed point of T, and intervals on $\mathbb{T}^1$ are defined in the usual way).

4. The Set of Points, Whose Orbit Is Not Dense

Throughout this section let $T\colon[0,1]\to[0,1]$ be an expanding piecewise monotonic map. We show that for every $t\in[0,1]$ there exists a closed, T-invariant $R_t\subseteq[0,1]$ with $\mathrm{HD}\,(R_t)=t$. Furthermore we show $\mathrm{HD}\,\big(\{x\in[0,1]:\omega(x)\neq[0,1]\}\big)=1$.

As T is expanding it follows from [**1**], that T has periodic points. Hence fix an $x_0\in[0,1]$ and an $n\in\mathbb{N}$ with $T^nx_0=x_0$. Set $K:=\mathrm{card}\,\big(\{x_0,Tx_0,T^2x_0,\dots,T^{n-1}x_0\}\cup\{0,1\}\big)-1$, and choose $c_0<c_1<\dots<c_K$, such that $\{c_0,c_1,\dots,c_K\}=$

$\{x_0, Tx_0, \dots, T^{n-1}x_0\} \cup \{0,1\}$. For every $j \in \{1,2,\dots,K\}$ we choose a $b_j \in (c_{j-1}, c_j)$. Let $s \in [0,1]$. Define for $j \in \{1,2,\dots,K\}$

$$\begin{aligned} a_{2j-1}(s) &:= \max\{c_{j-1}, b_j - s\}\,, \\ a_{2j}(s) &:= \min\{c_j, b_j + s\}\,, \end{aligned} \tag{4.1}$$

and set

$$Q_s := \big(a_1(s), a_2(s), \dots, a_{2K-1}(s), a_{2K}(s)\big)\,. \tag{4.2}$$

Then $Q_s \in \mathcal{Q}_K$ and $\{x_0, Tx_0, \dots, T^{n-1}x_0\} \in R(Q_s)$ for every $s \in [0,1]$. If $s_1, s_2 \in [0,1]$ and $|s_1 - s_2| < \varepsilon$, then Q_{s_2} is ε-close to Q_{s_1}. Furthermore we have $R(Q_0) = [0,1]$ and $\{x_0, Tx_0, \dots, T^{n-1}x_0\} \subseteq R(Q_1) \subseteq \{x_0, Tx_0, \dots, T^{n-1}x_0\} \cup \{0,1\}$. Hence $\mathrm{HD}\,\big(R(Q_0)\big) = 1$ and $\mathrm{HD}\,\big(R(Q_1)\big) = 0$. Therefore we have the following result.

Lemma 4. *The function $s \mapsto \mathrm{HD}\,\big(R(Q_s)\big)$ defined on $[0,1]$ is continuous and decreasing, and satisfies $\mathrm{HD}\,\big(R(Q_0)\big) = 1$ and $\mathrm{HD}\,\big(R(Q_1)\big) = 0$.*

Now we can prove the following result.

Theorem 3. *Let $T\colon [0,1] \to [0,1]$ be an expanding piecewise monotonic map. Then for every $t \in [0,1]$ there exists a closed, T-invariant $R_t \subseteq [0,1]$ with $\mathrm{HD}\,(R_t) = t$.*

Proof. This is an easy consequence of Lemma 4 and the intermediate value theorem. $\square$

Remarks. (1) Using the results of [**1**] we can show that for every $t \in [0,1]$ there exists a topologically transitive, closed, T-invariant $R_t \subseteq [0,1]$ with $R_t = \bigcap_{j=0}^{\infty} \overline{F_t \setminus T^{-j} G_t}$, where F_t and G_t are finite unions of (not necessarily open) intervals, such that $\mathrm{HD}\,(R_t) = t$.

(2) If X is a finite union of closed intervals, $T\colon X \to \mathbb{R}$ is piecewise monotone with respect to $\mathcal{Z}$, such that $(T, \mathcal{Z})$ is of class E^1 as defined in [**4**], and $R(T)$ is defined as in [**4**], then for every $t \in \big[0, \mathrm{HD}\,\big(R(T)\big)\big]$ there exists a closed, T-invariant $R_t \subseteq R(T)$, which can be chosen as in (1), with $\mathrm{HD}\,(R_t) = t$.

Now we shall show that $\mathrm{HD}\,\big(\{x \in [0,1] : \omega(x) \neq [0,1]\}\big) = 1$. Observe that for $x \in [0,1]$ the condition $\omega(x) \neq [0,1]$ is equivalent to the condition that the orbit of x is not dense.

Theorem 4. *Let $T\colon [0,1] \to [0,1]$ be an expanding piecewise monotonic map. Then $\mathrm{HD}\,\big(\{x \in [0,1] : \omega(x) \neq [0,1]\}\big) = 1$.*

Proof. If $s > 0$, then $\emptyset \neq \big(a_1(s), a_2(s)\big) \subseteq [0,1] \setminus R(Q_s)$. Hence $R(Q_s) \subseteq \{x \in [0,1] : \omega(x) \neq [0,1]\}$ for every $s > 0$. By Lemma 4 there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $(0,1]$ with $\lim_{n\to\infty} \mathrm{HD}\,\big(R(Q_{s_n})\big) = 1$. Set $R := \cup_{n\in\mathbb{N}} R(Q_{s_n})$. Then

$R \subseteq \{x \in [0,1] : \omega(x) \neq [0,1]\}$ and $\mathrm{HD}(R) = \sup_{n\in\mathbb{N}} \mathrm{HD}\big(R(Q_{s_n})\big) = 1$, which implies the desired result. $\square$

Remark. If X is a finite union of closed intervals, $T\colon X \to \mathbb{R}$ is piecewise monotone with respect to $\mathcal{Z}$, such that $(T, \mathcal{Z})$ is of class E^1 as defined in [**4**], and $R(T)$ is defined as in [**4**], then $\mathrm{HD}\big(\{x \in R(T) : \omega(x) \neq R(T)\}\big) = \mathrm{HD}\big(R(T)\big)$.

If T is an expanding C^2-diffeomorphism of the circle, then Theorem 3 and Theorem 4 can be easily deduced from [**5**].

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