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A NOTE ON CONGRUENCE LATTICES OF DISTRIBUTIVE p–ALGEBRAS

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1. Introduction

T. Katriňák [**4**] (see also [**5**]) has characterized the congruence lattices of distributive p-algebras within the class of algebraic lattices using his triple construction of distributive p-algebras. In this note we give a short, virtually self-contained proof of his result based on some fundamental properties of principal congruences of distributive lattices and p-algebras.

2. Preliminaries

A (**distributive**) p-**algebra** is an algebra $\langle L; \vee, \wedge, *, 0, 1\rangle$ whose reduct $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and whose unary operation $*$ is characterized by $x \le a^*$ if and only if $a \wedge x = 0$. If L is a p-algebra, $B(L) = \{\, x \in L : x = x^{**} \,\}$ and $D^*(L) = \{\, x \in L : x^{**} = 1 \,\}$ then $\langle B(L); \cup, \wedge, 0, 1\rangle$ is a Boolean algebra when $a \cup b$ is defined to be $(a^* \wedge b^*)^*$, for any $a, b \in B(L)$, $D^*(L) = \{\, x \vee x^* : x \in L \,\}$ and is a filter of L.

A (**distributive**) **dual** p-**algebra** is an algebra $\langle L; \vee, \wedge, +, 0, 1\rangle$ whose reduct $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and whose unary operation $+$ is characterized by $x \ge a^+$ if and only if $a \vee x = 1$. In such an algebra, $D^+(L) = \{x \in L : x^{++} = 0\}$ is an ideal of L. A distributive p-algebra (dual p-algebra) L is said to be **of order** 3 if and only if $D^*(L)$ $(D^+(L))$ is relatively complemented. By a **congruence relation** of a p-algebra L we mean a lattice congruence θ of L preserving $*$ and, for $a \in L$, we denote $\{x \in L : x \equiv a(\theta)\}$ by $[a]\theta$. The relation φ defined on L by $(a, b) \in \varphi$ if and only if $a^* = b^*$ is a congruence called the **Glivenko congruence** of L, $L/\varphi \cong B(L)$ and $[1]\varphi = D(L)$. $\theta(a, b)(\theta_{\mathrm{lat}}(a, b))$ will denote the principal congruence of L (of the lattice reduct of L) collapsing a pair $a, b \in L$ and, for any filter F of L, $\Theta(F)$ $(\Theta_{\mathrm{lat}}(F))$ will denote the smallest congruence of L (of the lattice reduct of L) collapsing F. The congruence lattice of L will be denoted $\mathrm{Con}\,(L)$: it is distributive and algebraic and its join subsemilattice of compact elements will be denoted $\mathrm{Comp}\,(\mathrm{Con}\,(L))$.

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For all unexplained terminology and notation we refer the reader to [**1**] or [**3**].

3. The Theorem

The following well known description of principal congruences of distributive lattices is crucial to our proof of Katriňák's theorem.

For a distributive lattice L and $a, b \in L$ with $a \leq b$,

$$\theta_{\mathrm{lat}}(a,b) = \{(x,y) \in L^2 : x \wedge a = y \wedge a \text{ and } x \vee b = y \vee b\}.$$

Application of this result yields the **principal intersection formula**:

$$\theta_{\mathrm{lat}}(a,b) \wedge \theta_{\mathrm{lat}}(c,d) = \theta_{\mathrm{lat}}((a \vee c) \wedge (b \wedge d), b \wedge d),$$

which holds for any $a, b, c, d \in L$ with $a \leq b$ and $c \leq d$. Some fundamental properties of principal congruences of p-algebras which will be needed are contained in the following.

Lemma. *Let L be a p-algebra.*
(1) *A congruence of L is principal if and only if it is of the form $\theta(a,1) \vee \theta(c,d)$, for some $a \in B(L)$ and $c, d \in L$ with $c \leq d$ and $c^* = d^*$.*
(2) *If $\theta, \psi \in \mathrm{Con}\,(L)$ and $\psi \leq \varphi$ then*

$$\theta \vee \psi = \imath \Longleftrightarrow \theta = \imath\,.$$

(3) *For any $c, d \in L$ satisfying $c \leq d$ and $c^* = d^*$, $\theta(c,d) = \theta_{\mathrm{lat}}(c,d)$ and, in the event that L is distributive and $a \in L$, $\theta(c,d) = \theta(c \vee c^*, d \vee c^*)$ and $\theta(a,1) = \theta_{\mathrm{lat}}(a,1)$.*

Proof. Part (1) is proved in [**2**] (see also [**5**]). Part (2) follows from the observation that if there is a sequence $0 = x_0, x_1, \ldots, x_n = 1$ in L with $x_{i-1} \equiv x_i(\theta \cup \psi)$ for all $i \in \{1, \ldots, n\}$ then $0 \equiv 1(\theta)$ is witnessed by the sequence $0 = x_0^{**}, x_1^{**}, \ldots, x_n^{**} = 1$ since, for any $k \in \{1, \ldots, n\}$ with $x_{k-1} \equiv x_k(\psi)$, $x_{k-1}^{**} \equiv x_k^{**}(\varphi)$ and therefore $x_{k-1}^{**} = x_k^{**}$. The very last part of (3) is a consequence of the fact that $\Theta([a)) = \Theta_{\mathrm{lat}}([a))$ holds for any a in any distributive p-algebra (see [**1**]) and the remainer is proved in [**2**]. □

We are now ready to give our proof.

Theorem (T. Katriňák). *An algebraic lattice is the congruence lattice of a distributive p-algebra if and only if its join subsemilattice of compact elements is a distributive dual p-algebra of order 3.*

Proof. Let L be a distributive p-algebra. Henceforth, let us write $\mathscr{D} = \{\theta_{\mathrm{lat}}(c,d) : d \geq c \in D^*(L)\}$. The join subsemilattice $K = \mathrm{Comp}\,(\mathrm{Con}\,(L))$

of $\mathrm{Con}\,(L)$ is closed under meets. Indeed, K, being the set of finite joins of principal congruences of L, consists of all congruences of the form $\theta_{\mathrm{lat}}(a,1) \vee \bigvee \mathscr{E}$, where $a \in B(L)$ and $\mathscr{E}$ is a finite subset of $\mathscr{D}$, by parts (1) and (3) of the lemma in conjunction with the fact that $\theta(a,1) \vee \theta(b,1) = \theta(a \wedge b, 1)$, for any $a, b \in L$. By the distributivity of $\mathrm{Con}\,(L)$, the meet of two members of K is a finite join of congruences: one being of the form $\theta_{\mathrm{lat}}(a,1) \wedge \theta_{\mathrm{lat}}(b,1)$ while the rest are of the form $\theta_{\mathrm{lat}}(a,1) \wedge \theta$ or $\theta \wedge \psi$, where $a, b \in B(L)$ and $\theta, \psi \in \mathscr{D}$. However, $\theta_{\mathrm{lat}}(a,1) \wedge \theta_{\mathrm{lat}}(b,1) = \theta_{\mathrm{lat}}(a \vee b, 1)$ and the remaining congruences in question belong to $\mathscr{D}$; by the principal intersection formula, the fact that congruences of the second and third type are below φ, and part (3) of the lemma. Thus, K is a sublattice of $\mathrm{Con}\,(L)$. Next, for convenience sake, we imitate Katriňák's proof in [**5**] of the fact that K is dually pseudocomplemented. To this end, let θ, $\psi \in K$. Then there exist $a, b \in B(L)$ and finite subsets $\mathscr{E}$, $\mathscr{F}$ of $\mathscr{D}$ such that

$$\theta = \theta_{\mathrm{lat}}(a,1) \vee \mathscr{E} \quad \text{and} \quad \psi = \theta_{\mathrm{lat}}(b,1) \vee \mathscr{F}.$$

Now, if $\theta \vee \psi = \imath$ then $\theta_{\mathrm{lat}}(a \wedge b, 1) = \imath$, by part (2) of the lemma, so that $a \wedge b = 0$ and therefore $b \leq a^*$. Consequently, $\psi \geq \theta_{\mathrm{lat}}(b,1) \geq \theta_{\mathrm{lat}}(a^*,1)$. Furthermore,

$$\theta \vee \theta_{\mathrm{lat}}(a^*,1) \geq \theta_{\mathrm{lat}}(a,1) \vee \theta_{\mathrm{lat}}(a^*,1) = \theta_{\mathrm{lat}}(a \wedge a^*, 1) = \theta_{\mathrm{lat}}(0,1) = \imath.$$

Thus, θ^+ exists in K and is $\theta_{\mathrm{lat}}(a^*,1)$. It is now easy to show that $D^+(K)$ consists of the joins of all finite subsets of $\mathscr{D}$. To show that $D^+(K)$ is relatively complemented it is enough to show that every interval of the form $[\omega, \theta]$ in $D^+(K)$ is Boolean and for this it suffices to show that any $\theta_{\mathrm{lat}}(c,d) \leq \theta$, with $d \geq c \in D^*(L)$, has a complement in the interval $[\omega, \theta]$ of $D^+(K)$. We note that $\theta'(c,d) = \theta_{\mathrm{lat}}(0,c) \vee \theta_{\mathrm{lat}}(d,1)$ is the complement of $\theta_{\mathrm{lat}}(c,d)$ in the congruence lattice of the lattice reduct of L and claim that $\overline{\theta}(c,d) = \theta'(c,d) \wedge \theta$ is the complement of $\theta_{\mathrm{lat}}(c,d)$ in $[\omega, \theta]$. Obviously we need only show that $\overline{\theta}(c,d) \in D^+(K)$. Recall that $\theta \in D^+(K)$ and so is the join of a finite subset of $\mathscr{D}$. Therefore $\overline{\theta}(c,d)$ is the join of a finite subset of the union $\mathscr{C}$ of

$$\{\theta_{\mathrm{lat}}(0,c) \wedge \theta : c \in D^*(L), \theta \in \mathscr{D}\} \quad \text{and} \quad \{\theta_{\mathrm{lat}}(d,1) \wedge \theta : d \in D^*(L), \theta \in \mathscr{D}\}.$$

However, the members of $\mathscr{C}$ are below φ and so the principal congruence formula in conjunction with part (3) of the lemma shows that $\mathscr{C} = \mathscr{D}$. Thus, $\overline{\theta}(c,d) \in D^+(K)$.

Conversely, let us suppose that A is an algebraic lattice whose compact elements form a distributive dual p-algebra K of order 3. Then the dual of the lattice reduct of K, construed as a distributive p-algebra L, is of order 3. We show that $A \cong \mathrm{Con}\,(L)$ for which it suffices to show that $K \cong \mathrm{Comp}\,(\mathrm{Con}\,(L))$. Observe that if $d \geq c \in D^*(L)$ then there exists $e \in D^*(L)$ such that the interval $[c,d]$ of L transposes up to $[e,1]$, since $D^*(L)$ is relatively complemented, and so $\theta(c,d) = \theta(e,1)$. Therefore $\mathrm{Comp}\,(\mathrm{Con}\,(L)) = \{\theta(x,1) : x \in L\}$, by part (1) of the lemma.

Finally, note that, for $k, \ell \in L$, $k \leq \ell \Leftrightarrow \theta(\ell, 1) \leq \theta(k, 1)$. Indeed, if $\theta(\ell, 1) \leq \theta(k, 1)$ then $[\ell) = [1]\theta_{\mathrm{lat}}(\ell, 1) = [1]\theta(\ell, 1) \subseteq [1]\theta(k, 1) = [1]\theta_{\mathrm{lat}}(k, 1) = [k)$, by part (3) of the lemma, and so $k \leq \ell$. Thus, $A \cong \mathrm{Con}\,(L)$. □

Concluding remarks.

It is known that distributive p-algebras of order 3 are, in fact, Heyting algebras (see [**4**]). Furthermore, other characterizations of distributive p-algebras (dual p-algebras) of order 3 are known. Indeed, for a distributive p-algebra L, the following are equivalent. (1) L is of order 3, (2) L is congruence permutable, (3) there is no 3-element chain in the poset of prime ideals of L, (4) given any $x, y \in L$ with $x \leq y$, there exist $x', y' \in L$ such that $0 = x \wedge x'$, $x \vee x' = y \wedge y'$ and $y \vee y' = 1$. For these and related results the reader is referred to [**2**] and the references therein.

References

1. Balbes R. and Dwinger P., *Distributive Lattices*, University of Missouri Press, Columbia, Missouri, 1974.

2. Beazer R., *Principal congruence properties of some algebras with pseudocomplementation*, Portugal. Math. **50** (1993), 75–86.

3. Grätzer G., *General Lattice Theory*, Birkhäuser Verlag, Basel and Stuttgart, 1978.

4. Katriňák T., *Congruence lattices of distributive p-algebras*, Algebra Univ. **7** (1977), 265–271.

5. ———, *Congruence lattices of finite p-algebras*, preprint.

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