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NONASSOCIATIVE ALGEBRAS WITH SUBMULTIPLICATIVE BILINEAR FORM

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ABSTRACT. In this paper algebras with identity, over ordered fields, which possess a (weak) submultiplicative positive bilinear form are studied. They are called (weak) subdecomposition algebras. It is proved that every weak subdecomposition algebra is a simple quadratic algebra with no nontrivial nilpotents or idempotents. If a weak subdecomposition algebra is also flexible, then it is a noncommutative Jordan algebra and an algebraic involution can be defined in a natural way. Every subdecomposition algebra is automatically flexible. There exist simple examples of weak subdecomposition algebras which are not flexible. Alternative weak subdecomposition algebras can be completely classified. Some structure theorems for weak subdecomposition algebras with large nucleus are also given.

INTRODUCTION

In this paper we are concerned with the structure of nonassociative algebras with identity over an ordered field, which possess a submultiplicative positive nondegenerate bilinear form. This algebras are generalization of well-known composition algebras, i.e. algebras in which the bilinear form is multiplicative. Therefore we shall call them subdecomposition algebras.

There exist many examples of weak subdecomposition algebras and it is probably very difficult to classify all of them but many nice results can be obtained even in the general case. Alternative weak subdecomposition algebras can be completely classified. This is described in Section 2.

There are three groups of papers which motivated our study of subdecomposition algebras.

The first group consists of papers [**2**], [**8**], [**9**] and [**13**]. The results of [**8**], [**9**] and [**13**] can be summarized in the following

Theorem A. *Let $\mathcal{A}$ be a real Hilbert space which is also an associative algebra with identity e. Suppose that $\|e\| = 1$ and $\|xy\| \le \|x\| \cdot \|y\|$. Then $\mathcal{A}$ is isomorphic to the field of real numbers, complex numbers or quaternions.*

In [**2**] one of the results (Theoreme 4) goes as follows:

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Theorem B. *Let $\mathcal{A}$ be a complex Jordan H^*-algebra (see the paragraph before Corollary 1 for definition) with the identity e such that $\|e\| = 1$ and $\|xy\| \leq \|x\| \cdot \|y\|$. Then $\mathcal{A}$ is isomorphic to the field of complex numbers.*

Proofs and methods in these papers belong to the analysis. We obtain better results (even in the case of real algebras) with entirely algebraic methods.

The second group consists of papers [**1**], [**5**] and [**6**] which concern the Cayley-Dickson process. Let $\mathcal{A}$ be some algebra with involution $*$ over a field $\mathcal{F}$. Take a nonzero $\gamma \in \mathcal{F}$ and a symbol $i \notin \mathcal{F}$. We shall denote by $\mathcal{A}_{CD}(\gamma)$ the Cayley-Dickson extension of $\mathcal{A}$, i.e. $\mathcal{A} \oplus i\mathcal{A}$ with the following multiplication

$$(a+ib)(c+id) = (ac+\gamma db^*) + i(a^*d+cb)$$

and new involution

$$(a+ib)^* = a^* - ib$$

(see [**12**, p. 45]). The algebra $\mathcal{A}$ can be embedded into its Cayley-Dickson extension as $\mathcal{A} \oplus i\{0\}$. If we begin with a field $\mathcal{F}$ with the identity involution and with some sequence $\{\gamma_n\} \subset \mathcal{F}$ of nonzero constants, we obtain the **Cayley-Dickson sequence** $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$, ... such that $\mathcal{F}_{n+1}$ is the Cayley-Dickson extension of $\mathcal{F}_n$. If $\mathbb{R}$ denotes the field of real numbers, then $\mathbb{R}_1$ is isomorphic to the field of complex numbers, $\mathbb{R}_2$ to the field of quaternions and $\mathbb{R}_3$ to the algebra of octonions.

If $\mathcal{F}$ is an ordered field, then all elements of its Cayley-Dickson sequence can be turned into a weak subdecomposition algebra over $\mathcal{F}$. Even more: if $\mathcal{A}$ is some weak subdecomposition algebra then its Cayley-Dickson extension is also a weak subdecomposition algebra.

The third group consists of papers [**10**] and [**14**]. A real Banach algebra $\mathcal{A}$ is called an *absolute-valued algebra* if $\|xy\| = \|x\| \cdot \|y\|$ holds for all $x, y \in \mathcal{A}$. In real weak subdecomposition algebras $\|x^2\| = \|x\|^2$ holds for all $x \in \mathcal{A}$ and thus our work is connected with absolute-valued algebras with identity. For some results concerning absolute-valued algebras without identity see [**10**].

In the sequel we shall use the notation $[x,y,z] = xy \cdot z - x \cdot yz$ for the **associator** in a nonassociative algebra $\mathcal{A}$.

1. Preliminaries

Let $(\mathcal{K}, \leq)$ be some (linearly) ordered field. This means that

(i) $\mathcal{K} = \mathcal{P} \cup \{0\} \cup -\mathcal{P}$ for some subset $\mathcal{P}$ which does not contain 0.
(ii) $\alpha, \beta \in \mathcal{P}$ implies $\alpha+\beta$, $\alpha\beta \in \mathcal{P}$.
(iii) $\alpha \leq \beta$ if and only if $\alpha - \beta \in \mathcal{P} \cup \{0\}$.

We shall denote the elements of $\mathcal{K}$ with small Greek letters.

Let $\mathcal{A}$ be some (nonassociative) algebra over $\mathcal{K}$ with the identity element e (in the sequel the letter e automatically means the identity element). Let $B\colon \mathcal{A} \times$

$\mathcal{A} \longrightarrow \mathcal{K}$ be a symmetric $\mathcal{K}$-bilinear form. We shall say that the pair $(\mathcal{A}, B)$ is an **subdecomposition algebra** if

(i) B is **positive** i.e. $B(x,x) \geq 0$ holds for all $x \in \mathcal{A}$,
(ii) B is **nondegenerate** i.e. $B(x,x) = 0$ implies $x = 0$,
(iii) B is **submultiplicative** i.e. $B(xy, xy) \leq B(x,x)B(y,y)$ holds for all $x, y \in \mathcal{A}$,
(iv) $B(e,e) = 1$.

The pair $(\mathcal{A}, B)$ will be called a **weak subdecomposition algebra** if we substitute (iii) by a weaker statement

(iii') $B(x^2, x^2) \leq B(x,x)^2$ holds for all $x \in \mathcal{A}$.

From now on we avoid notation $(\mathcal{A}, B)$ and simply say that $\mathcal{A}$ is a (weak) subdecomposition algebra. It is obvious that $\mathcal{K}$ itself is an subdecomposition algebra with the bilinear form $B(\alpha, \beta) = \alpha\beta$.

Remark. If we start with a partially ordered field $\mathcal{K}$, then the existence of some weak subdecomposition algebra over $\mathcal{K}$ implies that the square of any element of $\mathcal{K}$ is positive (see (i) and (iv)). In this case the order of $\mathcal{K}$ can be extended to a linear order. Therefore we assume that $\mathcal{K}$ is linearly ordered from the beginning.

We say that a, b are **orthogonal** if $B(a,b) = 0$. If $\mathcal{S}$ is a subset of $\mathcal{A}$ it shall be called an **orthogonal subbase** if all its elements are nonzero and pairwise orthogonal. If $\mathcal{A}$ is spanned by $\mathcal{S}$, then we call $\mathcal{S}$ an **orthogonal base**. Every element $a \in \mathcal{A}$ can be uniquely decomposed into a sum $a = \alpha e + x$ where x is orthogonal to e.

In the sequel we shall need some simple but very useful observations on ordered fields. We collect them into

Observation 1. *Let $\mathcal{K}$ be an ordered field and $\alpha, \beta, \gamma \in \mathcal{K}$. Then the following holds:*

(i) *The square of any element of $\mathcal{K}$ is positive. Thus $1 \geq 0$.*
(ii) *If $\alpha \geq 0$, then $\alpha^{-1} \geq 0$.*
(iii) *If $\beta\xi + \gamma \leq 0$ for all $\xi \in \mathcal{K}$, then $\beta = 0$.*
(iv) *If $\alpha\xi^2 + \beta\xi + \gamma \geq 0$ for all $\xi \in \mathcal{K}$, then $\alpha \geq 0$ and $\beta^2 \leq 4\alpha\gamma$.*
(v) *Let $\mathcal{A}$ be a weak subdecomposition algebra over $\mathcal{K}$. Then $B(x,y)^2 \leq B(x,x)B(y,y)$ holds for all $x, y \in \mathcal{A}$.*

Proof. (iii) Suppose for a moment that $\beta \neq 0$. Take $\xi_1 = 0$ and $\xi_2 = -2\gamma\beta^{-1}$. This gives us $\gamma \leq 0$ and $-\gamma \leq 0$ and so $\gamma = 0$ must hold. Thus $\beta\xi \leq 0$ for all $\xi \in \mathcal{F}$. If we take $\xi = \pm 1$, we obtain $\beta = 0$ which is a contradiction.

(iv) If $\alpha = 0$, we arrive in the situation described in (iii). Thus $\beta = 0$ also holds and there is nothing to prove. Suppose that α is nonzero. By replacing ξ with $-\xi$, we obtain that $\alpha\xi^2 + \gamma \geq 0$ holds for all $\xi \in \mathcal{K}$. From $\xi = 0$ we get $\gamma \geq 0$. Now suppose that $\alpha \leq 0$. Then $\xi^2 \leq -\gamma\alpha^{-1}$. But this is a contradiction since we can take $\xi = 1 - \gamma(2\alpha)^{-1}$. Therefore $\alpha \geq 0$. If we insert now $\xi = -\beta(2\alpha)^{-1}$ into $\alpha\xi^2 + \beta\xi + \gamma \geq 0$, we get $\beta^2 \leq 4\alpha\gamma$.

(v) Take $x, y \in \mathcal{A}$ and $\xi \in \mathcal{K}$. From

$$0 \leq B(x + \xi y, x + \xi y) = B(x, x) + 2\xi B(x, y) + \xi^2 B(y, y)$$

we obtain, using (iv) $B(x, y)^2 \leq B(x, x)B(y, y)$. In the case of real weak subdecomposition algebras this is a well-known Cauchy-Schwartz inequality for the inner product. □

2. General Theory of Weak Subdecomposition Algebras

The following lemma is of fundamental importance for the theory of subdecomposition algebras.

Lemma 1. *Let $\mathcal{A}$ be a weak subdecomposition algebra. Suppose that x is orthogonal to the identity element e. Then $x^2 = -B(x, x)e$.*

Proof. Take some $\xi \in \mathcal{K}$ and form an element $a = \xi e + x$. From $B(a^2, a^2) \leq B(a, a)^2$ we obtain

$$\begin{aligned} B(\xi^2 e + 2\xi x + x^2, \xi^2 e + 2\xi x + x^2) &\leq (\xi^2 + B(x, x))^2, \\ \xi^4 + 4\xi^2 B(x, x) + 2\xi^2 B(e, x^2) + 4\xi B(x, x^2) + B(x^2, x^2)& \\ &\leq \xi^4 + 2\xi^2 B(x, x) + B(x, x)^2 \end{aligned}$$

and finally

$$\xi^2(2B(x, x) + 2B(e, x^2)) + 4\xi B(x, x^2) + B(x^2, x^2) - B(x, x)^2 \leq 0.$$

From Observation 1(v) we obtain

$$B(e, x^2)^2 \leq B(e, e)B(x^2, x^2) = B(x^2, x^2) \leq B(x, x)^2.$$

Observation 1(iv) also tells us that $B(x, x) + B(e, x^2) \leq 0$. This implies that $-B(e, x^2)$ is positive and hence $B(x, x)^2 \leq B(e, x^2)^2$. Therefore $B(x, x)^2 = B(x^2, x^2) = B(e, x^2)^2$. Finally we get

$$\begin{aligned} 0 \leq B(x^2 + B(x, x)e, x^2 + B(x, x)e) &= B(x^2, x^2) + B(x, x)^2 \\ + 2B(x, x)B(e, x^2) &= 2B(x, x)(B(x, x) + B(e, x^2)) \leq 0\,. \end{aligned}$$

Since B is positive and nondegenerate, the proof is concluded. □

Define $\mathcal{A}_0 = \{\, x \in \mathcal{A};\ B(e, x) = 0 \,\}$. $\mathcal{A}_0$ is not a subalgebra of $\mathcal{A}$ but we shall make it an algebra by defining a new product in $\mathcal{A}_0$ with

$$x \circ y = xy - B(xy, e)e\,.$$

This is obviously a bilinear mapping since B is bilinear, and obviously we have $x \circ y \in \mathcal{A}_0$.

We shall also define a mapping $*\colon \mathcal{A} \longrightarrow \mathcal{A}$ with $(\alpha e + a)^* = \alpha e - a$ for $\alpha \in \mathcal{K}$ and $a \in \mathcal{A}_0$. Note that $*$ is not necessarily an algebraic involution of $\mathcal{A}$.

Proposition 1. *Let $\mathcal{A}$ be a weak subdecomposition algebra. Then $(\mathcal{A}_0, \circ)$ is anticommutative.*

Proof. Take some $x, y \in \mathcal{A}_0$. Since $x + y \in \mathcal{A}_0$, we get

$$\begin{aligned}(x+y)^2 &= -B(x+y, x+y)e = -B(x,x)e - 2B(x,y)e - B(y,y)e \\ &= x^2 + y^2 - 2B(x,y)e\,.\end{aligned}$$

Thus $xy + yx = -2B(x,y)e$ holds. Next we have

$$\begin{aligned}x \circ y + y \circ x &= xy - B(xy,e)e + yx - B(yx,e)e \\ &= -2B(x,y)e - B(xy,e)e - B(yx,e)e\,.\end{aligned}$$

Since $x \circ y + y \circ x$ is orthogonal to e, this finally gives us $x \circ y = -y \circ x$. □

Observation 2. From this proof it also follows that

$$2B(x,y) = -B(xy,e) - B(yx,e)$$

for all $x, y \in \mathcal{A}_0$. We shall use this observation later.

Proposition 2. *Let $\mathcal{A}$ be a weak subdecomposition algebra. Then $B(x^2, x^2) = B(x,x)^2$ for every $x \in \mathcal{A}$.*

Proof. Decompose $x = \alpha e + a$ with $a \in \mathcal{A}_0$. Then we have first

$$x^2 = (\alpha^2 - B(a,a))e + 2\alpha a\,.$$

Hence

$$\begin{aligned}B(x^2, x^2) &= \alpha^4 - 2\alpha^2 B(a,a) + B(a,a)^2 + 4\alpha^2 B(a,a) \\ &= \alpha^4 + 2\alpha^2 B(a,a) + B(a,a)^2 = (\alpha^2 + B(a,a))^2 \\ &= B(\alpha e + a, \alpha e + a)^2 = B(x,x)^2.\end{aligned}$$ □

Proposition 3. *Let $\mathcal{A}$ be a weak subdecomposition algebra. Then the following holds:*

(i) *$\mathcal{A}$ is a quadratic algebra: $x^2 - 2B(e,x)x + B(x,x)e = 0$.*
(ii) *The only nilpotent in $\mathcal{A}$ is 0.*
(iii) *$\mathcal{A}$ contains only two idempotents, 0 and e.*
(iv) *Every left (right) ideal of $\mathcal{A}$ is trivial.*
(v) *Every nonzero subalgebra contains the identity e and is closed under the mapping $*\colon \mathcal{A} \longrightarrow \mathcal{A}$. Thus every subalgebra of a weak subdecomposition algebra is again a weak subdecomposition algebra.*
(vi) *There is one to one correspondence between the subalgebras of $\mathcal{A}$ and subalgebras of $(\mathcal{A}_0, \circ)$.*

Proof. (i) Take some $x \in \mathcal{A}$ and decompose $x = \alpha e + a$ with $a \in \mathcal{A}_0$. Using Lemma 1 we obtain

$$x^2 - 2B(e,x)x + B(x,x)e = \alpha^2 e - B(a,a)e + 2\alpha a - 2\alpha(\alpha e + a) + (\alpha^2 + B(a,a))e = 0\,.$$

(ii) Suppose that $x^2 = 0$ for some $x \in \mathcal{A}$. If we decompose $x = \alpha e + a$, we obtain $2\alpha a = 0$ and $\alpha^2 - B(a,a) = 0$. This obviously implies $\alpha = 0$ and $a = 0$ and thus $x = 0$. Since $\mathcal{A}$ is power-associative, there are no nontrivial nilpotents in $\mathcal{A}$.

(iii) This can be proved in a similar way as (ii).

(iv) is obvious.

(v) Let $\mathcal{B}$ be some nontrivial subalgebra of $\mathcal{A}$ and $x \in \mathcal{B}$ a nonzero element. From

$$B(x,x)^{-1}(x^2 - 2B(e,x)x) = e$$

it follows that e belongs to $\mathcal{B}$. From

$$x^* = 2B(e,x)e - x$$

it follows that x^* also belongs to $\mathcal{B}$.

(vi) Let $\mathcal{B}$ be some subalgebra of $\mathcal{A}$. It is obvious that $(\mathcal{B} \cap \mathcal{A}_0, \circ)$ is a subalgebra of $(\mathcal{A}_0, \circ)$. Therefore it remains to prove that every subalgebra $(\mathcal{C}, \circ)$ of $(\mathcal{A}_0, \circ)$ is of the form $\mathcal{B} \cap \mathcal{A}_0$ for some subalgebra $\mathcal{B}$ of $\mathcal{A}$. In fact we can define $\mathcal{B} = \mathcal{K}e \oplus \mathcal{C}$. For $\alpha, \beta \in \mathcal{K}$ and $x, y \in \mathcal{C}$ we have

$$\begin{aligned}(\alpha e + x)(\beta e + y) &= \alpha\beta e + \alpha y + \beta x + xy \\ &= \alpha\beta e + \alpha y + \beta x + x \circ y + B(xy, e)e \\ &= (\alpha\beta + B(xy,e))e + \alpha y + \beta x + x \circ y \in \mathcal{K}e \oplus \mathcal{C}.\end{aligned}$$

Thus $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\mathcal{C} = \mathcal{B} \cap \mathcal{A}_0$ obviously holds. $\square$

More can be said about weak subdecomposition algebras if they are flexible. This means that $[x,y,x] = 0$ for all $x, y \in \mathcal{A}$.

Example 1. Take $\mathcal{A} = \mathbb{R}^3$ with bilinear form

$$B((\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2)) = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2.$$

Denote $e = (1,0,0)$, $x = (0,1,0)$ and $y = (0,0,1)$. Consider the following multiplication table for $\mathcal{A}$:

$\bullet$	e	x	y
e	e	x	y
x	x	$-e$	x
y	y	$-x$	$-e$

The reader can easily verify that $\mathcal{A}$ is a weak subdecomposition algebra. $\mathcal{A}$ is not flexible since

$$[x, y, x] = xy \cdot x - x \cdot yx = x^2 + x^2 = -2e \neq 0\,.$$

$\mathcal{A}$ is also not an subdecomposition algebra, since

$$\begin{aligned} &B((e+x)(e+y),(e+x)(e+y)) = B(e+2x+y, e+2x+y) = 6,\\ &\quad B(e+x,e+x)B(e+y,e+y) = 2 \cdot 2 = 4\,. \end{aligned}$$

Theorem 1. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra. Then the following holds:*

(i) *$\mathcal{A}$ is a noncommutative Jordan algebra.*
(ii) *The mapping $*\colon \mathcal{A} \longrightarrow \mathcal{A}$ is an algebra involution.*
(iii) *If $x, y \in \mathcal{A}_0$ and $\alpha, \beta \in \mathcal{K}$ then*

$$(\alpha e + x)(\beta e + y) = (\alpha\beta - B(x,y))e + \alpha y + \beta x + x \circ y.$$

(iv) *$\mathcal{A}$ is a (commutative) Jordan algebra if and only if $x \circ y = 0$ for all $x, y \in \mathcal{A}_0$.*
(v) *If we define $T(x,y) = B(x,y^*)$, then T is a trace form i.e. $T(xy,z) = T(x,yz)$ holds for all $x,y,z \in \mathcal{A}$ (see* [**12**, page 24]*).*

Proof. (i) Take some $x, y \in \mathcal{A}$ and decompose $x = \alpha e + a$ and $y = \beta e + b$ with $a, b \in \mathcal{A}_0$. Then we have

$$[x^2, y, x] = [(\alpha^2 - B(a,a))e + 2\alpha a, \beta e + b, \alpha e + a] = 2\alpha[a,b,a] = 0\,.$$

(ii), (iii) Take some $x, y \in \mathcal{A}_0$ and suppose for a moment that $x \neq 0$. From $[x,y,x] = 0$ we obtain

$$\begin{aligned} (x \circ y + B(xy,e)e)x &= x(y \circ x + B(yx,e)e),\\ (x \circ y) \circ x + B((x \circ y)x, e)e + B(xy,e)x &\\ &= x \circ (y \circ x) + B(x(y \circ x), e)e + B(yx,e)x\,. \end{aligned}$$

Since $\mathcal{A}_0$ is anticommutative, it follows $x \circ (y \circ x) = (x \circ y) \circ x$. Since e and x are linearly independent, $B(xy,e) = B(yx,e)$ holds. From Observation 1 it follows that

$$B(xy,e) = B(yx,e) = -B(x,y). \tag{1}$$

But (1) also holds for $x = 0$ and hence (1) is valid for all $x, y \in \mathcal{A}_0$. From (1) and the definition of the product $\circ$, (iii) easily follows.

Now we must verify that

$$((\alpha e + x)(\beta e + y))^* = (\beta e + y)^*(\alpha e + x)^* = (\beta e - y)(\alpha e - x)$$

holds for all $\alpha, \beta \in \mathcal{K}$ and $x, y \in \mathcal{A}_0$. First we have

$$(\beta e - y)(\alpha e - x) = \alpha\beta e - \alpha y - \beta x + y \circ x - B(x,y)e\,.$$

Next we have

$$(\alpha\beta e + \alpha y + \beta x + x \circ y - B(x,y)e)^* = \alpha\beta e - \alpha y - \beta x - B(x,y)e - x \circ y$$

and since $x \circ y = -y \circ x$ we finally establish (ii).

(iv) This follows directly from (iii).

(v) Take some $x, y, z \in \mathcal{A}_0$. Then we have, using Lemma 1,

$$(x \circ y + z)^2 = -B(x \circ y + z, x \circ y + z)e\,.$$

If we expand this, we obtain

$$(x \circ y)z + z(x \circ y) = -2B(x \circ y, z).$$

Since $x \circ y = xy + B(x,y)e$, we get

$$xy \cdot z + x \cdot yz + 2B(x,y)z = -2B(x \circ y, z)e\,. \tag{2}$$

If we take adjoints in (2), we get

$$-z \cdot yx - yx \cdot z - 2B(x,y)z = -2B(x \circ y, z)e$$

and so

$$xy \cdot z + z \cdot xy - z \cdot yx - yx \cdot z = -4B(x \circ y, z)e\,. \tag{3}$$

With a cyclic permutation we get from (3):

$$yz \cdot x + x \cdot yz - x \cdot zy - zy \cdot x = -4B(y \circ z, x)e\,. \tag{4}$$

If we subtract (4) from (3), we obtain

$$\begin{aligned}[x,y,z] + [z,y,x] + x \cdot zy + z \cdot xy - yx \cdot z - yz \cdot x \\ = -4B(x \circ y, z)e + 4B(y \circ z, x)e\,.\end{aligned}$$

Since $\mathcal{A}$ is flexible, $[x, y, z] + [z, y, x] = 0$. Next we have

$$\begin{aligned} x \cdot zy - yz \cdot x &= x(z \circ y - B(z, y)e) - (y \circ z - B(z, y)e)x \\ &= x(z \circ y) + (z \circ y)x = -2B(z \circ y, x)e \end{aligned}$$

and

$$\begin{aligned} z \cdot xy - yx \cdot z &= z(x \circ y - B(x, y)e) - (y \circ x - B(x, y)e)z \\ &= z(x \circ y) + (x \circ y)z = -2B(x \circ y, z)e\,. \end{aligned}$$

Finally we get

$$-2B(z \circ y, x) - 2B(x \circ y, z) = -4B(x \circ y, z) + 4B(y \circ z, x)$$

and thus $B(x \circ y, z) = B(y \circ z, x)$. Since $\mathcal{A}_0$ is anticommutative, we obtain

$$B(x \circ y, z) = -B(x, z \circ y). \tag{5}$$

Now take $a, b, c \in \mathcal{A}$ and decompose $a = \alpha e + x$, $b = \beta e + y$ and $c = \gamma e + z$. We must prove that

$$B(ab, c) = B(a, cb^*).$$

Hence we must verify

$$B((\alpha e + x)(\beta e + y), \gamma e + z) = B(\alpha e + x, (\gamma e + z)(\beta e - y)).$$

If we expand this equality, we see that $B(x \circ y, z) = -B(x, z \circ y)$ must hold. Since this is (5), the proof of the theorem is completed. $\square$

In [**4**] nonassociative real H^*-algebras are studied. This are real Hilbert algebras with an involution such that

$$\langle xy, z\rangle = \langle x, zy^*\rangle = \langle y, x^*z\rangle$$

holds for all $x, y, z \in \mathcal{A}$.

Corollary 1. *Let $\mathcal{A}$ be a real flexible algebra with the identity e which is also a real Hilbert space with the norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. Suppose that $\|e\| = 1$ and that $\|x^2\| \leq \|x\|^2$ holds for all $x \in \mathcal{A}$. Then $\mathcal{A}$ is a quadratic noncommutative Jordan real H^*-algebra.*

The assumption of $\mathcal{A}$ being flexible can be removed in the case of subdecomposition algebras as the following tells us.

Proposition 4. *Let $\mathcal{A}$ be a subdecomposition algebra. Then $\mathcal{A}$ is flexible.*

Proof. Take $x, y \in \mathcal{A}_0$. For all $\xi \in \mathcal{K}$ we have

$$\begin{aligned} B(\xi x + xy, \xi x + xy) &\leq B(\xi e + y, \xi e + y) B(x, x) \\ &= \xi^2 B(x, x) + B(x, x) B(y, y) \end{aligned}$$

and so

$$2\xi B(x, xy) + B(xy, xy) - B(x, x,) B(y, y) \leq 0.$$

Thus, using Observation 1(iii), we get $B(x, xy) = 0$. In a similar way we prove $B(y, xy) = 0$.

Now we shall use the inequality

$$B(e + \xi x + y + \xi xy, e + \xi x + y + \xi xy) \leq B(e + \xi x, e + \xi x) B(e + y, e + y).$$

If we expand both sides, we obtain

$$\xi^2 (B(xy, xy) - B(x, x) B(y, y)) + 2\xi (B(e, xy) + B(x, y)) \leq 0.$$

By Observation 1, this implies $B(e, xy) = -B(x, y)$. In a similar way we prove $B(e, yx) = -B(x, y)$ and thus $xy = x \circ y - B(x, y)e$ holds. Finally

$$\begin{aligned} [x, y, x] &= (x \circ y - B(x, y)e)x - x(y \circ x - B(x, y)e) = (x \circ y)x + x(x \circ y) \\ &= (x \circ y) \circ x - B(x \circ y, x)e + x \circ (x \circ y) - B(x \circ y, x)e \\ &= 2B(x \circ y, x)e = 2B(xy + B(x, y)e, x)e = 0\,. \end{aligned}$$

□

3. Alternative Weak Subdecomposition Algebras

In this section we describe the structure of alternative weak subdecomposition algebras. Our result is that every flexible weak subdecomposition algebra which satisfies the identity $xy \cdot yx = x \cdot y^2 \cdot x$ is alternative and is isomorphic to one of the Cayley-Dickson algebras $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ or $\mathcal{K}_3$. This identity is a weak form of well-known Moufang identity $xy \cdot zx = x(yz)x$ which holds in any alternative algebra (see [**12**, page 28]).

Observation 3. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra and $\alpha \in \mathcal{K}$ a negative element. Then the Cayley-Dickson extension $\mathcal{A}_{CD}(\alpha)$ of the algebra $\mathcal{A}$ can be turned into a weak subdecomposition algebra.*

Proof. Let $(\mathcal{A}, B)$ be a flexible weak subdecomposition algebra. According to Theorem 1, $\mathcal{A}$ has an involution $*$. Define a new bilinear form on $\mathcal{A} \oplus i\mathcal{A}$ by

$$B_1(a + ib, c + id) = B(a, c) - \alpha B(b, d).$$

Since α is negative, B_1 is positive and nondegenerate. Decompose $a = \beta e + x$ with $x \in \mathcal{A}_0$ and $\beta \in \mathcal{K}$. Then we have

$$\begin{aligned}(a+ib)^2 &= (a^2 + \alpha bb^*) + i(a+a^*)b \\ &= \beta^2 e - B(x,x)e + \alpha B(b,b)e + 2\beta x + i(2\beta b).\end{aligned}$$

It can now easily be verified that

$$B_1((a+ib)^2, (a+ib)^2) = B_1(a+ib, a+ib)^2$$

holds. □

Lemma 2. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra which satisfies the identity $xy \cdot yx = xy^2x$. Then $\mathcal{A}$ is alternative and the form B is multiplicative i.e. $B(xy,xy) = B(x,x)B(y,y)$.*

Proof. Take some $x, y \in \mathcal{A}_0$ which are orthogonal. First we have $x^2 y = -B(x,x)y$. Now we shall compute

$$B([x,x,y],[x,x,y]) = B(x \cdot xy + B(x,x)y, x \cdot xy + B(x,x)y).$$

Since x and y are orthogonal, we have $xy = -yx \in \mathcal{A}_0$. Thus

$$\begin{aligned}B(xy,xy)e &= -(xy)^2 = xy \cdot yx = xy^2 x \\ &= -B(y,y)x^2 = B(x,x)B(y,y)e\,.\end{aligned} \tag{6}$$

From $B(x,xy) = B(x^2,y) = -B(x,x)B(e,y) = 0$ it also follows

$$B(x \cdot xy, x \cdot xy) = B(x,x)B(xy,xy) = B(x,x)^2 B(y,y)$$

and so we finally obtain

$$\begin{aligned}B([x,x,y],[x,x,y]) &= 2B(x,x)^2 B(y,y) + 2B(x,x)B(y, x \cdot xy) \\ &= 2B(x,x)^2 B(y,y) - 2B(x,x)B(xy,xy) = 0\,.\end{aligned}$$

Now take arbitrary $x, y \in \mathcal{A}_0$. If $x = 0$, there is nothing to prove, so we may suppose that $x \neq 0$. Define

$$z = y - B(x,y)B(x,x)^{-1}x\,. \tag{7}$$

It is straightforward to verify that x and z are orthogonal. Thus

$$0 = [x,x,z] = [x,x,y] - B(x,y)B(x.x)^{-1}[x,x,x] = [x,x,y]$$

and since $\mathcal{A} = \mathcal{K}e \oplus \mathcal{A}_0$ it follows that $\mathcal{A}$ itself is left alternative. In a similar way we prove that $\mathcal{A}$ is right alternative.

From (6) and (7) we get the validity of (6) for all $x, y \in \mathcal{A}_0$. Now, if $x = \alpha e + a$, $y = \beta e + b$, we can verify (6) with an easy computation. □

Theorem 2. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra which satisfies the identity $xy \cdot yx = xy^2x$. Then $\mathcal{A}$ is isomorphic to one of the following four examples:*

(i) *The field $\mathcal{K}$.*
(ii) *A two-dimensional associative commutative field $\mathcal{K}_1$.*
(iii) *A four-dimensional associative noncommutative division quaternion algebra over $\mathcal{K}$.*
(iv) *An eight-dimensional octonion algebra over $\mathcal{K}$.*

The proof of this result can be made with standard methods using Lemma 1.

Remark. In [**7**, Theorem 4.1] Elduque described the structure of all alternative quadratic algebras. They are of six different types and so the above result is also a consequence of his classification. We only have to observe which quadratic alternative algebras admit a symmetric bilinear form.

As we have already noted, in every alternative weak subdecomposition algebra the form B is actually multiplicative. The converse is also true.

Proposition 5. *Let $\mathcal{A}$ be an subdecomposition algebra in which $B(xy, xy) = B(x, x)B(y, y)$ holds. Then $\mathcal{A}$ is alternative.*

Proof. It is sufficient to verify that $[x, x, y] = 0$ holds for all $x, y \in \mathcal{A}_0$. First we have $x^2y = -B(x, x)y$. Using the fact that B is nondegenerate, Theorem 1 and the following calculation

$$\begin{aligned}
&B(x \cdot xy + B(x,x)y, x \cdot xy + B(x,x)y) \\
&\qquad = B(x \cdot xy, x \cdot xy) + 2B(x,x)B(x \cdot xy, y) + B(x,x)^2 B(y,y) \\
&\qquad = 2B(x,x)^2 B(y,y) - 2B(x,x)B(xy,xy) = 0,
\end{aligned}$$

we obtain $x^2y = x \cdot xy$. □

Proposition 6. *Let $\mathcal{G}$ be a field which is a proper extension of an ordered field $\mathcal{K}$. Let $\mathcal{A}$ be a (nonassociative) algebra with the identity element e over $\mathcal{G}$. Suppose that $\mathcal{A}$ is a weak subdecomposition algebra over $\mathcal{K}$. Then $\mathcal{A}$ is isomorphic to the field $\mathcal{G}$.*

Proof. The field $\mathcal{G}$ itself is also a weak subdecomposition algebra over $\mathcal{K}$ since $\mathcal{G}e \subset \mathcal{A}$. Since $\mathcal{G}$ is associative and commutative, $\mathcal{G}$ is isomorphic to $\mathcal{K}_1$ and is 2-dimensional as an algebra over $\mathcal{K}$.

Suppose that $\mathcal{A}$ is not isomorphic to $\mathcal{G}$. Then there exists a nonzero $a \in \mathcal{A}$ which is orthogonal to $\mathcal{G}e$. Take some nonzero $g \in \mathcal{G}e$ such that $g^2 = -B(g, g)e$.

First we have $(ga)^2 = g^2a^2 = B(a,a)B(g,g)e$. Decompose $ga = \alpha e + y$ with $y \in \mathcal{A}_0$. Then $(ga)^2 = \alpha^2 e - B(y,y)e + 2\alpha y$. Therefore $\alpha y = 0$ must hold. If $\alpha = 0$, then $-B(y,y) = B(a,a)B(g,g)$ and from the fact that B is positive it follows $a = 0$ which is contradiction. So $y = 0$ and from $ga = \alpha e$, $g^2 a = \alpha g$ (note that $\mathcal{A}$ is algebra over $\mathcal{G}$) we finally obtain $a = -\alpha B(g,g)^{-1} g$. This is also not possible, since we assumed that a and g are orthogonal. $\square$

Corollary 2. *Let $\mathcal{A}$ be a complex algebra with identity e which is also a pre-Hilbert space such that $\|e\| = 1$ and $\|x^2\| \le \|x\|^2$ holds for all $x \in \mathcal{A}$. Then $\mathcal{A}$ is isomorphic to the field of complex numbers.*

4. Weak Subdecomposition Algebras with Large Nucleus

If $\mathcal{A}$ is some nonassociative algebra then the **left, middle and right nucleus** are defined by

$$\begin{aligned} \mathcal{N}_L &= \{\, x \in \mathcal{A};\ [x,\mathcal{A},\mathcal{A}] = (0)\,\},\\ \mathcal{N}_M &= \{\, x \in \mathcal{A};\ [\mathcal{A},x,\mathcal{A}] = (0)\,\},\\ \mathcal{N}_R &= \{\, x \in \mathcal{A};\ [\mathcal{A},\mathcal{A},x] = (0)\,\}. \end{aligned}$$

Their intersection $\mathcal{N} = \mathcal{N}_L \cap \mathcal{N}_M \cap \mathcal{N}_R$ is called a **nucleus** of $\mathcal{A}$. The nucleus is always an associative subalgebra of $\mathcal{A}$. If $\mathcal{A}$ is a weak subdecomposition algebra then it is obvious that the nucleus of $\mathcal{A}$ is an associative subdecomposition algebra and therefore by Theorem 2 it is 1, 2 or 4-dimensional. By the phrase "large nucleus" we mean 2 or 4-dimensional nucleus.

Observation 4. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra. Then $\mathcal{N}_L = \mathcal{N}_M = \mathcal{N}_R = \mathcal{N}$.*

Proof. This follows from Theorem 1. First we have

$$\begin{aligned} a \in \mathcal{N}_M \quad &\text{iff} \quad [\mathcal{A},a,\mathcal{A}] = (0) \quad \text{iff} \quad [\mathcal{A},a,\mathcal{A}]^* = (0) \quad \text{iff}\\ &\text{iff} \quad [\mathcal{A},a^*,\mathcal{A}] = (0) \quad \text{iff} \quad a^* \in \mathcal{N}_M. \end{aligned}$$

Next we can observe, using the fact that B is nondegenerate,

$$\begin{aligned} a \in \mathcal{N}_L \quad &\text{iff} \quad [a,\mathcal{A},\mathcal{A}] = (0) \quad \text{iff} \quad B([a,\mathcal{A},\mathcal{A}],\mathcal{A}) = (0)\\ &\text{iff} \quad B(\mathcal{A},[\mathcal{A},a^*,\mathcal{A}]) = (0) \quad \text{iff} \quad [\mathcal{A},a^*,\mathcal{A}] = (0)\\ &\text{iff} \quad a^* \in \mathcal{N}_M \quad \text{iff} \quad a \in \mathcal{N}_M. \end{aligned}$$

Thus $\mathcal{N}_L = \mathcal{N}_M$. In a similar way we prove that $\mathcal{N}_R = \mathcal{N}_M$. $\square$

Theorem 3. *Let $\mathcal{A}$ be a weak subdecomposition algebra and suppose that the nucleus of $\mathcal{A}$ is isomorphic to the quaternion algebra. Then $\mathcal{A}$ itself is a quaternion algebra and therefore associative.*

Proof. Suppose that $\mathcal{A} \neq \mathcal{N} = \mathcal{K}_2$. Let $\{e, x, y, z = xy\}$ be a (usual) orthogonal base of $\mathcal{N}$. There exists some nonzero $b \in \mathcal{A}$ orthogonal to $\mathcal{N}$. Using

$$(x+b)^2 = -B(x+b, x+b)e = -B(x,x)e - B(b,b)e = x^2 + b^2$$

we obtain $bx = -xb$. In a similar way we see that $by = -yb$ and $bz = -zb$. But this implies

$$\begin{aligned} b &= (-B(z,z)^{-1})(-B(z,z))b = -B(z,z)^{-1}z^2b = -B(z,z)^{-1}zxy \cdot b \\ &= -B(z,z)^{-1}zx \cdot yb = B(z,z)^{-1}zx \cdot by = -B(z,z)^{-1}z \cdot bxy \\ &= B(z,z)^{-1}b \cdot zxy = B(z,z)^{-1}bz^2 = -B(z,z)^{-1}B(z,z)b = -b \end{aligned}$$

which contradicts the assumption that $b \neq 0$. □

Lemma 3. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra and suppose that the nucleus $\mathcal{N}$ of $\mathcal{A}$ is 2-dimensional. Suppose that b is orthogonal to $\mathcal{N}$ and nonzero. Then the subalgebra, generated with $\mathcal{N}$ and b, is isomorphic to the quaternion algebra.*

Proof. Write $\mathcal{N} = \mathcal{K}e \oplus \mathcal{K}a$ where a is orthogonal to e. Let $c = ab$. From

$$\begin{aligned} B(c,e) &= B(ab,e) = -B(b,a) = 0, \\ B(c,a) &= B(ab,a) = -B(b,a^2) = B(a,a)B(b,e) = 0, \\ B(c,b) &= B(ab,b) = -B(a,b^2) = B(b,b)B(a,e) = 0 \end{aligned}$$

we obtain that the set $\{e, a, b, c\}$ spans a 4-dimensional subspace since

$$ac = a \cdot ab = a^2b = -B(a,a)b$$

implies that $c \neq 0$. Using the fact that a belongs to the nucleus of $\mathcal{A}$, we can easily compute the following multiplication table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	$-B(a,a)e$	c	$-B(a,a)b$
b	b	$-c$	$-B(b,b)e$	$B(b,b)a$
c	c	$B(a,a)b$	$-B(b,b)a$	$-B(a,a)B(b,b)e$

which is a multiplication table of quaternions. □

Lemma 4. *Let $\mathcal{A}$ be a flexible weak subdecomposition algebra. Suppose that the nucleus $\mathcal{N} = \mathcal{K}e \oplus \mathcal{K}a$ is 2-dimensional and that the set $\{e, a, b_1, c_1 = ab_1, b_2\}$ is an orthogonal subbase of $\mathcal{A}$. Then $\{e, a, b_1, c_1, b_2, c_2 = ab_2\}$ is also an orthogonal subbase of $\mathcal{A}$. The linear subspace generated with this set is a subalgebra of $\mathcal{A}$ with the following multiplication table:*

$\bullet$	e	a	b_1	c_1	b_2	c_2
e	e	a	b_1	c_1	b_2	c_2
a	a	$-\alpha e$	c_1	$-\alpha b_1$	c_2	$-\alpha b_2$
b_1	b_1	$-c_1$	$-\beta_1 e$	$\beta_1 a$	0	0
c_1	c_1	αb_1	$-\beta_1 a$	$-\alpha\beta_1 e$	0	0
b_2	b_2	$-c_2$	0	0	$-\beta_2 e$	$\beta_2 a$
c_2	c_2	αb_2	0	0	$-\beta_2 a$	$-\alpha\beta_2 e$

where $\alpha = B(a, a)$, $\beta_1 = B(b_1, b_1)$ and $\beta_2 = B(b_2, b_2)$.

Proof. From Lemma 3 we already know everything except that $b_1 b_2 = b_1 c_2 = c_1 b_2 = c_1 c_2 = 0$. Using the fact that a belongs to the nucleus of $\mathcal{A}$ we first obtain

$$c_1 c_2 = ab_1 \cdot ab_2 = -b_1 a \cdot ab_2 = -b_1 a^2 \cdot b_2 = \alpha b_1 b_2.$$

Next we have

$$a(b_1 + c_2)^2 = a(b_1 + c_2) \cdot (b_1 + c_2).$$

If we expand both sides of the above equality, we get, using the fact that b_1 and c_2 are orthogonal,

$$\begin{aligned}(-\beta_1 - B(c_2, c_2))a &= (c_1 - \alpha b_2)(b_1 + c_2)\\ &= -\beta_1 a - \alpha b_2 b_1 + c_1 c_2 - \alpha\beta_2 a\\ &= -\beta_1 a + 2\alpha b_1 b_2 - \alpha\beta_2 a\,.\end{aligned}$$

Thus $2\alpha b_1 b_2 = 0$ and so $b_1 b_2 = c_1 c_2 = 0$. Now $b_1 c_2 = c_1 b_2 = 0$ easily follows. $\square$

Theorem 4. *Let $\mathcal{A}$ be a finite-dimensional flexible weak subdecomposition algebra with 2-dimensional nucleus $\mathcal{N} = \mathcal{K}e \oplus \mathcal{K}a$. Then $\mathcal{A}$ can be decomposed into an orthogonal sum $\mathcal{A} = \mathcal{N} \oplus \mathcal{B} \oplus a\mathcal{B}$ where $\mathcal{B} \circ \mathcal{B} = a\mathcal{B} \circ a\mathcal{B} = (0)$. The dimension of $\mathcal{A}$ is even but different from 4.*

The proof easily follows from Lemma 4 and the finite-dimensionality of $\mathcal{A}$.

A problem with infinite-dimensional subdecomposition algebras is that they are not necessarily orthogonally complemented i.e. given a subspace $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B} \oplus \mathcal{B}^{\perp}$ may be a proper subset of $\mathcal{A}$.

For real weak subdecomposition algebras which are complete with respect to the norm which is generated with the inner product, a similar result can be proved.

Theorem 5. *Let $\mathcal{A}$ be a real flexible weak subdecomposition algebra which is a Hilbert space in the topology of the inner product B. Suppose that the nucleus $\mathcal{N} = \mathbb{R}e \oplus \mathbb{R}a$ is 2-dimensional. Then $\mathcal{A}$ can be decomposed into an orthogonal sum $\mathcal{A} = \mathcal{N} \oplus \mathcal{B} \oplus a\mathcal{B}$ where $\mathcal{B} \circ \mathcal{B} = a\mathcal{B} \circ a\mathcal{B} = (0)$.*

Proof. Consider the family Φ of those subspaces $\mathcal{B} \subset \mathcal{N}^\perp$ for which $a\mathcal{B} \subset \mathcal{B}^\perp \cap \mathcal{N}^\perp$ holds. From Theorem 1 it follows that the operators $L_y(x) = yx$ and $R_y(x) = xy$ are continuous since $L_y^* = L_{y^*}$. Note that it is well-known that an everywhere defined operator T acting on a Hilbert space is continuous if and only if it has an adjoint T^*. This also implies that the multiplication of $\mathcal{A}$ is jointly continuous (see [**3**, Theorem 49.6]).

If $\overline{\mathcal{B}}$ denotes the closure of $\mathcal{B}$ we have

$$a\overline{\mathcal{B}} = L_a(\overline{\mathcal{B}}) \subset \overline{\mathcal{B}^\perp \cap \mathcal{N}^\perp} = \mathcal{B}^\perp \cap \mathcal{N}^\perp = (\overline{\mathcal{B}})^\perp \cap \mathcal{N}^\perp$$

for all $\mathcal{B} \in \Phi$. Thus $\mathcal{B} \in \Phi$ implies $\overline{\mathcal{B}} \in \Phi$. Using the Zorn lemma, we see that Φ contains some maximal element which we again denote with $\mathcal{B}$. From the maximality of $\mathcal{B}$ it easily follows that $\mathcal{B}$ is closed and that $a\mathcal{B} = \mathcal{B}^\perp \cap \mathcal{N}^\perp$ (see Lemma 3 and Lemma 4). From the continuity of the product it also easily follows that $\mathcal{B} \circ \mathcal{B} = (0)$ since every element of $\mathcal{B}$ can be represented as a limit of finite linear combinations of some orthonormal base of $\mathcal{B}$ which we can always choose. The products of base elements are zero because of Lemma 4. □

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