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ARC–TRANSITIVE NON–CAYLEY GRAPHS FROM REGULAR MAPS

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ABSTRACT. We prove that the underlying graphs of p-gonal r-valent orientably regular maps are arc-transitive but non-Cayley if $r \geq 3$ and p is a prime greater than $r(r-1)$.

1. Introduction

Orientable maps, i.e., embeddings of graphs in orientable surfaces, have been studied in various contexts. They are interesting not only from the point of view of embeddings themselves, but also as a tool for extracting information about the embedded graphs and their automorphism groups. For example, the underlying graph of an orientably regular map (that is, a map with the largest possible number of orientation-preserving map automorphisms) possesses a relatively rigid structure; it must be arc-transitive. In this note we prove that if we require further that the map be p-gonal and r-valent, where p, $r \geq 3$ and p is prime, then in its underlying graph the number of closed p-walks emanating from a fixed vertex must satisfy a certain congruence relation.

As an application of this result, we exhibit for every $r \geq 3$ a construction of an infinite class of maps whose underlying graphs are r-regular, arc-transitive, but non-Cayley. This is interesting and surprising, as even constructions of vertex-transitive non-Cayley graphs seem to be difficult to find. Note that the problem of constructing vertex-transitive non-Cayley graphs is equivalent to the (extensively studied) existence of certain permutation groups that do not have regular subgroups. Only a few families of vertex-transitive non-Cayley graphs were known in the late eighties [**W**], but there has been a lot of activity in the field since then (the most recent progress is well documented in [**MP**]).

2. Preliminaries

For the purpose of this note, a **map** is a cellular embedding of a graph in an orientable surface (= closed 2-manifold). Let M be a map and let G be the

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corresponding graph (often called the **underlying** graph of M). A permutation representation of M can be obtained in the following standard way (cf. e.g. [**GT**]). Fix an orientation of the ambient surface. For each vertex v of G let P_v be the permutation that cyclically permutes the arcs (= edges with direction) emanating from v in accordance with the chosen orientation of the surface. The product $P = \prod_{v \in V(G)} P_v$ is called **rotation** of G. Further, let L be the fixed-point-free involution that assigns to each arc its reverse. Both P and L are permutations of the set $D(G)$ of arcs of G (note that $|D(G)| = 2|E(G)|$). The group $\langle P, L \rangle$ generated by P and L is a transitive permutation group on the set D, and carries a complete information about the map M; we therefore write $M = M(P, L)$. Indeed, given a set D and two permutations P and L of D such that L is a fixed-point-free involution and the group $\langle P, L \rangle$ is transitive on D, the topological structure of M can be recovered easily. Vertices and edges of the underlying graph are orbits of the permutations P and L. The boundary walks of faces of M correspond to orbits of the permutation PL. Incidence between vertices, edges and faces is given by a nonempty intersection of the corresponding orbits.

An **automorphism** of a map $M(P, L)$ is a permutation of $D = D(G)$ commuting with both P and L. The set of automorphisms of M will be denoted by Aut M. This set is clearly a group under compositions of permutations. Since $\langle P, L \rangle$ is transitive on D, an automorphism $A \in \text{Aut } M$ is uniquely determined by its value at a fixed arc $f \in D$. Thus $|\text{Aut } M| \le |D|$. The maps for which the equality is attained are called **orientably regular**. Such maps have been widely studied; one of the best references here is [**JS**]. We will need the following well known fact: If $M(P, L)$ is orientably regular then the groups Aut M and $\langle P, L \rangle$ are isomorphic. Consequently, in an orientably regular map, if $Rf = f$ for some arc $f \in D$ and some $R \in \langle P, L \rangle$ then $R = 1$, the identity permutation. (This observation will be repeatedly used in the next section.) It is readily seen that if a map is orientably regular then there are numbers n, m such that each face of the map is bounded by a walk of length n and each vertex of the underlying graph has valency m. In such case we say that the map is of **type** $\{n, m\}$.

We conclude with a few facts about Cayley graphs. Let H be a finite group and let S be a **symmetric** subset of H, that is, $1 \notin S$ and $s \in S \Leftrightarrow s^{-1} \in S$ for all $s \in H$. The **Cayley graph** $C(H, S)$ **of** H **with respect to** S is defined as follows. The set of vertices of $C(H, S)$ is the set of all elements of H, and $h_1, h_2 \in H$ are joined in $C(H, S)$ by an edge if and only if $h_1^{-1} h_2 \in S$ ($\Leftrightarrow h_2^{-1} h_1 \in S$). One may observe that every vertex in $C(H, S)$ has valency $|S|$, and the graph $C(H, S)$ is connected if and only if S is a (symmetric) generating set for the group H. Also, note that Cayley graphs are automatically vertex-transitive; the converse is not true.

An interesting property of Cayley graphs, based on counting closed walks, was obtained in [**FRS**] . Recall that for an arbitrary graph G, a **closed oriented walk**

of length n in G (starting at a vertex v_0) is a sequence $f_1, f_2, \dots, f_n$ of arcs of G with f_i emanating from a vertex v_{i-1} and terminating at a vertex v_i for $1 \le i \le n$ (here, of course, $v_n = v_0$).

Theorem 1. [**FRS**] *Let H be a group and let S be a symmetric subset of H. Let p be an odd prime. Then in the Cayley graph $C(H,S)$ the number of closed oriented walks of length p starting at an (arbitrary but fixed) vertex v is congruent* $(\text{mod } p)$ *to the number of elements of order p in S.*

This theorem will play a key role in our subsequent considerations.

3. Results

First we show that a result similar to Theorem 1 also holds true for underlying graphs of regular maps.

Theorem 2. *Let $p \ge 3$ and $r \ge 3$, with p prime. Let M be an orientably regular map of type $\{p, r\}$ and let v be a vertex of the underlying graph G of M. Then, the number of closed oriented walks in G of length p, starting at v, is congruent to kr* $(\text{mod } p)$ *for some k such that $2 \le k \le r-1$.*

Proof. Let $M = M(P,L)$ be an orientably regular map with the required properties (consequently, its underlying graph contains no loops). Consider an oriented closed walk $W = f_1, f_2, \dots, f_p, f_{p+1} = f_1$ of length p in G, starting at the vertex v. (The repeated use of the first arc $f_{p+1} = f_1$ in our notation has auxiliary reasons only.) By regularity of M, for each i, $1 \le i \le p$ there exists a unique element R_i in the group $\langle P, L\rangle$ such that $f_{i+1} = R_i f_i$. The fact that W is closed readily implies (invoking regularity again) that $R_p R_{p-1} \dots R_2 R_1 = 1$. Moreover, since the terminal vertex of the arc f_i coincides with the initial vertex of the arc f_{i+1}, the latter can be obtained by reversing the direction of f_i and rotating the reverse (i.e., the arc Lf_i) a suitable number of times in accordance with the chosen orientation of the map surface; in other terms, $R_i = P^{j_i} L$ for some j_i, $0 \le j_i \le r-1$. Thus, with the walk W as above we may associate the p-tuple $(j_1, j_2, \dots, j_p)$, $0 \le j_i \le r-1$ for which

$$P^{j_p} L P^{j_{p-1}} L \dots P^{j_1} L = 1\,. \tag{1}$$

Conversely, given a p-tuple $(j_1, j_2, \dots, j_p)$ with the above property, one can check that the sequence $f_1, R_1 f_1, R_2 R_1 f_1, \dots, R_p R_{p-1} \dots R_1 f_1$, where $R_i = P^{j_i} L$, is an oriented closed walk in G that starts at v and uses f_1 as its first arc. Thus we have a $1-1$ correspondence between the set of closed oriented walks of length p starting at v and using f_1 as its first arc, and the set I_p of p-tuples $(j_1, j_2, \dots, j_p)$, $0 \le j_i \le r-1$ which satisfy (1). It also follows that there are exactly $r|I_p|$ oriented closed walks in G of lenght p starting at v (namely, f_1 can be any of the r arcs emanating from v).

The key to the proof is the following observation: $P^{j_p}LP^{j_{p-1}}L\dots P^{j_1}L = 1$ if and only if $P^{j_1}LP^{j_p}LP^{j_{p-1}}L\dots P^{j_2}L = 1$. Therefore, the mapping ϕ defined on I_p and given by $\phi(j_1, j_2, \dots, j_p) = (j_2, j_3, \dots, j_p, j_1)$ is a permutation of the set I_p, and defines an action of the cyclic group Z_p on I_p. As p is prime, the orbits of this action are either of size p or of size 1. Another consequence of the primeness of p is the fact that the orbits of size 1 are exactly those for which $j_1 = j_2 = \dots = j_p$. Let k be the number of orbits of size 1, i.e., $k = |\{j : 0 \le j \le r-1, (P^jL)^p = 1\}|$, and let s be the number of orbits of length p. The preceding discussion implies that $|I_p| = ps + k$, and so $|I_p| \equiv k \pmod p$. We saw earlier that $r|I_p|$ is the number of oriented closed walks in G of length p, starting at v; therefore this number is congruent to $kr \pmod p$, as claimed in the theorem. The last thing to do is to establish the inequality $2 \le k \le r-1$. Observe that $(PL)^p = 1$ because the map M is of type $\{p, r\}$. But then also $(P^{r-1}L)^p = (P^{-1}L)^p = 1$, which shows that $k \ge 2$. At last, $(P^0L)^p = L^p = L \ne 1$, and so $k \le r-1$. $\square$ $\square$

Theorems 1 and 2 enable us to prove our main result stating that underlying graphs of certain orientably regular maps cannot be Cayley graphs.

Theorem 3. *Let M be an orientably regular map of type $\{p, r\}$ where p is prime, $r \ge 3$ and $p > r(r-1)$. Then the underlying graph of M is not a Cayley graph.*

Proof. Fix a vertex v in the underlying graph G of the map M. By Theorem 2, the number of closed oriented walks in G starting at v and having length p is congruent to $kr \pmod p$ for some $2 \le k \le r-1$. Now, assume that G is a Cayley graph $C(H, S)$ for some group H and some symmetric generating subset S of H. Then, by Theorem 1, the number t of elements of order p contained in the set S would have to satisfy the congruence relation $t \equiv kr \pmod p$. As $p > r(r-1)$ and $k \le r-1$, we have $t \ge kr$; in particular, $t \ge 2r$ because $k \ge 2$. On the other hand, since M is of type $\{p, r\}$, the graph G is regular of valency $r = |S|$, and so $t \le r$, a contradiction. $\square$

4. Conclusions

The preceding result suggests the question of whether or not there are enough ingredients for it, that is, if there are infinitely many orientably regular maps with the required properties. The question turns out to be equivalent with the existence of finite groups generated by two elements, say, x and y, that satisfy the relations $x^r = y^2 = (xy)^p = \dots = 1$. This was the way how Vince [**V**] (originally) and Gray and Wilson [**GW**] (later, a simplified version) proved that orientably regular maps of type $\{q, r\}$ exist for all pairs $q, r \ge 2$ (settling thereby Grünbaum's conjecture). Applying Surowski's method [**S**] of lifting maps by means of the canonical voltage assignment taken in the first homology group of the surface yields immediately the following stronger result: For each $q \ge 2$ and $r \ge 2$ (except the Platonic solids) there exist infinitely many orientably regular maps of type $\{q, r\}$. Since the underlying graph of any orientably regular map is automatically arc-transitive, we have the following consequence of Theorem 3 and the above discussion.

Theorem 4. *For every $r \geq 3$ there exist infinitely many arc-transitive r-regular non-Cayley graphs.*

Recently, quite a lot of attention has been devoted to Cayley maps. Recall that a map $M(P, L)$ is a **Cayley map** if its underlying graph is a Cayley graph $C(H, S)$ and the rotation P satisfies the following condition: There exists a cyclic permutation σ of the set S such that for every arc f in $C(H, S)$ emanating from h and terminating at hs, the arc Pf emanates from h and terminates at $h\sigma(s)$. (Roughly speaking, the rotation P is at every vertex given by the same cyclic permutation of generators.) For a recent deep study of orientable regularity of Cayley maps we refer to [**J**]; it is worth noting that most of the known "small" orientably regular maps are Cayley maps. However, as another immediate consequence of Theorem 3 and the stronger version of the solution of Grünbaum's conjecture we have:

Theorem 5. *Let $r \geq 3$, $p > r(r-1)$ and let p be prime. Then there exist infinitely many orientably regular maps of type $\{p, r\}$ which are not Cayley maps.*

Finally, we remark that Theorem 2 admits futher group-theoretic generalizations [**JŠ**].

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