Acta Math. Univ. Comenianae
Vol. LXIII, 2(1994), pp. 259–265

WEAK ISOMETRIES IN PARTIALLY ORDERED GROUPS

M. JASEM

Abstract. In this paper the author gives necessary and sufficient conditions under which to a stable weak isometry f in a directed group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$. Further, some results on weak isometries in partially ordered groups are established.

Isometries in an abelian lattice ordered group (l-group) have been introduced and investigated by Swamy [**16**], [**17**]. Jakubík [**4**] proved that for every stable isometry f in an l-group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$. Isometries in non-abelian l-groups were also studied in [**2**] and [**5**]. Weak isometries in l-groups were introduced by Jakubík [**6**]. Rachůnek [**14**] generalized the notion of the isometry for any partially ordered group (po-group). Isometries and weak isometries in some types of po-groups have been investigated in [**7**], [**8**], [**9**], [**10**], [**12**], [**13**], [**14**]. In [**11**] it was proved that each stable weak isometry in a directed group is an involutory group automorphism (hence each weak isometry in a directed group is an isometry).

First we recall some notions and notations used in the paper.

Let G be a po-group. The group operation will be written additively. We denote $G^+ = \{x \in G; x \geq 0\}$. If a, b are elements of G, then we denote by $U(a,b)$ and $L(a,b)$ the set of all upper bounds and the set of all lower bounds of the set $\{a,b\}$ in G, respectively. If for a, $b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a,b\}$ in G, then it will be denoted by $a \vee b$ ($a \wedge b$). For each $a \in G$, $|a| = U(a,-a)$.

A partially ordered semigroup (po-semigroup) P with a neutral element is said to be the direct product of its po-subsemigroups P_1 and P_2 (notation: $P = P_1 \times P_2$) if the following conditions are fulfilled:

(1) If $a \in P_1$, $b \in P_2$, then $a + b = b + a$.
(2) Each element $c \in P$ can be uniquely represented in the form $c = c_1 + c_2$ where $c_1 \in P_1$, $c_2 \in P_2$.
(3) If $g, h \in P$, $g = g_1 + g_2$, $h = h_1 + h_2$ where $g_1, h_1 \in P_1$, $g_2, h_2 \in P_2$, then $g \leq h$ if and only if $g_1 \leq h_1$, $g_2 \leq h_2$.

Received September 16, 1993; revised February 24, 1994.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 06F15.

In this case it is also spoken about the direct decomposition of the po-semigroup P.

The direct decomposition of a po-group is defined analogously.

If $G = P \times Q$ is a direct decomposition of a po-group G, then for $x \in G$ we denote by $x(P)$ and $x(Q)$ the components of x in the direct factors P and Q, respectively. Analogous denotation of components is also applied for the direct decomposition of the semigroup G^+.

If G is a po-group, then a mapping $f\colon G \to G$ is called a weak isometry if $|f(x) - f(y)| = |x - y|$ for each x, $y \in G$. A weak isometry f is called a stable weak isometry if $f(0) = 0$. A weak isometry f is called an isometry if f is a bijection.

A po-group G is called directed if $U(x,y) \neq \emptyset$ and $L(x,y) \neq \emptyset$ for each $x, y \in G$.

In [**12**] and [**13**] the above mentioned Jakubík's result concerning stable isometries and directed decompositions of l-groups was extended to Riesz and distributive multilattice groups.

The following example shows that this result cannot be extended to all directed groups.

Example. Let G be the additive group of all complex numbers $x+iy$ such that x and y are integers. An element $z = x + iy \in G$ is positive if and only if $y \geq x$ and $y \geq -x$. Then G is an abelian non-distributive multilattice group. If we put $f(x+iy) = y+ix$ for each $x+iy \in G$, then f is a stable weak isometry in G. If there exists a direct decomposition $G = A \times B$ of G such that $f(z) = z(A) - z(B)$ for each $z \in G$ then $z + f(z) = 2z(A)$ for each $z \in G$. For the element $a = 1$ we have $a+f(a) = 1+i$, but there does not exist an element b in G such that $a+f(a) = 2b$. Hence there does not exist the above mentioned direct decomposition of G.

Now we are going to establish necessary and sufficient conditions under which to a stable weak isometry f in a directed group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$.

First we establish the following theorem.

1. Theorem. *Let f be a stable weak isometry in a* po-*group G, $A_1 = \{x \in G^+, f(x) = x\}$, $B_1 = \{x \in G^+, f(x) = -x\}$. Then the following conditions are equivalent:*

(i) *For each $x \in G^+$ there exists the least upper bound of the set $\{0, f(x)\}$ in G^+.*

(ii) *For each $x \in G^+$, there exists $x_1 \in G^+$ such that $0 \leq x_1 \leq x$, $f(x) \leq x_1 \leq f(x) + x$.*

(iii) *G^+ is the direct product of the* po-*semigroup A_1 and the commutative* po-*semigroup B_1 and $f(x) = x(A_1) - x(B_1)$ for each $x \in G^+$.*

Proof. (i) $\Longrightarrow$ (ii). Let $x \in G^+$ and let $x_1 = 0 \vee f(x)$. From $|x| = |f(x)|$ we get $x \geq f(x) \geq -x$. Thus $0 \leq x_1 \leq x$. Further, we have $-x \leq 0 \wedge f(x)$. Hence $-f(x) \vee 0 = -(f(x) \wedge 0) \leq x$. This implies $0 \vee f(x) \leq f(x) + x$. Therefore $f(x) \leq x_1 \leq f(x) + x$.

(ii) $\Longrightarrow$ (iii). Let $x \in G^+$ and let $0 \leq x_1 \leq x$. $f(x) \leq x_1 \leq f(x) + x$ for some $x_1 \in G^+$. From $|x| = |f(x)|$ we get $x = -f(x) \vee f(x)$. Let $x_2 = x - x_1$. Then $x = x_2 + x_1$, $x \geq x_2 \geq 0$. Since $x_1 \leq f(x) + x$, we have $-f(x) \leq x_2$. Hence $x_1 \in U(0, f(x))$, $x_2 \in U(0, -f(x))$. Let $t \in U(0, f(x))$. Then $x_2 + t \in U(f(x), -f(x)) = |f(x)| = |x| = U(x_2 + x_1)$. This implies $t \geq x_1$. Therefore $x_1 = f(x) \vee 0$. Analogously we can show that $x_2 = -f(x) \vee 0$.

Let $z \in U(x_1, x_2)$. Then $z \in U(f(x), -f(x)) = |f(x)| = |x| = U(x)$. Since $x \in U(x_1, x_2)$, we have $x = x_1 \vee x_2$. Then clearly $x_1 \wedge x_2 = 0$ and $x_1 + x_2 = x_2 + x_1$.

Since $-x_2 = f(x) \wedge 0$, $f(x) = f(x) \vee 0 + f(x) \wedge 0$, we have $f(x) = x_1 - x_2$. From $|x_1| = |f(x_1)|$ we obtain $x_1 \geq f(x_1)$, $x_1 \geq -f(x_1)$. Then $f(x_1) + x_2 \geq -x_1 + x_2 = x_2 - x_1 = -f(x)$. From $|x_2| = |x - x_1| = |f(x) - f(x_1)| = |x_1 - x_2 - f(x_1)|$ we get $x_2 \geq x_1 - x_2 - f(x_1)$. This implies $f(x_1) + x_2 \geq -x_2 + x_1 = x_1 - x_2 = f(x)$. Hence $f(x_1) + x_2 \geq -f(x) \vee f(x) = x = x_1 + x_2$. This yields $f(x_1) \geq x_1$. Therefore $f(x_1) = x_1$.

From $|x_2| = |f(x_2)|$ we have $x_2 \geq f(x_2)$, $x_2 \geq -f(x_2)$. Then $-f(x_2) + x_1 \geq -x_2 + x_1 = x_1 - x_2 = f(x)$. From $|x_1| = |x - x_2| = |f(x) - f(x_2)|$ we obtain $x_1 \geq f(x_2) - f(x)$. Thus $-f(x_2) + x_1 \geq -f(x)$. Hence $-f(x_2) + x_1 \geq x = x_2 + x_1$. This implies $-f(x_2) \geq x_2$. Therefore $f(x_2) = -x_2$.

Thus $x = x_1 + x_2$, where $x_1 \in A_1$, $x_2 \in B_1$. By Lemmas 1.8 and 1.9 [**13**], A_1 is a semigroup and B_1 is a commutative semigroup. Let $x = a + b$, where $a \in A_1$, $b \in B_1$. From Theorem 1.13 [**13**] it follows that $f(a + b) = a - b$. Then $b \in U(0, -f(x))$. Hence $b \geq x_2$. Since $a + b = x_1 + x_2$, $a - b = x_1 - x_2$, we obtain $x_2 - b = -x_2 + b \geq 0$. Therefore $x_2 = b$. Then $x_1 = a$. From this also follows that $c + d = d + c$ for each $c \in A_1$, $d \in B_1$.

Let $u, v \in G^+$, $u \leq v$. Let $u = u_1 + u_2$, $v = v_1 + v_2$, $v - u = (v - u)_1 + (v - u)_2$ where u_1, v_1, $(u - v)_1 \in A_1$, u_2, v_2, $(u - v)_2 \in B_1$. From $v - u = v_1 + v_2 - u_2 - u_1$ we get $(v - u)_1 + u_1 + (v - u)_2 + u_2 = v_1 + v_2$. Then we have $v_1 - u_1 = (v - u)_1 \geq 0$, $v_2 - u_2 = (v - u)_2 \geq 0$. Hence $v_1 \geq u_1$, $v_2 \geq u_2$. Therefore G^+ is the direct product of partially ordered semigroups A_1 and B_1.

(iii) $\Longrightarrow$ (i). Let G^+ be the direct product of the po-semigroup A_1 and the commutative po-semigroup B_1 and $f(z) = z(A_1) - z(B_1)$ for each $z \in G^+$. Let $x \in G^+$. Then $x(A_1) \in U(0, f(x))$. Let $y \geq 0$, $y \geq f(x)$. Thus $y(B_1) \geq 0$ $y(A_1) + y(B_1) + x(B_1) \geq x(A_1)$. Since $G^+ = A_1 \times B_1$, from the last inequality we get $y(A_1) \geq x(A_1)$ Then $y = y(A_1) + y(B_1) \geq x(A_1)$. Therefore $x(A_1) = 0 \vee f(x)$.□

2. Theorem. *Let f be a stable isometry in a directed group G. Let $A_1 = \{x \in G^+, f(x) = x\}$, $B_1 = \{x \in G^+, f(x) = -x\}$, $A = A_1 - A_1$, $B = B_1 - B_1$. Then the following conditions are equivalent:*

(i) *For each* $x \in G^+$ *there exists the least upper bound of* $\{0, f(x)\}$ *in* G^+.
(ii) *For each* $x \in G^+$ *there exists* $x_1 \in G^+$ *such that* $0 \le x_1 \le x$, $f(x) \le x_1 \le f(x) + x$.
(iii) G *is the direct product of the* po-*group* A *and the abelian* po-*group* B *and* $f(z) = z(A) - z(B)$ *for each* $z \in G$.

Proof. In view of 1 it suffices to verify that (ii) implies (iii) and (iii) implies (i).

(ii) $\Longrightarrow$ (iii). In [**13**] it was proved that A is a group, $A^+ = A_1$ [Lemma 1.8], B is an abelian group, $B^+ = B_1$ [Lemma 1.9] and $f(a+b) = a - b$ for each $a \in A$, $b \in B$ [Theorem 1.13]. By 1, $G^+ = A_1 \times B_1$. Then from Theorem 2.3 [**3**] it follows that $G = A \times B$.

(iii) $\Longrightarrow$ (i). Since $A^+ = A_1$ and $B^+ = B_1$, we get $G^+ = A_1 \times B_1$. Then the desired result follows from 1. □

3. Theorem. *Let* G *be a directed group. Let for each* $x \in G^+$ *there exists* $y \in G^+$ *such that* $x = 2y$. *Then for each stable isometry* f *in* G *there exists a direct decomposition* $G = A \times B$ *of* G *with* B *abelian such that* $f(z) = z(A) - z(B)$ *for each* $z \in G$.

Proof. Let $x \in G^+$. Let $y \in G^+$ such that $x = 2y$. From Theorem 1 [**11**] it follows that $y + f(y) = 0 \vee f(x)$. Then the required statement follows from 2. □

Throughout the rest of this paper let f be a stable weak isometry in a po-group G and let S be the subgroup of G generated by G^+. It is clear that S is a directed convex subgroup of G^+. In [**15**] Shimbireva proved that S is a normal subgroup of G.

4. Theorem. (i) $f(x+y) = f(x) + f(y)$ *for each* $x, y \in S$.
(ii) $f^2(x) = x$ *for each* $x \in S$.
(iii) $f(S) = S$.

Proof. The proof of (i) and (ii) is the same as the proof of analogous propositions in Theorem 3 [**11**] concerning directed groups.

(iii) Let $x \in S$. Then $x = a - b$ for some $a, b \in G^+$. By (i), $f(x) = f(a) - f(b)$. Since $a, b \in G^+$, from the relations $|a| = |f(a)|$, $|b| = |f(b)|$ we get $a \ge f(a)$, $b \ge -f(b)$. Hence $a + b \ge f(a) - f(b) = f(x)$, $a + b \ge 0$. Then $f(x) = (a+b) - [-f(x) + (a+b)] \in S$. In view of (ii) we have $f(S) = S$. □

5. Corollary. *Restriction of a stable weak isometry* f *in a* po-*group* G *to the subgroup* S *is an involutory group automorphism.*

Proof. It is easy to see that each stable weak isometry in G is an injection. Then 4 ends the proof. □

6. Theorem. *Let $x \in G^+$. Then*

(i) $f([0,x]) \subseteq [-x,x] \cap [-x+f(x), x+f(x)]$,
(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $f([0,x]) \subseteq [0 \wedge f(x), 0 \vee f(x)]$.

Proof. (i) Let $y \in [0,x]$. From $|x-y| = |f(x)-f(y)|$ we get $x-y \geq f(y)-f(x)$, $x-y \geq f(x)-f(y)$. Then $f(x)-f(y)+x \geq y \geq 0$, $f(y)-f(x)+x \geq y \geq 0$. Hence $x+f(x) \geq f(y) \geq -x+f(x)$. Since $x \in |y| = |f(y)|$, we have $x \geq f(y) \geq -x$.

(ii) From the assumption it follows that there exist $0 \wedge f(x)$ and $0 \vee -f(x)$ in G, too. From $|x| = |f(x)|$ we have $x = -f(x) \vee f(x)$. Then $[0 \wedge f(x)] + [0 \vee -f(x)] = 0 \vee [f(x) \vee -f(x)] = 0 \vee x = x$. This implies $0 \vee f(x) = x - [0 \vee -f(x)] = x + [0 \wedge f(x)] = x \wedge (x+f(x))$, $0 \wedge f(x) = -[0 \vee -f(x)] = -x + [0 \vee f(x)] = -x \vee [-x+f(x)]$. Then (i) ends the proof. ☐

7. Theorem. *Let $x \in G^+$. Then*

(i) $[-x,x] \cap [-x+f(x), x+f(x)] \subseteq f([0,x])$,
(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $[0 \wedge f(x), 0 \vee f(x)] \subseteq f([0,x])$.

Proof. (i) Let $z \in G$ such that $-x \leq z \leq x$, $-x+f(x) \leq z \leq x+f(x)$. Then $0 \leq z+x \leq 2x$, $0 \leq z-f(x)+x \leq 2x$. According to 4, $f(x) \in S$. By Theorem 1 [**11**], $x+f(x) = 0 \vee f(2x)$, $-x+f(x) = 0 \wedge f(2x)$. Since x, $f(x)$, z are elements of S, from 4 and 6(ii) we get $f(z)+f(x) = f(z+x) \leq x+f(x)$, $f(z)-x+f(x) = f(z-x+f(x)) \geq -x+f(x)$. Thus $0 \leq f(z) \leq x$. In view of 4 we have $f^2(z) = z \in f([0,x])$.

(ii) By the same way as in the proof of 6(ii) we can prove that $0 \vee f(x) = x \wedge [x+f(x)]$, $0 \wedge f(x) = -x \vee [-x+f(x)]$. Then (i) completes the proof. ☐

From 6 and 7 we immediately obtain

8. Theorem. *Let $x \in G^+$. Then*

(i) $f([0,x]) = [-x,x] \cap [-x+f(x), x+f(x)]$,
(ii) *if there exists the least upper bound of the set* $\{0, f(x)\}$ *in* G, *then* $f([0,x]) = [0 \wedge f(x), 0 \vee f(x)]$.

If C is a normal convex subgroup of a po-group H then the factor group H/C can be partially ordered by the induced order. See [**1**, p. 20].

9. Theorem. *The factor group G/S of a po-group G with respect to S is trivially ordered with regard to induced order.*

Proof. Let $S+a \leq S+b$ for some $a,b \in G$. Then there exist $h,g \in S$ such that $h+a \leq g+b$. This yields $0 \leq -h+g+b-a$. Thus $-h+g+b-a \in S$ and hence $b-a \in S$. This implies that $b = (b-a)+a \in S+a$. Therefore $S+a = S+b$. ☐

10. Theorem. *Let $a \in G$. Then $f(x) = \overline{h}(x - a) + f(a)$ for each $x \in S + a$, where $\overline{h}$ is a stable weak isometry in S.*

Proof. Let $h(z) = f(z + a) - f(a)$ for each $z \in G$. Then h is a stable weak isometry in G. By 4, $h(S) = S$. Thus the restriction $\overline{h}$ of h to S is a stable weak isometry in S. Let $x \in S + a$. Hence $x = y + a$ for some $y \in S$. Then $\overline{h}(y) = \overline{h}(x - a) = f(x) - f(a)$. From this we get $f(x) = \overline{h}(x - a) + f(a)$. ☐

11. Theorem. *Let $a \in G$. Then $f(S + a) = S + f(a)$.*

Proof. From 10 it follows that $f(S+a) \subseteq S+f(a)$. Let $z = t+f(a)$ where $t \in S$. By 10, $f(x) = \overline{h}(x-a)+f(a)$ for each $x \in S+a$ where $\overline{h}$ is a stable weak isometry in S. Since $z - f(a) \in S$, then there exists $u \in S$ such that $\overline{h}(u) = z - f(a)$. In view of 10 we have $f(u + a) = \overline{h}(u) + f(a) = z$. Therefore $S + f(a) \subseteq f[S + a]$.☐

12. Theorem. *Let $S + a$, $S + b \in G/S$ such that $S + a \neq S + b$. Then $f(S + a) \neq f(S + b)$.*

Proof. Since f is an injection, it follows from 11. ☐

13. Theorem. *If the factor group G/S is finite, then f is a bijection.*

Proof. Since f is an injection, the desired assertion follows from 12. ☐

Remark. There exist a nontrivially ordered group H and a stable weak isometry in H which is not a bijection.

References

1. Fuchs L., *Partially ordered algebraic systems*, Pergamon Press, Oxford, 1963.

2. Holland Ch., *Intrinsic metrics for lattice ordered groups*, Algebra Universalis **19** (1984), 142–150.

3. Jakubík J., *Direct decompositions of partially ordered groups II.*, Czech. Math. J. **11** (1961), 490–515. (Russian)

4. ______, *Isometries of lattice ordered groups*, Czech. Math. J. **30** (1980), 142–152.

5. ______, *On isometries of non-abelian lattice ordered groups*, Math. Slovaca **31** (1981), 171–175.

6. ______, *Weak isometries of lattice ordered groups*, Math. Slovaca **38** (1988), 133–138.

7. Jakubík J. and Kolibiar M., *Isometries of multilattice groups*, Czech. Math. J. **33** (1983), 602–612.

8. Jasem M., *Isometries in Riesz groups*, Czech. Math. J. **36** (1986), 35–43.

9. ______, *On weak isometries in multilattice groups*, Math. Slovaca **40** (1990), 337–340.

10. ______, *On isometries in partially ordered groups*, Math. Slovaca **43** (1993), 21–29.

11. ______, *Weak isometries in directed groups*, Math. Slovaca, (to appear).

12. ______, *Isometries in non-abelian multilattice groups*, (submitted).

13. ______, *Weak isometries and isometries in partially ordered groups*, (submitted).

14. Rachůnek J., *Isometries in ordered groups*, Czech. Math. J. **34** (1984), 334–341.

15. Shimbireva H., *On the theory of partially ordered groups*, Mat. Sb. **20**(62) (1947), 145–178. (Russian)

16. Swamy K. L. N., *Isometries in autometrized lattice ordered groups*, Algebra Universalis **8** (1978), 59–64.

17. ______, *Isometries in autometrized lattice ordered groups II*, Seminar Notes Kobe Univ. **5** (1977), 211–214.

M. Jasem, Department of Mathematics, Faculty of Chemical Technology, Slovak Technical University, Radlinského 9, 812 37 Bratislava, Slovakia