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STRICTLY ORDER PRIMAL ALGEBRAS

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Partial orders and the clones of functions preserving them have been thoroughly studied in recent years. The topic of this papers is strict orders which are irreflexive, asymmetric and transitive subrelations of partial orders. We call an algebra $\mathcal{A} = (A, \Omega)$ strictly order primal if for some strict order $(A, <)$ the term functions are precisely the functions which preserve this strict order. Our approach has some parallels to the theory of order primal algebras [**8**], [**2**]. We present new examples of congruence distributive varieties and of strict orders without near unanimity operations. Then we give a series of new examples showing that there are varieties which are $(n+1)$-permutable but not n-permutable. Furthermore the dual category of strict chains is described by the methods from B. Davey and H. Werner [**3**]. Throughout we use the notations of Grätzer [**4**] and assume a knowledge of Davey-Werner [**3**] for the last section.

1. Notation

Definition 1.1. A binary relation $<$ on A is called a strict order if the following properties hold.

(i) $a \not< a$ for every $a \in A$ (irreflexivity)
(ii) if $a < b$ then $b \not< a$ for all $a, b \in A$ (asymmetry)
(iii) if $a < b$ and $b < c$ then $a < c$ for all $a, b, c \in A$ (transitivity).

To every strict order we can define a partial order $a \le b$ iff $a < b$ or $a = b$ and vice versa from every partial order we can define a strict order. But we would like to call the attention of the reader to the fact that the product of strict orders is defined according to the principles in clone theory.

Definition 1.2. Let $(A; <_A)$ and $(B; <_B)$ be strict orders. Then the strict order $(A \times B; <)$ is defined componentwise in the following way. $(a_1, b_1) < (a_2, b_2)$ if and only if $a_1 <_A a_2$ and $b_1 <_B b_2$.

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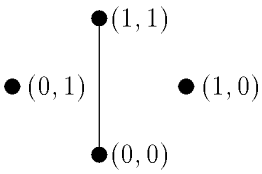
We present as an example the strict order $D_2^2 = (\{0,1\}^2; <)$ by its Hasse diagram

D_2^2



Definition 1.3. A function $f\colon A \longrightarrow B$ from a strict order $(A; <_A)$ into a strict order $(B; <_B)$ is called strictly monotone if from $a_1 <_A a_2$ it follows $f(a_1) <_B f(a_2)$.

One can observe in the above example four strictly monotone functions $f\colon D_2^2 \longrightarrow D_2$ which can be presented by the four term functions x, y, xy, xy of the lattice connected to $D_2 = (\{0,1\}; <)$. We will write $a \prec b$ in $(A; <)$ if $a < b$ in $(A; <)$ and there exists no $c \in A$ with $a < c < b$.

$Pol_n <$ is the set of all strictly monotone functions $f\colon A^n \longrightarrow A$ and $Pol <= \cup_{n \in \mathbb{N}} Pol_n <$ is the clone of all strictly monotone functions of $(A; <)$. Given an algebra $\mathcal{A} = (A, \Omega)$ we write $T_n(\mathcal{A})$ for the set of all n-place term functions of $\mathcal{A}$ and $T(\mathcal{A})$ for the clone of term functions of $\mathcal{A}$.

2. Strictly Monotone Funtions on a Chain

Notation 2.1. Assume that A has no infinite chains. The length $\ell(a,b)$ for $a, b \in A$ with $a < b$ is defined to be one less than the number of elements in a chain of maximum size from a to b. Extend ℓ to a function $\ell\colon A^2 \longrightarrow \mathbb{N}_0$ by defining $\ell(a,b) = 0$ whenever $a \not< b$. A function $f\colon A^n \longrightarrow A$ is called length preserving if $\ell(a,b) \leq \ell(f(a), f(b))$ for all $a, b \in A^n$.

Lemma 2.2. *Assume that A has no inifinte chains. The function $f\colon A^n \longrightarrow A$ is strictly monotone on $(A; <)$ if and only if f is length preserving.*

Proof. Let $f \in Pol <$, $a < b$ and $\ell(a,b) = r$. Then there exists a maximal chain $a = a_0 \prec a_1 \prec \ldots \prec a_r = b$ implying $f(a) < f(a_1) < \ldots < f(b)$. Hence $\ell(f(a), f(b)) \geq r$. On the other hand if f is length preserving and $a < b$ then $1 \leq \ell(a,b) \leq \ell(f(a), f(b))$ and hence $f(a) < f(b)$. $\square$

From now on we assume that $(A; <)$ is a finite chain where $A = \{0, 1, \ldots, k\}$ and $0 \prec 1 \prec \ldots \prec k$.

Proposition 2.3. *The length function ℓ on A^n satiesfies $\ell(0, a) = \min\{a_1, \ldots, a_n\}$ where $a = (a_1, \ldots a_n)$.*

Proof. One observes that for $a_m = \min\{a_1, \dots, a_n\}$ the chain $a_m \succ a_{m-1} \succ \dots \succ 0$ corresponds to a maximal chain for $0 < a$. □

Notation 2.4. Denote the top element of A^n by $\overline{k}$. Let $c \in A^n$, $c = (c_1, \dots, c_n)$ and $r \in A$ with $\ell(0,c) \leq r \leq k - \ell(c, \overline{k})$. We define the function $g_c^r \colon A^n \longrightarrow A$ by

(i) $g_c^r(x) = \max\{r + \ell(c,x), \ell(0,x)\}$ whenever $x > c$ or $x = c$
(ii) $g_c^r(x) = \ell(0,x)$ whenever $x \not> c$ and $x \neq c$.

Note that $g_c^r(x) \colon A^n \longrightarrow A$ is strictly monotone.

Lemma 2.5. *Every strictly monotone function $f \colon A^n \longrightarrow A$ can be obtained by composing the function* max *and g_c^r for $c \in A^n$ and $r \in A$.*

Proof. We consider $C = \{c \in A^n \mid f(c) > \ell(0,c)\}$ as an index set to define $h(x) = \max\{g_c^r(x) \mid c \in C, r = f(c)\}$.

We have to show that $h(x) = f(x)$ for all $x \in A^n$. Consider an n-tuple $x \in A^n$. We have $\ell(0,x) \leq f(x)$ and hence $g_c^r(x) \leq f(x)$ whenever $x \not> c$ and $x \neq c$. For $x = c$ we have $g_c^r(x) = f(x)$ and for $x > c$ we have

$$\begin{aligned} g_c^r(x) &= \max\{r + \ell(c,x), \ell(0,x)\} \\ &= \max\{f(c) + \ell(c,x), \ell(0,x)\} \\ &\leq \max\{f(c) + \ell(f(c), f(x)), \ell(0, f(x))\} \\ &\leq f(x) \end{aligned}$$

as f is length preserving.

Altogether we have $g_c^r(x) \leq f(x)$ and for some c we have $g_c^r(x) = f(x)$. □

Lemma 2.6. *Every function $g_c^r \colon A^n \longrightarrow A$ can be composed by functions $g_d^r \colon A^3 \longrightarrow A$ and* min.

Proof. We assume $n \geq 3$ and we put for $c = (c_1, \dots, c_n)$

$$D := \{d \in \{c_1, \dots, c_n\}^3 \mid \ell(0,d) \leq r \leq k - \ell(d, \overline{k})\}$$

We define a function h by

$$h(x) = \min\{g_c^r(\overline{x}) \mid d \in D\}$$

with $\overline{x} = (x_i, x_j, x_l)$ corresponding to $d = (c_i, c_j, c_l)$.

Our aim is to prove that $h(x) = g_c^r(x)$ for all $x \in A^n$.

Consider an n-tuple $x \in A^n$. If we have $\overline{x} > d$ or $\overline{x} = d$ then we have

$$\begin{aligned} g_d^r(\overline{x}) &= \max\{r + \ell(d, \overline{x}), \ell(0, \overline{x})\} \\ &\geq \max\{r + \ell(c,x), \ell(0,x)\} \\ &\geq g_c^r(x) \end{aligned}$$

If we have $\overline{x} \not> d$ and $\overline{x} \neq d$ then we have $x \not> c$ and $x \neq c$ and we get

$$g_d^r(\overline{x}) = \ell(0, \overline{x}) \geq \ell(0, x) = g_c^r(x).$$

It remains to show that for some d we have

$$g_d^r(\overline{x}) = g_c^r(x).$$

Case: $x = c$.
In this case we have $\overline{x} = d$ and so $g_d^r(\overline{x}) = g_c^r(x)$.

Case: $x > c$.
Let $c = a_0 \prec a_1 \prec \ldots \prec a_q = x$ be a maximal chain from c to x. Then for some component m we have a chain $c_m = a_{0m} \prec a_{1m} \prec a_{2m} \prec \ldots \prec a_{qm} = x_m$ and for all other components there exist chains of equal or larger length. Let i be an index such that $x_i = \ell(0, x)$. We now take an index j with $c_j = \ell(0, c)$ if $c_i \geq r$ and $c_j = k - \ell(c, \overline{k})$ else. Obviously $d = (c_i, c_j, c_m) \in D$. We have

$$\begin{aligned} g_d^r(\overline{x}) &= \max\{r + \ell(d, \overline{x}), \ell(0, \overline{x})\} \\ &= \max\{r + \ell(c, x), \ell(0, x)\} \\ &= g_c^r(x). \end{aligned}$$

Case: $x \not> c$ and $x \neq c$.
In this case there exist $h, l \in \{1, \ldots, n\}$ such that $x_h \not> c_h$ and $x_l \neq c_l$. Let s be an index such that $x_s = \min\{x_1, \ldots, x_n\}$. For $c_s \geq r$ let $c_i = \min\{c_1, \ldots, c_n\}$ else $c_i = \max\{c_1, \ldots, c_n\}$. If $c_s \neq x_s$ then we choose $d = (c_s, c_h, c_i)$ else $d = (c_s, c_l, c_i)$. We have also $d \in D$,$d \not< \overline{x}$ and $d \neq \overline{x}$. Hence

$$g_d^r(\overline{x}) = x_s = g_c^r(x).$$

□

3. Strictly Order Primal Algebras

Definition 3.1. The algebra $\mathcal{A} = (A, \Omega)$ is called n-sop for $n \in \mathbb{N}$ if $T_n(\mathcal{A}) = Pol_n <$ for some strict order $<$. We call $\mathcal{A}$ a sop (: = **s**trictly **o**rder **p**rimal) algebra if $\mathcal{A}$ is n-sop for every $n \in \mathbb{N}$.

Examples.

3.2. Every lattice L where L is a chain is 1-sop. Observe that the identity function id is the only function which preserves this strict order.

3.3. A non trivial lattice L is 2-sop if and only if L is isomorphic to the two-element distributive lattice D_2. D_2 is not 3-sop.

3.4. Consider an algebra $G = (\{0, 1\}; \wedge, \vee, q)$ where $q(x_1, x_2, x_3) = x_1 + x_2 + x_3$ with the addition mod 2. The algebra G is 3-sop.

The following lemma can be proved in a similar way as the analogous lemma in [**8**].

Lemma 3.5. *If $\mathcal{A}$ is n-sop then $\mathcal{A}$ is k-sop for $1 \leq k \leq n$.*

Using Lemma 2.6 above we have the following result:

Theorem 3.6. *Let $\mathcal{A}$ be an algebra with $T(\mathcal{A}) \subseteq Pol <$ where $>$ is a finite chain. Then $\mathcal{A}$ is sop if and only if $\mathcal{A}$ is 3-sop.*

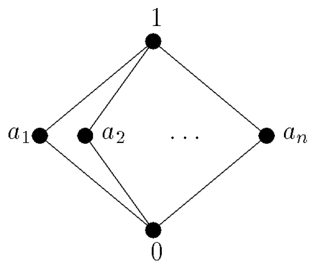
Examples.

3.7. The algebra $G = (\{0,1\}; \wedge, \vee, q)$ is sop.

3.8. The algebra $G = (\{0,1,\dots,k\}; Pol_3 <)$ is sop.

3.9. By a similar method one can show for the "projective line" $\mathcal{M}_n$; $n \geq 2$, that $\mathcal{M}_n = (M_n; Pol_3 <)$ is sop.

$(M_n; <)$



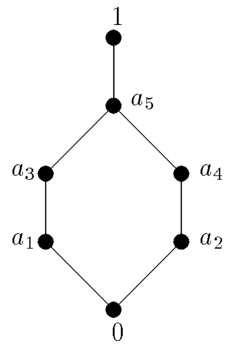
Remark 3.10. In an unpublished manuscript [**6**] it has been shown that for the strict orders $(\mathbb{Q}; <)$ and $(\mathbb{R}; <)$ the clone $Pol <$ is locally preprimal. $Pol <$ includes the following operations: $x + y$, $c \cdot x$ with $c \in \mathbb{Q}^+$ or $c \in \mathbb{R}^+$ respectively, and $\min\{x, y\}$, $\max\{x, y\}$. These algebras are called locally sop.

4. Congruence Distributivity and n-permutability

In our examples the existence of near unanimity operations play an important role. Therefore we would like to present two examples of strict orders which cause major obstacles to proofs like those above.

Examples.

4.1. The strict order which is induced by the following lattice order does not admit a majority function. This is the smallest example with this property.



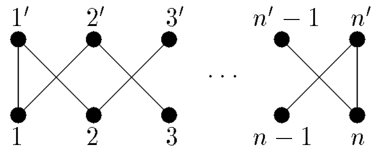
Observe that for a majority function h we would have that

$$h(a_3, a_4, a_5) < h(a_5, a_5, 1\ ) = a_5$$
$$h(a_3, a_4, a_5) > h(a_1, a_2, a_1) = a_1$$
$$h(a_3, a_4, a_5) > h(a_1, a_2, a_2) = a_2$$

in contradiction to the fact that there is no element x with $a_1, a_2 < x < a_5$.

4.2. Let $<$ be the strict order induced by the lattice order $(B_3; <)$ where B_3 is the Boolean lattice with 8 elements. Then there exists no near unanimity function in $Pol <$. This again is the smallest example with this property.

Notation 4.3. We consider the following strict order "zig zag" (D, D'), $D = \{1, \ldots, n\}$, $D = \{1', \ldots, n'\}$, presented by the Hasse diagram



A zig zag line is a sequence $a_0 < a_1 > a_2 < a_3 \ldots \lessgtr a_n$ where $a_i < a_{i+1}$ is a lower respectively $a_i > a_{i+1}$ is an upper neighbor of a_{i+1} in the zig zag (D, D'), $i = 1, \ldots, n-1$.

Theorem 4.4. *If $\mathcal{A}$ is a sop algebra with a zig zag (D, D'), $n \geq 3$ as a subalgebra then $\mathcal{A}$ generates a variety which is not congruence distributive.*

Proof. We carry out the proof for n even; the case when n is odd can be treated similary. Let n be an even integer, $n \geq 4$. Then there is a unique shortest zig zag line from 1 to n' and it has n elements. We show the following conditions for terms $d, t \in T(\mathcal{A})$.

$$\alpha)\quad \left.\begin{aligned} d(1,1,2) &= 1\\ d(1',1',2') &= 1'\\ d(x,y,x) &= x \end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned} d(1',2',2') &= 1'\\ d(1,2,2) &= 1 \end{aligned}\right.$$

$$\beta)\quad \left.\begin{aligned} t(1',2',2') &= 1'\\ t(1,2,2) &= 1\\ t(x,y,x) &= x \end{aligned}\right\} \quad \text{implies} \quad \left\{\begin{aligned} t(1,1,2) &= 1\\ t(1',1',2') &= 1' \end{aligned}\right.$$

From $1 = d(1,1,2) < d(2',1',3') > d(3,2,4) < d(4',3',5') > \ldots > d(n-1,n-2,n) < d(n',n'-1,n') = n'$ we conclude that $d(i,i-1,i+1) = i$ for $1 < i < n$ or respectively $1' < i < n'$. Especially we have $d(2',1',3') = 2'$ and furthermore $d(1,2,2) < d(2',1',3') = 2'$, $d(1',1',1') = 1'$. Hence we have $d(1,2,2) = 1$. In the same we get $d(1',2',2') = 1'$. Hence α) is proved.

Again we use that there is a unique shortest zig zag line from $1'$ to n and it has n elements and consider $1' = t(1',2',2') > t(2,1,3) < t(3',2',4') < \ldots > t(n,n-1,n) = n$ which implies $t(2,1,3) = 2$. Furthermore we have $t(1',1',2') > t(2,1,3) = 2$ and $t(1,1,1) = 1$. We conclude that $t(1',1',2') = 1'$ and in the same way that $t(1,1,2) = 1$. Hence β) is proved.

Now we assume that the variety generated by $\mathcal{A}$ is congruence distributive. Then there are ternary terms $t_0, \ldots, t_k$ with $t_0(x,y,z) = x$, $t_k(x,y,z) = z$, $t_i(x,y,x) = x$ $(0 \le i \le k)$, $t_i(x,x,z) = t_{i+1}(x,x,z)$ for i even and $t_i(x,z,z) = t_{i+1}(x,z,z)$ for i odd. Because of the implications α) and β) we have $t_{k-1}(1,2,2) = 1$ for $k-1$ odd or respectively $t_{k-1}(1,1,2) = 1$ for $k-1$ even. This contradicts the conditions for congruence distributivity.

Corollary 4.5. *If $\mathcal{A}$ is a sop algebra with a zig zag (D, D'), $n \ge 3$, as a subalgebra then the variety generated by $\mathcal{A}$ has no near unanimity term.*

Lemma 4.6. *Let $<$ be a bounded strict order on A with a maximal chain $0 \prec 1 \prec 2 \ldots \prec n$. If $\mathcal{A}$ is a sop algebra with $T(\mathcal{A}) = Pol <$ then the variety generated by $\mathcal{A}$ is not n-permutable.*

Proof. If $\mathcal{A}$ generates a variety with n-permutable congruences then there exist ternary terms $p_0, p_1, \ldots, p_n$ such that $p_0(x,y,z) = x$, $p_n(x,y,z) = z$ and $p_i(x,x,y) = p_{i+1}(x,y,y)$ $(0 \le i \le n)$. For $x, y \in \{0, \ldots, n\}$ and $n > x > y$ we have $p_i(x,x,y) = p_{i+1}(x,y,y) < p_{i+1}(x+1,x+1,y+1)$. Hence we have $1 = p_0(1,1,0) < p_1(2,2,1) < p_2(3,3,2) < \ldots < p_{n-1}(n-2,n-2,n-1) = p_n(n-2,n-1,n-1) = n-1$. This contradicts $\ell(1,n-1) = n-2$. □

Lemma 4.7. *Let $\mathcal{A}$ be an algebra on $A = \{0, 1, \ldots, n\}$. If $\mathcal{A}$ is sop with respect to the strict order of the chain $0 \prec 1 \prec \ldots \prec n$, then the variety generated by $\mathcal{A}$ is $(n+1)$-permutable.*

Proof. Let $A_i = \{(t,s,s) \mid s = i-1,\ t \geq 1\}$ and $B_i = \{(u,u,w) \mid u = n-i,\ w \geq n-i+1\}$. By Lemma 2.5 the function $p_i(x,y,z) = \max\{g_c^r(x,y,z) \mid c = (c_1,c_2,c_3) \in A_i \cup B_i,\ r = \max\{c_1,c_3\}\}$ is a term function of $\mathcal{A}$ for $i = 1,\dots,n$. This function p_i has the following properties

$$p_i(x,y,y) = \begin{cases} x & x > y \geq i-1 \quad 1) \\ y & y > x > n-i \quad 2) \\ \min\{x,y\} & \text{else} \quad 3) \end{cases}$$
$$p_i(x,x,y) = \begin{cases} x & x > y > i-1 \quad 4) \\ y & y > x \geq n-i \quad 5) \\ \min\{x,y\} & \text{else} \quad 6) \end{cases}$$

0) We note that for $x \leq y$ we have $x \leq p_i(x,y,y)$, $p_i(x,x,y) \leq y$ as p_i is a term function or respectively for $y \leq x$, $y \leq p_i(x,y,y)$, $p_i(x,x,y) \leq x$.

1) For $x > y \geq i-1$ there exists $(t,s,s) \in A_i$ such that$((t = x)$ and $(s = y))$ or $((t = x-1)$ and $(s < y))$. Hence we have $g^t_{(t,s,s)}(x,y,y) = x \leq p_i(x,y,y)$ which implies $p_i(x,y,y) = x$ by 0).

2) For $y > x > n-i$ there exists $(u,u,w) \in B_i$ such that $u < x$ and $w = y-1$. Therefore $g^w_{(u,u,w)}(x,y,y) = y \leq p_i(x,y,y)$ which implies $p_i(x,y,y) = y$ by 0).

3) If neither $x > y \geq i-1$ nor $y > x > n-i$ holds then there exists no element $c \in A_i \cup B_i$ such that $(x,y,y) \geq c$. Then $g_c^r(x,y,y) = \min\{x,y\}$ for every $c \in A_i \cup B_i$. Hence $p_i(x,x,y) = \min\{x,y\}$.

4) For $x > y > i-1$ there exists $(t,s,s) \in A_i$ such that $t = x-1$ and $s < y$. This implies $g^t_{(t,s,s)}(x,x,y) = x \leq p_i(x,x,y)$ which implies $p_i(x,x,y) = x$ by 0).

5) For $y > x \geq n-i$ there exists $(u,u,w) \in B_i$ such that $((x = u)$ and $(y = w))$ or $((u < x)$ and $(w = y-1))$. Hence we have $g^w_{(u,u,w)}(x,x,y) = y \leq p_i(x,x,y)$ which implies $p_i(x,x,y) = y$.

6) If neither $x > y > i-1$ nor $y > x \geq n-i$ hold then there exists no element $c \in A_i \cup B_i$ such that $(x,y,y) \geq c$. Then $g_c^r(x,y,y) = \min\{x,y\}$ for every $x \in A_i \cup B_i$. Hence $p_i(x,x,y) = \min\{x,y\}$.

Theorem 4.8. *Let $<$ be a bounded strict order on $A = \{0,1,\dots,n\}$ with a maximal chain $0 \prec 1 \prec 2 \prec \dots \prec n$. If $\mathcal{A}$ is a sop algebra with $T(\mathcal{A}) = Pol <$ then the variety generated by $\mathcal{A}$ is $(n+1)$-permutable but not n-permutable.*

G. Grätzer asks for examples of varieties which show that n-permutability and $(n+1)$-permutability are not equivalent. (E. T. Schmidt [**7**]) The above theorem provides a new series of such examples.

5. A Duality for a Finite Sop Algebra

In this section we use notions and methods which were developed in B. Davey and H. Werner [**3**]. We consider the finite sop algebra $\mathcal{A}$ where $(A;<)$ is the strict order induced by a k-element chain. Since $\mathcal{A}$ has a ternary near-unanimity term, the NU-duality theorem from [**3**] guarantees that the prevariety generated by $\mathcal{A}$ has a duality given by relations of arity at most two. Theorem 5.1 isolates an appropriate set of relations.

Theorem 5.1. *Let $<$ be the strict order which is induced by a k-element chain, $A = \{0, \dots, k-1\}$, $\mathcal{A} = (A, \Omega)$ with $T(\mathcal{A}) = Pol <$, $\mathcal{L} = ISP(\mathcal{A})$, $\tilde{A} = (A; \tau, <, < \circ <, \dots, \underbrace{< \circ < \circ \dots \circ <}_{k-1}, 0, 1, \dots k-1)$ and $\mathcal{R}_0 = ISP(\tilde{A})$. Then the protoduality is a full duality between $\mathcal{L}$ and $\mathcal{R}_0$.*

Proof. By [**3**] (Davey, Werner) we have to show that $\mathcal{A}$ is injective in $\mathcal{R}_0$ (INJ) and fulfills condition (E3F).

(INJ) Let $\tilde{X} \subseteq \tilde{A}^n$ and let $\varphi \colon \tilde{X} \longrightarrow \tilde{A}$ be a morphism. Then φ preserves the relations $<, < \circ <, \dots, \underbrace{< \circ < \circ \dots \circ <}_{k-1}, 0, 1, \dots k-1$. Note that $(\overline{x}, \overline{y}) \in \underbrace{< \circ \dots \circ <}_{i}$ iff $\ell(\overline{x}, \overline{y}) \geq i$ in $(A^n, <)$. Hence φ has the following properties

(i) for $\overline{x} = (x_1, \dots, x_n) \in X$ we have: if $\min\{x_1, \dots, x_n\} = x_i \neq 0$ then $x_i - 1 = \varphi(x_i - 1, x_i - 1, \dots, x_i - 1) < \varphi(x_1, \dots, x_n)$.
Hence $\min\{x_1, \dots, x_n\} \leq \varphi(x_1, \dots, x_n)$. Similary we get $\varphi(x_1, \dots, x_n) \leq \max\{x_1, \dots, x_n\}$.

(ii) for $\overline{x} = (x_1, \dots, x_n)$, $\overline{y} = (y_1, \dots, y_n) \in X$, $\overline{x} < \overline{y}$ and $\ell(\overline{x}, \overline{y}) = r$ in $(A^n, <)$ we have $(\overline{x}, \overline{y}) \in \underbrace{< \circ \dots \circ <}_{r}$ and hence $\varphi(x_1, \dots, x_n) + r \leq \varphi(y_1, \dots, y_n)$.

Consequently we can extend φ by Lemma 2.5 to a length function $\psi \colon A^n \longrightarrow A$ in the following way

$$\psi(\overline{x}) = \begin{cases} \varphi(\overline{x}) & \text{if } \overline{x} \in X \\ \max\{\ell(\overline{y}, \overline{x}) + \varphi(\overline{y}, \min\{x_1, \dots x_n\})\} & \text{if } \overline{x} \notin X \text{ and } \exists\, \overline{y} \in X : \overline{y} < \overline{x} \\ \min\{x_1, \dots, x_n\} & \text{otherwise} \end{cases}$$

(E3F) Let $X \subset Y \subseteq A^n$, $a \in Y \setminus X$. Then the morphisms $\varphi, \psi \colon Y \longrightarrow A$ defined by

$$\varphi(\overline{x}) = \begin{cases} \min\{x_1, \dots, x_n\} & \text{for } \overline{x} \ngeq a \\ \max\{x_1, \dots, x_n\} & \text{for } \overline{x} \geq a \end{cases}$$
$$\psi(\overline{x}) = \begin{cases} \min\{x_1, \dots, x_n\} & \text{for } \overline{x} \not> a \\ \max\{x_1, \dots, x_n\} & \text{for } \overline{x} > a \end{cases}$$

are such that $\varphi|X = \psi|X$ but $\varphi \neq \psi$. $\square$

Remark. Since we have a ternary near unanimity function h with $h(x, y, z) = \max\{\min\{x, y\}, \min\{x, z\}, \min\{y, z\}\}$ in $T(\mathcal{A})$ the variety $V(\mathcal{A})$ is congruence distributive. Furthermore $\mathcal{A}$ has only simple subalgebras. Hence by B. Jónsson's theorem we obtain $V(\mathcal{A}) = ISP(\mathcal{A})$, i.e. the full duality of Theorem 5.1 for the quasivariety generated by $\mathcal{A}$ is also a full duality for $V(\mathcal{A})$.

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