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ADJOINTS OF SOLUTION SEMIGROUPS AND IDENTIFIABILITY OF DELAY DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. The paper deals with semigroups of operators associated with delay differential equation:

$$\dot{x}(t) = Ax(t) + L_1 x(t-h) + L_2 x_t,$$

where A is the infinitesimal generator of an analytic semigroup on a Hilbert space X and L_1, L_2 are densely defined closed operators in X and $L^2(-h,0;X)$ respectively.

The adjoint semigroup of the solution semigroup of the delay differential equation is characterized. Eigenspaces of the generator of the adjoint semigroup are studied and the identifiability of parameters of the equation is given.

1. Introduction

The purpose of this paper is to consider the delay differential equation (DDE) of the form:

$$\begin{aligned} &\dot{x} = Ax(t) + A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s)\, ds, \qquad t > 0 \\ &x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad \text{a.e. on } [-h, 0) \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup on a Hilbert space X, A_1 and A_2 are densely defined closed operators in X and $a(\cdot)$ is scalar valued function.

Equations of this type were considered by Di Blasio, Kunisch and Sinestrari in [**5**], [**6**]. They have given results on existence, uniqueness and stability of the solution. In the study of DDE (1.1) in $\mathbb{R}^n$ Bernier, Delfour and Manitius have introduced so-called structural operators in order to describe the evolution of the trajectories as well as to characterize the adjoint semigroup of the solution semigroup of (1.1); see [**3**], [**4**]. Their results have been later generalized to infinite-dimensional spaces, see [**8**], [**13**], [**15**], [**19**] for example. The adjoint semigroup for

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the DDE (1.1) where $a(\cdot)$ is Hoelder's continuous and A is defined by a sesquilinear form was first characterized by Tanabe in [**19**]; (see also [**7**] and [**8**]). In [**16**] Nakagiri and Yamamoto solved the identifiability problem for (1.1) where A_1, A_2 are bounded operators in X. It is an inverse problem, its objective is to show the injectivity of the parameter to the solution mapping. In [**7**] Yeong showed the identifiability of parameters for the case where $A = A_1 = A_2$.

In this paper the characterization of the adjoint semigroup of DDE (1.1) is obtained in a different way by using Yosida — type approximations. Here it is assumed that $a(\cdot)$ is a square integrable function and A generates a bounded analytic semigroup on X. In the second part of the paper, eigenspaces of the infinitesimal generator of the adjoint semigroup are considered. It is shown that the identifiability results of Nakagiri and Yamamoto [**16**] can be generalized for the case where A_1 and A_2 are unbounded operators in X.

The notation is as follows. Let X denote a complex Banach space with norm $\|\cdot\|_X$. For real numbers $a < b$, $L^2(a, b; X)$ denotes the vector space (of equivalence classes) of strongly measurable functions x from $[a, b]$ to X such that $t \to \|x(t)\|_X^2$ is Lebesgue integrable on $[a, b]$. If X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ then $L^2(a, b; X)$ is a Hilbert space with inner product

$$\langle x, y \rangle_{L^2(a,b;X)} = \int_a^b \langle x(t), y(t) \rangle \, dt.$$

The space of continuous functions on $[a, b]$ with values in X is denoted by $C(a, b; X)$ and $W^{1,2}(a, b; X)$ denotes the space of absolutely continuous functions f from $[a, b]$ to X with $\dot{f} \in L^2(a, b; X)$. Given a function x from $[-h, \tau]$ to X and $t \in [0, \tau]$, a segment of a trajectory x is defined by $x_t(s) := x(t + s)$ for $s \in [-h, 0]$.

If Y is a Banach space, then $\mathscr{L}(X, Y)$ is the space of linear bounded operators from X to Y and the inclusion $X \hookrightarrow Y$ means that X is continuously and densely embedded in Y. If $A\colon D(A) \subset X \to X$ is a closed linear operator, $D(A)$ will be regarded as a normed space equipped with the graph norm $\|x\|_{D(A)} := (\|x\|_X^2 + \|Ax|_X^2)^{1/2}$. As usual $R(\lambda, A) = (\lambda I - A)^{-1}$ for every $\lambda \in \rho(A)$ — the resolvent set of A. The spectrum and the kernel of A are denoted by $\sigma(A)$ and $\ker(A)$ respectively.

If A is the infinitesimal generator of an analytic semigroup $S(t)$, then the intermediate vector space V between $D(A)$ and X is defined as follows

$$V := \left\{ v \in X \mid \int_0^\infty \|AS(t)v\|^2 \, dt < \infty \right\} \tag{1.2}$$

with norm

$$\|v\|_V := \left(\|v\|^2 + \int_0^\infty \|AS(t)v\|^2 \, dt \right)^{1/2}. \tag{1.3}$$

The following relations are satisfied: $D(A) \hookrightarrow V \hookrightarrow X$. It is known that the space $L^2(0,\tau;D(A)) \cap W^{1,2}(0,\tau;X)$ is continuously embedded in $C(0,\tau;V)$ so that there exists c_0 such that

$$\|x\|_{C(0,\tau;V)} \le c_0 \|x\|_{L^2(0,\tau;D(A))\cap W^{1,2}(0,\tau;X)} \tag{1.4}$$

for each $x \in L^2(0,\tau;D(A)) \cap W^{1,2}(0,\tau;X)$. For the details see e.g. [**10**, p. 23].

2. Solution Semigroups

Let X be a Hilbert space with norm $\|\cdot\|$ and let $A\colon D(A) \subset X \to X$ be the infinitesimal generator of a bounded analytic semigroup $\{S(t); t \ge 0\}$ on X. Let $A_i \in \mathscr{L}(D(A), X)$ for $i = 1,2$, and let V be the intermediate space, $D(A) \subset V \subset X$, defined by (1.2) and (1.3).

We consider the delay differential equation (DDE):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s)\, ds \\ x(0) &= \phi^0, \quad x(s) = \phi^1(s) \quad \text{a.e. on } [-h,0), \end{aligned} \tag{2.1}$$

for almost every $t \in (0,\tau)$ and $\phi = (\phi^0, \phi^1)$ an element from the product space $M^2 := X \times L^2(-h,0;X)$. We assume $h > 0$, $\tau > 0$ and $a \in L^2(-h,0;\mathbb{R})$.

This form of the DDE (2.1) was studied by Di Blasio, Kunish and Sinestrari in [**5**], [**6**]. By Theorems 3.3 and 4.1 in [**5**] we have the following result: For every $\phi \in Z := V \times \mathscr{H} := V \times L^2(-h,0;D(A))$ the solution x of (2.1) exists uniquely and the following estimate holds

$$\|x\|_{L^2(0,\tau;D(A))\cap W^{1,2}(0,\tau;X)} \le c_1 (\|\phi^0\|_V + \|\phi^1\|_{\mathscr{H}}),$$

for some constant c_1 dependent of τ. Moreover, the family of operators $\{T(t); t \ge 0\}$ defined by

$$T(t)\phi := (x(t), x_t)$$

is a strongly continuous semigroup on Z.

In order to obtain the characterization of the adjoint semigroup $T^*(t)$ of the solution semigroup $T(t)$, we introduce approximative DDEs of the equation (2.1).

We define approximating bounded operators by using the resolvent $R(\lambda, A)$ of the operator A:

$$R_\lambda = \lambda R(\lambda, A), \quad A_{1\lambda} = A_1 R_\lambda \quad \text{and} \quad A_{2\lambda} = A_2 R_\lambda,$$

for $\lambda > 0$. Let us note that $\|R_\lambda\|_{\mathscr{L}(X)} \le M < \infty$ and $\lim_{\lambda \le \infty} \|R_\lambda x - x\| = 0 \; \forall\, x \in X$. Thus the following estimates hold:

$$\lim_{\lambda\to\infty} \|A_{i\lambda} x - A_i x\| = 0 \quad \text{for } x \in D(A) \quad \text{and} \quad i = 1,2. \tag{2.2}$$

We consider the following approximative delay differential equation (ADDE):

$$\begin{aligned} (2.3) \qquad & \dot{x}_\lambda(t) = Ax_\lambda(t) + A_{1\lambda}x_\lambda(t-h) + \int_{-h}^{0} a(s)A_{2\lambda}x_\lambda(t+s)\,ds \\ & x_\lambda(0) = \phi^0, \quad x_\lambda(s) = \phi^1(s) \quad \text{a.e. on } [-h,0). \end{aligned}$$

Since the operators in (2.3), excluding A, are bounded in X we can use known results on DDEs: For every $\phi \in M^2$ the mild solution x_λ of (2.3) exists uniquely; i.e. x_λ solves the integral equation

$$x_\lambda(t) = S(t)\phi^0 + \int_0^t S(t-s)\left[A_{1\lambda}x_\lambda(s-h) + \int_{-h}^{0} a(r)A_{2\lambda}x_\lambda(s+r)\,dr\right]\,ds\,.$$

Moreover, the family of operators $\{T_\lambda(t);\ t \geq 0\}$ defined by

$$T_\lambda(t)\phi = (x_\lambda(t), x_{\lambda t})$$

is a strongly continuous semigroup on M^2 for every $\lambda > 0$, (see e.g. [**13**], [**15**]).

Proposition 2.1. *i) The restriction of $T_\lambda(t)$ to Z is a strongly continuous semigroup on Z and*

$$\|T_\lambda(t)\phi\|_Z \leq c_2\|\phi\|_Z, \quad 0 \leq t \leq \tau,$$

for every $\phi \in Z$. The constant c_2 is independent of λ.

(ii) For every $\phi \in Z$ the following equation holds

$$(2.4) \qquad \lim_{\lambda\to\infty} \|T_\lambda(t)\phi - T(t)\phi\|_Z = 0\,.$$

uniformly over bounded time intervals.

Proof. The proposition can be proved analogously to Corollary 2.4 and Proposition 2.5 in [**12**] and [**13**], so we omit the details. □

The next objective is to introduce the so-called transposed semigroup which allows a characterization of the adjoint semigroup $\{T^*(t);\ t \geq 0\}$ of the solution semigroup $\{T(t);\ t \geq 0\}$ of DDE (2.1). It is well known that the adjoint operator A^* of A is the infinitesimal generator of the semigroup of adjoint operators $\{S^*(t);\ t \geq 0\}$ of $S(t)$. The spectrum of the adjoint A^* is just the conjugate of the spectrum of A. Therefore the operator A^* is itself the generator of a bounded analytic semigroup (see e.g. [**2**], [**17**]).

We consider the dual or transposed DDE associated with the equation (2.1)

$$\begin{aligned} (2.5) \qquad & \dot{y}(t) = A^*y(t) + A_1^*y(t-h) + \int_{-h}^{0} a(s)A_2^*y(t+s)\,ds \\ & y(0) = \psi^0, \quad y(s) = \psi^1(s) \quad \text{a.e. on } [-h,0), \end{aligned}$$

for a.e. $t \in (0,\tau)$. The initial value $\psi = (\psi^0, \psi^1)$ is an element of the product space $Z_* : = V_* \times L^2(-h, 0; D(A^*))$, where

$$V_* : = \left\{ w \in X \mid \int_0^\infty \|A^* S^*(t) w\|^2 \, dt < \infty \right\}$$

with norm

$$\|w\|_{V_*} = \left(\|w\|^2 + \int_0^\infty \|A^* S^*(t) w\|^2 \, dt \right)^{1/2}$$

is the intermediate space between $D(A^*)$ and X and $D(A^*)$ is equipped with the graph norm. We assume that the operators A_i^*, $i = 1, 2$ belong to $\mathscr{L}(D(A^*), X)$. Thus the operators appearing in (2.5) are of the same type as those given in the original equation (2.1). Therefore by Theorems 3.3 and 4.1 in [**5**] we have the following result:

For every $\psi \in Z_*$ there is a unique solution y of (2.5). Moreover, the family of operators $\{T^T(t);\ t \geq 0\}$ defined by

$$T^T(t)\psi : = (y(t), y_t)$$

is a strongly continuous semigroup on Z_*. It is called a transposed semigroup.

Let us denote the approximating adjoint operators by $R_\lambda^* = (R_\lambda)^*$, $A_{i\lambda}^* = (A_{i\lambda})^*$ $i = 1, 2$ for $\lambda > 0$. We note that $A_{i\lambda}^* y = (A_i R_\lambda)^* y = R_\lambda^* A_i^* y$ for every $y \in D(A^*)$ (see e.g. [**9**, p. 168]).

Consider the approximative dual DDE of equation (2.5):

$$\begin{aligned} &\dot{y}_\lambda(t) = A^* y_\lambda(t) + A_{1\lambda}^* y_\lambda(t-h) + \int_{-h}^0 a(s) A_{2\lambda}^* y_\lambda(t+s) \, ds \\ &y_\lambda(0) = \psi^0, \quad y_\lambda(s) = \psi^1(s) \quad \text{a.e. on } (-h, 0), \end{aligned} \tag{2.6}$$

for a.e. $t \in (0, \tau)$ and $\psi \in M^2$.

The approximative equation (2.6) is of the same type as ADDE (2.3) Therefore the unique mild solution y_λ of (2.6) exists for every $\psi \in M^2$ and $\lambda > 0$. Also the family of approximative transposed semigroups $\{T_\lambda^T(t);\ t \geq 0\}$ is defined by

$$T_\lambda^T(t)\psi : = (y_\lambda(t), y_{\lambda t}), \quad \forall\, \psi \in M^2.$$

By the same argument as given above for Proposition 2.1 we conclude: the restriction of $T_\lambda^T(t)$ to Z_* is a strongly continuous semigroup on Z_* and

$$\lim_{\lambda \to \infty} \|T_\lambda^T(t)\psi - T^T(t)\psi\|_{Z_*} = 0, \text{ for every } \psi \in Z_* \tag{2.7}$$

uniformly over bounded time intervals.

3. Structural Operators, Adjoint Semigroups and Generators

In this section we introduce structural operators which provide the essential connection between the adjoint and transposed semigroups associated with equations (2.1) and (2.3). First we define the structural operator $F\colon Z \to M^2$ associated with DDE (2.1):

$$F\phi : = (\phi^0, H\phi^1) \ \text{ for } \ \phi \in Z,$$

where

$$(H\phi^1)(s) : = A_1\phi^1(-h-s) + \int_{-h}^{s} a(r)A_2\phi^1(r-s)\,dr\,.$$

We note that $F \in \mathscr{L}(Z, M^2)$. Let Z' and Z'_* denote dual spaces of Z and Z_* respectively. Then the following relations hold:

$$Z \hookrightarrow M^2 = (M^2)' \hookrightarrow Z' \quad \text{and} \quad Z_* \hookrightarrow M^2 = (M^2)' \hookrightarrow Z'_*.$$

Therefore, it follows that $F \in \mathscr{L}(Z, Z'_*)$ and $F^* \in \mathscr{L}(Z_*, Z')$.

By a change of variables the following characterization of the adjoint operator F^* can be obtained (for the proof see e.g. [**12**], [**13**]):

$$F^*\psi = (\psi^0, H^*\psi^1) \quad \text{for } \psi \in Z_* \ \text{ and}$$
$$(H^*\psi^1)(s) = A_1^*\psi^1(-h-s) + \int_{-h}^{s} a(r)A_2^*\psi^1(r-s)\,dr\,.$$

Next we define structural operators $F_\lambda\colon M^2 \to M^2$ associated with ADDE (2.3):

$$F_\lambda\phi : = (\phi^0, H_\lambda\phi^1) \text{ where } H_\lambda\phi^1 = HR_\lambda\phi^1 \text{ and } \phi \in M^2.$$

We note that $F_\lambda \in \mathscr{L}(M^2)$ for $\lambda > 0$ and thus $F_\lambda^* \in \mathscr{L}(M^2)$.

The product space M^2 is a Hilbert space, so that the elements of the topological dual $(M^2)'$ can be identified with the elements of M^2 itself. Therefore the adjoint semigroup $\{T_\lambda^*(t);\ t \geq 0\}$ is a strongly continuous semigroup on M^2 for every $\lambda > 0$.

DDE (2.3) with bounded operators acting in the delays has the same form as the DDE studied in [**13**, Section 2]. Therefore the following characterization of the adjoint semigroup $T_\lambda^*(t)$ is a direct consequence of Theorem 2.1 in [**13**] (see also Theorem 4.2 in [**15**]):

Proposition 3.1. *For $\lambda > 0$ let $\{T_\lambda^*(t) : t \geq 0\}$ be the adjoint semigroup associated with DDE* (2.3) *and $\{T_\lambda^T(t);\ t \geq 0\}$ be the transposed semigroup associated with* (2.5)*. Then we have*

$$T_\lambda^*(t)F_\lambda^*\psi = F_\lambda^*T_\lambda^T(t)\psi, \quad t \geq 0, \tag{3.1}$$

for every $\psi \in M^2$.

By using relations (2.2) we can conclude that for $\phi \in Z$, $F_\lambda \phi$ converges to $F\phi$ and for $\psi \in Z_*$, $F_\lambda^* \psi$ converges to $F^* \psi$ in M^2. Therefore we get the following result:

Theorem 3.1. *Let $\{T^*(t);\ t \geq 0\}$ be the adjoint semigroup of DDE* (2.1) *and let $\{T^T(t);\ t \geq 0\}$ be the transposed semigroup associated with dual equation* (2.5). *Then the following equation holds for every $\psi \in Z_*$:*

$$T^*(t) F^* \psi = F^* T^T(t) \psi, \quad \text{for} \quad t \geq 0 . \tag{3.2}$$

Proof. The equation (3.2) follows from (2.4), (2.7) and equation (3.1). For the details see e.g. [**12**, Theorem 4.5]. □

Solution semigroups $\{T(t);\ t \geq 0\}$ and $\{T^T(t);\ t \geq 0\}$ are strongly continuous semigroups. We will denote their infinitesimal generators by $\mathscr{A}$ and $\mathscr{A}^T$ respectively. The following characterization of $\mathscr{A}$ is proved in [**5**, Theorem 4.2.]:

$$\begin{aligned} D(\mathscr{A}) = \Big\{ \phi \in Z \mid \phi^1(0) = \phi^0, \phi^1 \in W^{1,2}(-h, 0; D(A)) \quad \text{and} \\ (A\phi^0 + A_1 \phi^1(-h) + \int_{-h}^0 a(s) A_2 \phi^1(s)\, ds) \in V \Big\}, \\ \mathscr{A}\phi = \left(A\phi^0 + A_1 \phi^1(-h) + \int_{-h}^0 a(s) A_2 \phi^1(s)\, ds, \dot{\phi}^1 \right) . \end{aligned} \tag{3.3}$$

The operators appearing in the transposed equation (3.1) are of the same type as those in the equation (2.1), so we have an analogous characterization of the operator $\mathscr{A}^T$:

$$\begin{aligned} D(\mathscr{A}^T) = \Big\{ \psi \in Z_* \mid \psi^1(0) = \psi^0, \psi^1 \in W^{1,2}(-h, 0; D(A^*)) \quad \text{and} \\ (A^*\psi^0 + A_1^* \psi^1(-h) + \int_{-h}^0 a(s) A_2^* \psi^1(s)\, ds) \in V_* \Big\}, \\ \mathscr{A}^T\psi = (A^*\psi^0 + A_1^* \psi^1(-h) + \int_{-h}^0 a(s) A_2^* \psi^1(s)\, ds, \dot{\psi}^1). \end{aligned} \tag{3.4}$$

As noted the infinitesimal generator of the adjoint semigroup $\{T^*(t);\ t \geq 0\}$ is the adjoint operator $\mathscr{A}^*$ of $\mathscr{A}$. By the definition of the infinitesimal generators the following equation can be directly obtained from the equation (3.2):

$$\mathscr{A}^* F^* \psi = F^* \mathscr{A}^T \psi, \quad \text{for } \psi \in D(\mathscr{A}^T). \tag{3.5}$$

This means that the generator $\mathscr{A}^*$ can be characterized by the generator $\mathscr{A}^T$ and the structural operator F^*. In the next section this relation will be used for the characterization of eigenspaces of $\mathscr{A}^*$.

4. Eigenspaces and Identifiability

The description of the eigenvalues of $\mathscr{A}$ can be given by introducing a family of operators $\Delta(\omega)$. For $\omega \in C$ the operator $\Delta(\omega)\colon D(A) \to X$ is defined by

$$\Delta(\omega)x : = (\omega I - A)x - e^{-\omega h} A_1 x - \left(\int_{-h}^{0} a(s) e^{\omega s}\, ds \right) A_2 x \tag{4.1}$$

for every $x \in D(A)$. The equation $\Delta(\omega)x = 0$ is known to be the generalization of the characteristic equation for delay equations when $X = R^n$ (see e.g. [**6**], [**7**], [**15**]). The following result is readily obtained (see e.g. [**6**]):

Proposition 4.1. *Let $\omega \in C$. Then ϕ belongs to the set* $\ker(\omega I - \mathscr{A})$ *if and only if $\phi = (\phi^0, e^{\omega s}\phi^0)$ and $\Delta(\omega)\phi^0 = 0$.*

The description of the eigenvalues of the generator $\mathscr{A}^T$ can be given analogously to that of $\mathscr{A}$. We define the family of operators $\Delta^T(\omega)\colon D(A^*) \to X$ by

$$\Delta^T(\omega)x = (\omega I - A^*)x - e^{-\omega h} A_1^* x - \left(\int_{-h}^{0} a(s) e^{\omega s}\, ds \right) A_2^* x \tag{4.2}$$

for $x \in D(A^*)$ and $\omega \in C$. We have the following characterization of $\sigma_P(\mathscr{A}^T)$, the point spectrum of $\mathscr{A}^T$:

Proposition 4.2. *Let $\omega \in C$. Then $\psi \in \ker(\omega I - \mathscr{A}^T)$ if and only if $\psi = (\psi^0, e^{\omega s}\psi^0)$ and $\Delta^T(\omega)\psi^0 = 0$.*

Let us denote

$$\begin{aligned} M_\omega &= \ker(\omega I - \mathscr{A}), \qquad N_\omega^T = \ker(\omega I - \mathscr{A}^T), \\ N_\omega^* &= \ker(\omega I - \mathscr{A}^*), \quad \text{for } \omega \in C. \end{aligned} \tag{4.3}$$

We will consider the case where the eigenspaces of $\mathscr{A}$ are finite-dimensional. We will assume the following hypothesis:

$$\omega \in \sigma_P(\mathscr{A}) \quad \text{and} \quad \dim N_\omega = d < \infty\,. \tag{4.4}$$

Proposition 4.3. *Let $\omega \in C$ satisfy* (4.4) *and let $\overline{\omega}$ be its conjugate. Then $\overline{\omega} \in \sigma_P(\mathscr{A})^T$ and* $\dim N_{\overline{\omega}}^T = d$.

Proof. Let $\cong$ denote the linear isomorphism between two vector spaces. By Propositions 4.1 and 4.2 we have relations:

$$N_\omega \cong \ker \Delta(\omega) \qquad \text{and} \qquad N_{\overline{\omega}}^T \cong \ker \Delta^T(\overline{\omega}).$$

The operators defined by $R_\lambda A_i R_\lambda$, $i = 1,2$ are bounded in X for $\lambda > 0$. By assumption $A_i \in \mathscr{L}(D(A), X)$ and $A_i^* \in \mathscr{L}(D(A^*), X)$. Thus we have relations:

$$(R_\lambda A_i R_\lambda)^* = (R_\lambda(A_i R_\lambda))^* = (A_i R_\lambda)^* R_\lambda^* = R_\lambda^* A_i^* R_\lambda^*, \quad i = 1,2. \tag{4.5}$$

The action of the operator $\Delta(\omega)$ can be given by

$$\Delta(\omega)x = [(\omega I - A) - f_1(\omega)A_1 - f_2(\omega)A_2]\,x\,, \tag{4.6}$$

where

$$f_1(\omega) = e^{-\omega h}, \qquad f_2(\omega) = \int_{-h}^{0} a(s)e^{\omega s}\,ds\,.$$

We note that the operator $R_\lambda \Delta(\omega) R_\lambda$ is a sum of bounded operators on X. Thus we have following relations:

$$\begin{aligned}(R_\lambda \Delta(\omega) R_\lambda)^* &= [R_\lambda((\omega I - A) - f_1(\omega)A_1 - f_2(\omega)A_2)R_\lambda]^* \\ &= R_\lambda^*\,[(\overline{\omega} I - A^*) - f_1(\overline{\omega})A_1^* - f_2(\overline{\omega})A_2^*]\,R_\lambda^* = R_\lambda^* \Delta^T(\overline{\omega}) R_\lambda^*.\end{aligned}$$

Since $R_\lambda \Delta(\omega) R_\lambda$ is a bounded operator on X it follows that

$$\dim\ker(R_\lambda^* \Delta^T(\overline{\omega}) R_\lambda^*) = \dim\ker(R_\lambda \Delta(\omega) R_\lambda) = d\,,$$

see e.g. [**9**, p. 184]. By the fact that R_λ^* is bijective operator and $D(\Delta^T(\overline{\omega})) = D(A^*)$ we conclude that $\dim\ker \Delta^T(\overline{\omega}) = d$. By Proposition 4.2 then it follows: $\dim N_{\overline{\omega}}^T = d$ and $\overline{\omega} \in \delta_P(\mathscr{A}^T)$. ☐

Theorem 4.1. *Let $\omega \in C$ satisfy* (4.4). *Then the eigenspace $N_{\overline{\omega}}^*$ is characterized by:*

$$N_{\overline{\omega}}^* = F^*(N_{\overline{\omega}}^T). \tag{4.7}$$

Proof. By equation (3.5) the following relation holds:

$$(\overline{\omega} I - \mathscr{A}^*)F^*\psi = F^*(\overline{\omega} I - \mathscr{A}^T)\psi$$

for every $\psi \in D(\mathscr{A}^T)$. Hence we have the inclusion:

$$F^*(N_{\overline{\omega}}^T) \subset N_{\omega}^*\,. \tag{4.8}$$

Note that if $\psi \in N_{\overline{\omega}}^T$ and $F^*\psi = 0$, then it follows $\psi = 0$. This means that the operator F^* restricted to $N_{\overline{\omega}}^T$ is an injective linear operator.

By the operator theory (see [**9**, p. 184]) an by assumption (4.4) we also have $\overline{\omega} \in \sigma_P(\mathscr{A}^*)$ and $\dim N_{\overline{\omega}}^* = \dim N_\omega = d$. Thus we can conclude that $\dim N_{\overline{\omega}}^* =$

$\dim N_{\overline{\omega}}^T$. Hence by injectivity of F^* on $N_{\overline{\omega}}^T$ from the inclusion (4.8) the equation (4.7) follows. $\square$

By the operator theory we have the following representation of the projector P from the space Z to the finite-dimensional eigenspace N_ω:

$$Pz = \sum_{i=1}^{d} \langle z, \eta_i \rangle \, \phi_i, \qquad z \in Z,$$

where the set $\{\phi_i;\ i = 1, 2, \ldots, d\}$ forms a basis in N_ω, the set $\{\eta_j;\ j = 1, 2, \ldots, d\}$ forms a basis in $N_{\overline{\omega}}^*$ and the following identities hold:

$$\langle \phi_i, \eta_j \rangle_{Z \times Z'} = \delta_{i,j}.$$

For the details see e.g. [**9**, p. 25, p. 184].

By Theorem 4.1 there exists a basis $\{\psi_j;\ j = 1, 2, \ldots, d\}$ in $N_{\overline{\omega}}^T$ such that

$$Pz = \sum_{j=1}^{d} \langle z, F^* \psi_j \rangle \, \phi_j = \sum_{j=1}^{d} \langle Fz, \psi_j \rangle \, \phi_j \quad \text{for} \quad z \in Z. \tag{4.9}$$

Suppose that $\{g_i;\ i = 1, 2, \ldots, r\}$ is a finite set of elements of the space Z. Then the condition

$$\mathrm{Span}\, \{Pg_i;\ i = 1, 2, \ldots, r\} = N_\omega$$

can be equivalently expressed by the following rank condition:

$$\mathrm{rank} \left(\langle Fg_i, \psi_j \rangle \,, \begin{array}{l} i \to 1, 2, \ldots, r \\ j \downarrow 1, 2, \ldots, d \end{array} \right) = d\,. \tag{4.10}$$

In [**16**] Nakagiri and Yamamoto solved the identifiability problem for DDE (2.1) with A_1, $A_2 \in \mathscr{L}(X)$ provided that the set of initial functions satisfies rank conditions (4.10). By using Theorem 4.1 the identifiability problem for DDE (2.1) can be solved in the same way. More specifically, let us denote

$$\sigma_0(\mathscr{A}) = \{\omega \in C \mid \omega \in \sigma_P(\mathscr{A}),\ \dim M_\omega < \infty\},$$

where $M_\omega = \ker(\omega I - \mathscr{A})^k$ denotes the generalized eigenspace of $\mathscr{A}$ and where k is the order of the pole of the resolvent $R(\lambda, \mathscr{A})$ at $\lambda = \omega$. Let us assume that $\mathscr{A}$ satisfies the following hypothesis on the spectrum:

$$\sigma(\mathscr{A}) = \sigma_0(\mathscr{A}) \text{ is a countable set and} \tag{4.11}$$

$$\mathrm{Cl}\, (\mathrm{span}\, \{M_\omega; \omega \in \sigma(\mathscr{A})\}) = Z, \tag{4.12}$$

where Cl denotes the closure in Z. This means that the system of eigenfunctions of $\mathscr{A}$ is complete. For $\omega_n \in \sigma(\mathscr{A})$, $n = 1, 2, 3, \ldots$ we will denote by

$$\{\phi_{n,i};\ i = 1, 2, \ldots, d_n\}$$

the basis in N_{ω_n} and by

$$\{\psi_{n,j};\ j = 1, 2, \ldots, d_n\}$$

the basis in $N^T_{\overline{\omega}_n}$. Note that $N_{\omega_n} \subset M_{\omega_n}$.

We consider the problem of the identifiability of parameters of (2.1). We define the model equation $(2.1)^m$ as the equation (2.1) in which the parameters A, A_1 and $a(s)$ are replaced by A^m, A_1^m and $a^m(s)$ respectively. We assume that A_2 is a priori known operator, that is $A_2 = A_2^m$.

Let $\{g_i;\ i = 1, 2, \ldots, r\}$ be the finite set of initial values and let $x(t; g_i)$ and $x^m(t; g_i)$ be solutions of (2.1) and $(2.1)^m$ respectively. Then the operators A, A_1 and the function a are called identifiable when the following implication holds:

$$\text{(4.13)}\qquad \begin{array}{l} \text{If } x(t;g_i) = x^m(t;g_i) \ \text{ for } i = 1,2,\ldots,r \text{ and } t > 0, \text{ then} \\ Ax = A^m x,\ A_1 x = A_1^m x \ \text{ for } x \in D(A) \ \text{ and } \ a = a^m. \end{array}$$

Theorem 4.2. *Let $\mathscr{A}$ and $\mathscr{A}^m$ satisfy* (4.11) *and let $\mathscr{A}^m$ satisfy* (4.12). *If the initial functions $\{g_i;\ i = 1, 2, \ldots, r\}$ satisfy the following rank condition:*

$$\text{(4.14)}\qquad \operatorname{rank}\left(\left\langle F^m g_i, \psi^m_{n,j}\right\rangle,\ \begin{array}{l} i \to 1,2,\ldots,r \\ j \downarrow 1,2,\ldots,d_n \end{array}\right) = d_n\,,$$

for every $n \in N$, then A, A_1 a are identifiable.

Proof. By Proposition 3.1 in [**16**] and by Proposition 3 in [**18**] it follows that $\mathscr{A} = \mathscr{A}^m$. By the same argument as that given in the proof of Theorem 3.1 in [**16**] the identifiability of A, A_1, $a(s)$ can be readily obtained. ☐

Remark. In the case where A_1 and A_2 are bounded the hypothesis (4.11) of Theorem 4.2 are fulfilled, whenever A has a compact resolvent (see e.g. [**16**, p. 320]). When A_1, A_2 are unbounded an analogous result on identifiability under approximation can be obtained by using ADDE (2.3):

Let us denote by $\mathscr{A}_\lambda$ and $\mathscr{A}_\lambda^T$ infinitesimal generators of approximate semigroups $T_\lambda(t)$ and $T_\lambda^T(t)$ on M^2 for some $\lambda > 0$. Let us assume that A has a compact resolvent. Then

$$\sigma(\mathscr{A}_\lambda) = \sigma_0(\mathscr{A}_\lambda)$$

is a countable set and we can denote by

$$\{\psi_{\lambda,n,j};\ j = 1, 2, \ldots, d\}$$

the basis in

$$N^T_{\lambda,\overline{\omega}_n} = \ker(\overline{\omega}_n I - \mathscr{A}_\lambda^T).$$

We define the model equation $(2.3)^m$ as the equation (2.3) in which the parameters A, $A_{1\lambda}$ and $a(s)$ are replaced by A^m, $A^m_{1\lambda}$ and $a^m(s)$ respectively.

Theorem 4.3. *Let A and A^m have compact resolvents and let us assume that the system of generalized eigenfunctions of $\mathscr{A}^m_\lambda$ is complete in M^2 for some $\lambda > 0$. If the initial functions $\{g_i;\ i = 1, 2, \dots, r\}$ satisfy the following rank condition:*

$$\operatorname{rank}\left(\left\langle F^m_\lambda g_i, \psi^m_{\lambda,n,j}\right\rangle,\ \begin{matrix} i \rightarrow 1, 2, \dots, r \\ j \downarrow 1, 2, \dots, d_n \end{matrix}\right) = d_n\,, \tag{4.15}$$

for every $n \in N$, then $Ax = A^m x$, $A_1 x = A^m_1 x$ for $x \in \mathscr{D}(A)$ and $a = a^m$.

Proof. The operators $A_{1\lambda}$ and $A^m_{1\lambda}$ are bounded, so we can apply Theorem 3.1 in [**16**] to the equations (2.3) and $(2.3)^m$. We obtain identities $A = A^m$, $A_{1\lambda} = A^m_{1\lambda}$ and $a = a^m$. This means: $R(\lambda, A) = R(\lambda, A^m)$ and $A^m_1 R(\lambda, A) = A_1 R(\lambda, A)$. Hence it follows $A^m_1 x = A_1 x$ for every $x \in D(A)$. □

5. Example

We consider the partial functional differential equation of the form:

$$u_t(t,x) = c u_{xx}(t,x) + b u(t-h,x) + \int_{-h}^{0} a(s) u_x(t+s,x)\,ds \tag{5.1}$$

for $t > 0$, $x \in (0, 2\pi)$ with boundary conditions

$$\begin{aligned} u(t,0) &= u(t,2\pi), \quad u_x(t,0) = u_x(t,2\pi) \\ u(0,x) &= g^0(x), \qquad u(s,x) = g^1(s,x) \text{ a.e. } [-h,0] \times [0,2\pi], \end{aligned}$$

where $g^0 \in X = L^2(0,2\pi)$, $g^1 \in L^2(h,0;X)$, $c > 0$, $b \neq 0$ and $a(s) \in L^2(-h,0;\mathbb{R})$.

The space X is a Hilbert space with inner product $\langle f, g\rangle = \int_0^{2\pi} f(x) g(x)\,dx$. The operator $A\colon X \to X$ is defined by

$$\begin{aligned} &Af = cf'' \text{ for } f \in \mathscr{D}(A), \\ &\mathscr{D}(A) = \{f \in X \mid f, f' \in W^{1,2}(0,2\pi;\mathbb{R}) \text{ and } f(0) = f(2\pi), f'(0) = f'(2\pi)\} \end{aligned}$$

The operators A_i, $i = 1, 2$ are defined by

$$\begin{aligned} A_1 = bI \text{ and } A_2 f = f' \text{ for } f \in \mathscr{D}(A_2), \\ \mathscr{D}(A_2) = \{f \in X \mid f \in W^{1,2}(0,2\pi;\mathbb{R}) \text{ and } f(0) = f(2\pi)\}. \end{aligned}$$

Thus the partial functional differential equation (5.1) can be written in the form (2.1).

It is known that the operator A is self adjoint and that it is the infinitesimal generator of a compact analytic semigroup $S(t)$ in X, (see e.g. [**1**, p. 214], [**17**, p. 234]. The eigenvalues of A are given by $\mu_n = -cn^2$, $n = 0, 1, 2, \ldots$ and the eigenfunctions $\{\sin nx, \cos nx; n = 0, 1, 2, \ldots\}$ form a complete orthogonal system in X. Let $X_n = \operatorname{Span}\{\sin nx, \cos nx\}$ denote the associated eigenspace for $n = 0, 1, 2, \ldots$. If P_n is a projector from X to X_n, then

$$\sum_{n=0}^{\infty} P_n = I \tag{5.2}$$

and

$$S(t)f_n = e^{\mu_n t} f_n \quad \text{for} \quad f_n \in X_n. \tag{5.3}$$

By the compactness of the semigroup $S(t)$ it can be shown that the solution semigroup $T(t)$ of DDE (2.1) is compact for $t > h$, (see e.g. [**21**, p. 134]). This means that the spectrum of its infinitesimal generator $\mathscr{A}$ is countable and is equal to the point spectrum. Moreover, the generalized eigenspaces $M_\omega(\mathscr{A})$ of $\mathscr{A}$ are finite dimensional (see e.g. [**17**, p. 46], [**20**, p. 408]). For the characterization of generalized eigenvectors of $\mathscr{A}$ see e.g. [**14**, p. 97].

In order to show the completeness of generalized eigenfunctions of $\mathscr{A}$ we will proceed analogously to the proof of Theorem 2 in [**7**]. The restrictions of operators A and A_i to X_n will be denoted by A_n and A_{in}:

$$A_n = A\big|_{X_n} \qquad \text{and} \qquad A_{in} = A_i\big|_{X_n}.$$

The range of these restrictions is in X_n. Thus we can consider DDE (2.1) with operators A, A_i replaced by A_n and A_{in} for any initial value $\phi \in M_n^2 = X_n \times L^2(-h, 0; X_n)$. Let $T_n(t)$ denote the solution semigroup of this restriction initial value problem and let $\mathscr{A}_n$ be its infinitesimal generator. By (5.3) it follows

$$T_n(t) = T(t)\big|_{M_n^2} \qquad \text{and} \qquad \mathscr{A}_n = \mathscr{A}\big|_{\mathscr{D}(\mathscr{A}_n)},$$

which means that $\mathscr{M}_{\omega_{n,j}}(\mathscr{A}) = \mathscr{M}_{\omega_{n,j}}(\mathscr{A}_n)$ for every eigenvalue $\omega_{n,j}$ of A_n, $j \in \mathbb{N}$. Since X_n is finite dimensional and $b \neq 0$ Theorems 5.1 and 5.4 in [**11**] imply that elements of $\mathscr{M}_{\omega_{n,j}}(\mathscr{A}_n)$ form a complete system in M_n^2 for every $n = 0, 1, 2, \ldots$. Hence by (5.2) it follows that (4.12) holds and the assumptions of Theorem 4.3 are fulfilled. Therefore when the set of initial functions $\{g_i,\ i = 1, 2, \ldots, r\}$ satisfies rank conditions (4.14) the parameters c, b, $a(s)$ are identifiable.

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References

1. Balakrishnan A. V., *Applied functional analysis*, 2nd ed., Springer Verlag, New York, 1981.

2. Bensoussan A., Da Prato G., Delfour M. and Mitter S., *Representation and control of infinite-dimensional systems*, Vol. 1, Birkhauser, Boston, 1992.

3. Bernier C. and Manitius A., *On semigroups in $R^n \times L^p$ corresponding to differential equations with delays*, Can. J. Math. **30** (1978), 897–914.

4. Delfour M. C. and Manitius A., *The structural operator F and its role in the theory of retarded systems*, Part I: J. Math. Anal. Appl. **73** (1980), 466–490; Part II: J. Math. Anal. Appl. **74** (1980), 359–381.

5. DiBlasio G., Kunisch K. and Sinestrari E., *L^2-regularity for parabolic integrodifferential equations with delay in the highest order derivatives*, J. Math. Anal. Appl. **102** (1984), 38–57.

6. ______, *Stability for abstract linear functional differential equations*, Israel J. Math. **50** (1985), 231–263.

7. Jeong J. M., *Spectral properties of the operator associated with a retarded functional differential equation in Hilbert space*, Proc. Japan Accad. **65** A (1989), 98–101.

8. Jeong J. M., Nakagiri S. and Tanabe H., *Structural operators and semigroups associated with functional differential equations in Hilbert spaces*, (to appear).

9. Kato T., *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.

10. Lions J. L. and Magenes E., *Problemes aux Limites non Homogenes et Application*, Vol. I, Dunod, Paris, 1968.

11. Manitius A., *Completeness and F-completeness of eigenfunctions associated with retarded functional differential equations*, J. Diff. Eqs. **35** (1980), 1–29.

12. Mastinšek M., *Dual semigroups for delay differential equations with unbounded operators acting on the delays*, Differential and Integral Equations **7** (1994), 205–216.

13. ______, *Structural operators for abstract functional differential equations*, Semigroup Theory and Applications (P. Clement, S. Invernizzi, E. Mitidieri, I. I. Vrabie, eds.), Lecture Notes in Pure and Applied Mathematics 116, Marcel Dekker Inc., New York, 1989.

14. Milota J., *Stability and saddle point property for a linear autonomous functional parabolic equation*, Comm. Math. Univ. Carolinae **27** (1986), 87–101.

15. Nakagiri S., *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. **25** (1988), 353–398.

16. Nakagiri S. and Yamamoto M., *Identifiability of linear retarded systems in Banach spaces*, Funkc. Ekvacioj **31** (1988), 315–329.

17. Pazy A., *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

18. Suzuki T. and Yamamoto M., *Observability, controllability and stabilizability for evolution equations*, Part I, Japan J. Applied Math. **2** (1985), 211–228.

19. Tanabe H., *Structural operators for linear delay-differential equations in Hilbert space*, Proc. Japan Acad. **64**, Ser. A (1988).

20. Travis C. C. and Webb G. F., *Existence and stability for partial functional differential equations*, Trans. Am. Math. Soc. **200** (1974), 395–418.

21. ______, *Existence, stability and compactness in the α-norm for partial functional differential equations*, Trans. Am. Math. Soc. **240** (1978), 129–143.

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