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ON THE RELATIONSHIP BETWEEN PROJECTIVE DISTRIBUTIVE LATTICES AND BOOLEAN ALGEBRAS

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Abstract. The main result of this paper is the following theorem: If a projective Boolean algebra B is generated by its sublattice L, then there is a projective distributive lattice D which is a sublattice of L and generates B.

1. Preliminaries

The operations of Boolean algebras will be denoted by $\wedge$ (meet), $\vee$ (join), $'$ (complement), 0 (the least element) and 1 (the greatest element). If B is a Boolean algebra (we do not distinguish between the algebra and its underlying set) and $H \subseteq B$, then $\langle H\rangle$ denotes the subalgebra of B generated by H. For Boolean algebras A, C we will write $C \leq_{rc} A$ and say that C is relatively complete in A, if C is a subalgebra of A and for every $a \in A$ there exists the greatest $c \in C$ with $c \leq a$. We denote this element by a_C. If $C \leq_{rc} A$ then for each $a \in A$ there exists the least $c \in C$ with $a \leq c$. This element will be denoted by a^C. Thus, $a^C = (a'_C)'$. By $C \leq_{rc\omega} A$ we understand that $C \leq_{rc} A$ and $A = \langle C \cup X\rangle$ for some countable set X. The following statement is easy to prove (see [**8**]):

1.1. Lemma. *Let A and C be subalgebras of a Boolean algebra B.*

(i) *If $A \leq_{rc} B$, $A \subseteq C \subseteq B$, then $A \leq_{rc} C$.*
(ii) *If $A \leq_{rc} B$, $x \in B$, then $\langle A \cup \{x\}\rangle \leq_{rc} B$.*

A chain $\{A_\alpha \,|\, \alpha < \tau\}$ of Boolean algebras (where τ is an arbitrary ordinal number) is said to be continuous if $A_\lambda = \bigcup\{A_\alpha \,|\, \alpha < \lambda\}$ holds for each limit ordinal number $\lambda < \tau$.

For the sake of brevity, by a distributive lattice we understand in this paper a bounded lattice satisfying the well-known distributivity identities. Consequently, all lattice homomorphisms are assumed to preserve the universal bounds. Analogically, saying that C is a sublattice of a distributive lattice D we mean that C is closed under $\wedge$ and $\vee$ and contains the universal bounds of D. Free distributive lattices are the free objects in the category of bounded distributive lattices and $0,1$-preserving lattice homomorphisms. For every distributive lattice D

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there is unique (up to isomorphism) Boolean algebra $B(D)$ that contains D as a sublattice and $\langle D\rangle = B(D)$. Each $a \in B(D)$ can be expressed in the form $a = a_0 + a_1 + \cdots + a_n$, where $a_0, \dots, a_n \in D$, $a_0 \leq a_1 \leq \cdots \leq a_n$ and $+$ is the operation of symmetric difference (i.e. $x + y = (x' \wedge y) \vee (x' \wedge y)$). For every Boolean algebra B, the set B with the operations $+$, $\wedge$ is a ring satisfying the identities $a + a = 0$, $a \vee b = a + b + (a \wedge b)$, and $a' = a + 1$. Every homomorphism $f\colon D_1 \longrightarrow D_2$ of distributive lattices can be extended to a homomorphism $f^*\colon B(D_1) \longrightarrow B(D_2)$ of Boolean algebras. (See [**4**, Ch. II.4.])

An object P of a category $\mathcal{K}$ is $\mathcal{E}$-projective (where $\mathcal{E}$ is some class of epimorphisms) if, for every $e \in \mathcal{E}$, $e\colon A \longrightarrow B$ and every morphism $f\colon P \longrightarrow B$, there exists a morphism $g\colon P \longrightarrow A$ with $eg = f$. Injective objects are defined dually.

A projective Boolean algebra (distributive lattice) is a $\mathcal{E}$-projective object of the category $\mathcal{B}$ of Boolean algebras ($\mathcal{D}$ of distributive lattices) and their homomorphisms, where $\mathcal{E}$ is the class of all surjective homomorphisms. Basic facts about projective Boolean algebras are summarized in the following assertion. For the proofs see [**5**] and [**6**]. Recall that an object A is a retract of B if there are morphisms $f\colon A \longrightarrow B$, $g\colon B \longrightarrow A$ such that $gf = \mathrm{id}\,(A)$.

1.2. Theorem.

(i) *A Boolean algebra is projective iff it is a retract of some free Boolean algebra.*
(ii) *Any free product of projective Boolean algebras is projective.*
(iii) *Every retract of a projective Boolean algebra is projective.*
(iv) *Every countable Boolean algebra is projective.*

According to 1.2 every projective Boolean algebra is a subalgebra of a free Boolean algebra, hence it cannot contain an uncountable chain. We will use the following characterization of projective Boolean algebras proved by Koppelberg in [**8**]. Analogical result for Boolean topological spaces can be found in [**7**].

1.3. Theorem. *Let A be a Boolean algebra. The following statements are equivalent:*

(i) *The Boolean algebra A is projective.*
(ii) *There exists a continuous chain $\{A_\alpha \mid \alpha < \tau\}$ of subalgebras of A such that $A_0 = \{0,1\}$, $\bigcup\{A_\alpha \mid \alpha < \tau\} = A$ and $A_\alpha \leq_{rc\omega} A_{\alpha+1}$ holds for each α with $\alpha + 1 < \tau$.*
(iii) *There exists a family $\mathfrak{S}$ of subalgebras of A with the following properties:*
  (S1) $\{0,1\} \in \mathfrak{S}$ *;*
  (S2) *if $S \in \mathfrak{S}$ then $S \leq_{rc} A$;*
  (S3) *if $\mathfrak{C} \subseteq \mathfrak{S}$ is a non-empty chain under set inclusion then $\bigcup \mathfrak{C} \in \mathfrak{S}$;*
  (S4) *for each $S \in \mathfrak{S}$ and a countable subset X of A, there is $S' \in \mathfrak{S}$ such that $S \cup X \subseteq S'$ and $S \leq_{rc\omega} S'$.*

In the next theorem we summarize some facts about projective distributive lattices. Proofs of (i), (iii) and (iv) can be found in [**1**], [**2**], (ii) is contained in [**3**]. For a lattice L, let $J(L)$ and $M(L)$ denote the set of all non-zero $\vee$-irreducibles and the set of all non-unit $\wedge$-irreducibles respectively.

1.4. Theorem.

(i) *A distributive lattice is projective iff it is a retract of some free distributive lattice.*

(ii) *A distributive lattice D is projective iff it satisfies the following conditions:*
  (1) *$J(D)$ is a $\wedge$-subsemilattice of D;*
  (2) *both $J(D)$ and $M(D)$ generate D;*
  (3) *for each $a \in D$ there are two finite sets $A(a) \subseteq \{d \in D \mid d \geq a\}$ and $B(a) \subseteq \{d \in D \mid d \leq a\}$ such that $A(a) \cap B(b) \neq \emptyset$ for every $a \leq b$.*

(iii) *A finite $D \in \mathcal{D}$ is projective iff it satisfies* (1).

(iv) *A countable $D \in \mathcal{D}$ is projective iff it satisfies* (1) *and* (2).

The last four assertions of this section are technical lemmas about Boolean algebras. For the proof of 1.5 see [**4**, p. 73].

1.5. Lemma (Sikorski's extension criterion). *Let A and B be Boolean algebras, A generated by its subset G. Let f be a map of G into B. The map f can be extended to a homomorphism of A into B iff, for arbitrary $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in G$, the equality $x_1 \wedge \cdots \wedge x_n \wedge y_1' \wedge \ldots y_m' = 0$ implies $f(x_1) \wedge \cdots \wedge f(x_n) \wedge \cdots \wedge f(y_m)' = 0$.*

1.6. Lemma. *Let Boolean algebras A and C satisfy $C \leq_{rc} A$ and let $a \in A$, $b \in C$. Then $(a \vee b)_C = a_C \vee b$, $(a \wedge b)_C = a_C \wedge b$, $(a \vee b)^C = a^C \vee b$, $(a \wedge b)^C = a^C \wedge b$.*

Proof. We will show the first two equalities.

I. Denote $x = (a \vee b)_C$, $d = (a_C \vee b') \wedge x$. Clearly $a_C \vee b \leq x \leq a \vee b$, $a_C \leq d$. From $d \leq (a_C \vee b') \wedge (a \vee b) \leq a$ it follows that $d = a_C = \max\{c \in C \mid c \leq a\}$. We obtain that $a_C \vee b = d \vee b = ((a_C \vee b') \wedge x) \vee b = x$.

II. We have $a_C \wedge b \leq (a \wedge b)_C$, because $a \wedge b \geq a_C \wedge b \in C$. On the other hand, $(a \wedge b)_C \leq a_C$, $(a \wedge b)_C \leq b_C = b$, hence $(a \wedge b)_C \leq a_C \wedge b$. $\square$

1.7. Lemma. *Let Boolean algebras A and C satisfy $C \leq_{rc} A$. Let $x_1, x_2, \ldots x_n$ be distinct elements of A such that $(x_i)^C \leq (x_{i+1})_C$ for each $i = 1, 2, \ldots, n-1$. Let M and N be disjoint subsets of $\{1, 2, \ldots, n\}$. Denote $B = \{x_i \mid i \in M\} \cup \{x_i' \mid i \in N\}$. Then $(\bigwedge B)^C = \bigwedge\{x^C \mid x \in B\}$.*

Proof. Let us denote $j = \min(M)$ and $k = \max(N)$, provided $M \neq \emptyset$ and $N \neq \emptyset$ respectively. In the case $M = N = \emptyset$ the assertion is evident. (We set $\bigwedge \emptyset = 1$.) If $M = \emptyset$, $N \neq \emptyset$ (the case $M \neq \emptyset$, $N = \emptyset$ is analogous), we have $(\bigwedge B)^C = (x_k')^C = \bigwedge\{x^C \mid x \in B\}$. Finally, assume that $M, N \neq \emptyset$. We obtain $(\bigwedge B)^C = (x_j \wedge x_k')^C \leq (x_j)^C \wedge (x_k')^C = \bigwedge\{x^C \mid x \in B\}$. The inverse inequality is evident if $j < k$, because in this case $(x_j)^C \wedge (x'_k)^C \leq (x_k)_C \wedge (x'_k)^C = 0$. Now suppose that $j > k$. By 1.6 we have both $(x_j \wedge x'_k)^C \geq (x_j)^C \wedge (x'_k)_C$ and $(x_j \wedge x'_k)^C \geq (x_j)_C \wedge (x'_k)^C$. Since $(x_j)^C \vee (x'_k)^C \geq (x_j)_C \vee (x'_k)_C \geq (x_k)^C \vee (x'_k)_C = 1$, we get $(x_j \wedge x'_k)^C \geq ((x_j)^C \wedge (x'_k)_C) \vee ((x_j)_C \wedge (x'_k)^C) = (x_j)^C \wedge (x'_k)^C$. □

1.8. Lemma. *Let a Boolean algebra A be generated by its sublattice L. Let $a, b \in L$, $a \leq b$. Then the interval $[a, b]$ of A is generated (as a Boolean algebra) by its subset $[a, b] \cap L$.*

Proof. The algebra $[a, b]$ is a homomorphic image of A under the map $f(x) = (x \vee a) \wedge b$. Hence, it is generated by $f(L) \subseteq L \cap [a, b]$. □

2. The Main Results

2.1. Lemma. *Let D be a projective distributive lattice. Then $B(D)$ is a projective Boolean algebra.*

Proof. Let $f\colon B_1 \longrightarrow B_2$ be an epimorphism of Boolean algebras (i.e. surjective homomorphism) and $g\colon B(D) \longrightarrow B_2$ an arbitrary homomorphism. Then we have the lattice homomorphism $g^* = g \restriction D$ and, from the projectivity of D, a lattice homomorphism $h^*\colon D \longrightarrow B_1$ with $fh^* = g^*$, which can be extended to a homomorphism $h\colon B(D) \longrightarrow B_1 = B(B_1)$. The homomorphisms fh and g coincide on D, hence $fh = g$. □

Of course, a projective Boolean algebra can be generated by its non-projective sublattices as well. Notice that no Boolean algebra with more than two elements is a projective distributive lattice. Now we are going to prove that if for a distributive lattice L, $B(L)$ is a projective Boolean algebra, then there exists a projective sublattice D of L with $B(D) = B(L)$.

2.2. Lemma. *Let A be a projective Boolean algebra generated by its sublattice L. Suppose that $\mathfrak{S}$ is a family of subalgebras of A with the properties* (S1)–(S4). *Then for each $S \in \mathfrak{S}$ and countable $X \subseteq A$ there is a $S' \in \mathfrak{S}$ such that $S \cup X \subseteq S'$ and $S \leq_{rc\omega} S' = \langle S \cup Y \rangle$ for some countable $Y \subseteq L$.*

Proof. By induction we define an increasing chain $\{S_n \mid n < \omega\} \subseteq \mathfrak{S}$ such that $S \leq_{rc\omega} S_i$ for each $i < \omega$. By (S4) there is $S_0 \in \mathfrak{S}$ such that $S \cup X \subseteq S_0$ and $S \leq_{rc\omega} S_0$. Suppose now that we have defined $S_i \in \mathfrak{S}$ with $S_i = \langle S \cup \{s_k \mid k < \omega\} \rangle$. For each k there exists a finite set $Y^i_k \subseteq L$ such that $s_k \in \langle Y^i_k \rangle$. Denote $Y^i = \bigcup \{Y^i_k \mid k < \omega\}$. Using (S4) we get $S_{i+1} \in \mathfrak{S}$ such that $S \cup Y^i \subseteq S_{i+1}$, $S \leq_{rc\omega} S_{i+1}$. Moreover, the subalgebra of S_{i+1} generated by $S \cup Y^i$ contains S_i. Let us set $S' = \bigcup \{S_i \mid i < \omega\}$. We have $S' \in \mathfrak{S}$ (by (S3)), $S \leq_{rc} S'$ (by (S2) and 1.1), $X \subseteq S'$ and $S' = \langle S \cup Y \rangle$, where $Y = \bigcup \{Y^i \mid i < \omega\}$. □

2.3. Lemma. *Let A be a projective Boolean algebra generated by its sublattice L. Then there exists a continuous chain $\{A_\alpha \mid \alpha < \tau\}$ of subalgebras of A with the following properties:*

(i) $A_0 = \{0,1\}$;

(ii) $\bigcup\{A_\alpha \mid \alpha < \tau\} = A$;

(iii) *for each $\alpha < \tau$ there is $x \in L$ such that $A_\alpha \leq_{rc} A_{\alpha+1} = \langle A_\alpha \cup \{x\}\rangle$;*

(iv) $A_\alpha = \langle A_\alpha \cap L\rangle$ *holds for each $\alpha < \tau$.*

Proof. Let $\mathfrak{S}$ be a family of subalgebras of A with the properties (S1)–(S4). First we construct a continuous chain $\{B_\alpha \mid \alpha < \gamma\} \subseteq \mathfrak{S}$ satisfying (i), (ii), (iv) and

(iii') for each $\alpha < \gamma$ there is a countable $Y \subseteq L$ with $B_\alpha \leq_{rc} B_{\alpha+1} = \langle B_\alpha \cup Y\rangle$.

We proceed by induction. Let us set $B_0 = \{0,1\}$ and suppose that we have a chain $\{B_\alpha \mid \alpha < \lambda\} \subseteq \mathfrak{S}$. If $\bigcup\{B_\alpha \mid \alpha < \lambda\} = A$, we can set $\gamma = \lambda$. Otherwise we have $x \in A \setminus \bigcup\{B_\alpha \mid \alpha < \lambda\}$. For limit λ we set $B_\lambda = \bigcup\{B_\alpha \mid \alpha < \lambda\} \in \mathfrak{S}$. For $\lambda = \beta + 1$, 2.2 yields $B_\lambda \in \mathfrak{S}$ and a countable $Y \subseteq L$ with $B_\beta \cup \{x\} \subseteq B_\lambda$ and $B_\beta \leq_{rc} B_\lambda = \langle B_\beta \cup Y\rangle$. It is clear that the chain $\{B_\alpha \mid \alpha < \gamma\}$ has the desirable properties. Now we get the chain $\{A_\alpha \mid \alpha < \tau\}$ by inserting the algebras $\langle B_\alpha \cup \{y_1\}\rangle$, $\langle B_a \cup \{y_1, y_2\}\rangle$,..., (where $Y = \{y_i \mid i < \omega\} \subseteq L$, $B_{\alpha+1} = \langle B_\alpha \cup Y\rangle$) between B_α and $B_{\alpha+1}$. Validity of (i),(ii) and (iv) is evident, (iii) follows from (iii') and 1.1. $\square$

2.4. Lemma. *Let $x, a_0, a_1, \dots, a_{2n}$ be elements of a Boolean algebra A such that $x \geq a_0 + a_1 + \dots + a_{2n}$, $a_0 \leq \dots \leq a_{2n}$. Then $x \in \langle Y\rangle$, where $Y = \{a_0, \dots a_{2n}, x \vee a_{2n}, (x \wedge a_{2n-1}) \vee a_{2n-2}, \dots, (x \wedge a_1) \vee a_0\}$.*

Proof. Since x is the complement of a_{2n} in the interval $[x \wedge a_{2n}, x \vee a_{2n}]$, it suffices to prove that $x \wedge a_{2n} \in \langle Y\rangle$. By induction we show that $x \wedge a_i \in \langle Y\rangle$ for each $i = 0, 1, \dots, 2n$.

We have $x \wedge a_0 \geq (a_0 + \dots + a_{2n}) \wedge a_0 = a_0 + a_0 + \dots a_0 = a_0$, hence $x \wedge a_0 = a_0$, $x \wedge a_0 \in \langle Y\rangle$. Suppose now that $x \wedge a_{k-1} \in \langle Y\rangle$, $k \leq 2n$.

I. If k is odd, then $x \wedge a_k$ is the complement of a_{k-1} in the interval $[x \wedge a_{k-1}, (x \wedge a_k) \vee a_{k-1}]$ and $x \wedge a_{k-1} \in \langle Y\rangle$ implies that $x \wedge a_k \in \langle Y\rangle$.

II. If k is even, we get $a_k \geq a_{k-1} \vee (x \wedge a_k) \geq a_{k-1} \vee ((a_0 + \dots + a_{2n}) \wedge a_k) = a_{k-1} \vee (a_0 + \dots + a_k) = a_{k-1} + a_0 + a_1 + \dots + a_k + a_0 + a_1 + \dots + a_{k-2} + a_{k-1} + a_{k-1} = a_k$. Hence, $x \wedge a_k$ is the complement of a_{k-1} in $[x \wedge a_{k-1}, a_k]$, $x \wedge a_k \in \langle Y\rangle$. $\square$

2.5. Lemma. *Let $x, b_0, b_1, \dots b_{2n-1}$ be elements of a Boolean algebra A such that $x \leq b_0 + \dots + b_{2n-1}$, $b_0 \leq \dots \leq b_{2n-1}$. Then $x \in \langle Y\rangle$, where $Y = \{b_0, \dots, b_{2n-1}, (x \wedge b_1) \vee b_0, \dots, (x \wedge b_{2n-1}) \vee b_{2n-2}\}$.*

Proof. We have $x' \geq (b_0 + \dots + b_{2n-1})' = b_0 + \dots + b_{2n-1} + 1 = b'_{2n-1} + \dots + b'_0 + 1$. Now 2.4 yields that $x' \in \langle\{b'_0, \dots b'_{2n-1}, (x' \wedge b'_0) \vee b'_1, \dots, (x' \wedge b'_{2n-2}) \vee b'_{2n-1}\}\rangle = \langle Y\rangle$, which implies that $x \in \langle Y\rangle$. $\square$

2.6. Lemma. *Let K and L be sublattices of Boolean algebras C and A respectively, such that $C = \langle K \rangle$, $A = \langle L \rangle$, $K \subseteq L$, $C \leq_{rc} A$ and $A = \langle C \cup \{x\} \rangle$ for some $x \in L$. Then there exist $x_1, x_2, \dots x_m \in L$ with the properties*

(i) $(x_i)_C, (x_i)^C \in K$ *for each* $i = 1, 2, \dots m$*;*
(ii) $(x_i)^C \leq (x_j)_C$ *for each* $i < j$*;*
(iii) $A = \langle C \cup \{x_1, \dots, x_m\} \rangle$.

Proof. Let $x_C = a_0 + \cdots + a_k$, where $a_0, \dots, a_k \in K$, $a_0 \leq \cdots \leq a_k$. We can suppose that $k = 2n$ (otherwise we add 0 to the sum). By 2.4 we have $A = \langle C \cup \{y_0, \dots y_n\} \rangle$, where $y_n = x \vee a_{2n}$, $y_i = (x \wedge a_{2i+1}) \vee a_{2i}$ for $i = 0, \dots, n-1$. From 1.6 we get $(y_i)_C = ((a_0 + \cdots + a_{2n}) \wedge a_{2i+1}) \vee a_{2i} = a_{2i}$ (this holds also for $i = n$). Element y_i ($i = 0, \dots, n-1$) belongs to the interval $I_i = [a_{2i}, a_{2i+1}]$, $y_n \in I_n = [a_{2n}, 1]$. By 1.8, each I_i is, as a Boolean algebra, generated by $I_i \cap K$. Clearly $(y_i)^C \in I_i$, hence $(y_i)^C = b_0 * \cdots * b_q$, where $b_0, \dots, b_q \in I_i \cap K$, $b_0 \leq \cdots \leq b_q$ and $*$ is the addition in the Boolean algebra I_i. We can suppose that $q = 2p - 1$. Now 2.5 yields that $y_i \in \langle C \cup \{y_{i1}, \dots, y_{ip}\} \rangle$, where $y_{ij} = (y_i \wedge b_{2j-1}) \vee b_{2j-2}$. This holds in I_i as well as in A. (The complementation in I_i can be expressed by means of the operations of A.) We have $(y_{ij})_C = (a_{2i} \wedge b_{2j-1}) \vee b_{2j-2} \in K$ and $(y_{ij})^C = ((b_0 * \cdots * b_q) \wedge b_{2j-1}) \vee b_{2j-2} = b_{2j-1} \in K$. The set $\{x_1, \dots x_m\}$ will consist of all elements y_{ij}. □

2.7. Theorem. *Let a projective Boolean algebra A be generated by its sublattice L. Then there exists a projective distributive*

Proof. Let $\{A_\alpha \mid \alpha < \tau\}$ be the chain of subalgebras of A constructed in 2.3. By induction we find a sequence $\{F_\alpha \mid \alpha < \tau\}$ of free Boolean algebras (F_α with the free generating set M_α) and two sequences $\{f_\alpha \mid \alpha < \tau\}$ and $\{e_\alpha \mid a < \tau\}$ of homomorphisms ($f_\alpha \colon F_\alpha \longrightarrow A_\alpha$, $e_\alpha \colon A_\alpha \longrightarrow F_\alpha$) with the following properties:

(i) $f_\alpha e_\alpha = \mathrm{id}(A_\alpha)$, $f_\alpha(M_\alpha) \subseteq L$, $e_\alpha(f_\alpha(D_\alpha)) \subseteq D_\alpha$, for each $\alpha < \tau$, where D_α is the lattice generated by M_α in F_α;
(ii) $M_a \subseteq M_\beta$, $f_\alpha \subseteq f_\beta$, $e_\alpha \subseteq e_\beta$, for each $\alpha < \beta < \tau$.

We set $F_0 = \{0, 1\}$, $M_0 = \emptyset$ and define e_0 and f_0 by the obvious way. Let us suppose that we have constructed F_α , e_α , f_α for all $\alpha < \lambda < \tau$.

I. Let λ be a non-limit ordinal, $\lambda = \beta + 1$. Then we have $A_\beta \leq_{rc} A_\lambda = \langle A_\beta \cup \{x\} \rangle$ for some $x \in L \cap A_\lambda$, $A_\lambda = \langle L \cap A_\lambda \rangle$, $A_\beta = \langle f_\beta(D_\beta) \rangle$. We apply 2.6 with A_λ, A_β, $L \cap A_\lambda$ and $f_\beta(D_\beta)$ playing the roles of A, C, L and K respectively.Let $x_1, \dots x_m$ be the elements of $L \cap A_\lambda$ with the properties (i)–(iii) of 2.6. Take an arbitrary set $Z = \{z_1, \dots z_m\}$ of the cardinality m with $Z \cap A_\beta = \emptyset$. Let $F_\lambda \supseteq F_\beta$ be the free Boolean algebra with the free generating set $M_\lambda = M_\beta \cup Z$. Let $f_\lambda \colon F_\lambda \longrightarrow A_\lambda$ be the homomorphism uniquely determined by the conditions $f_\lambda \restriction F_\beta = f_\beta$ and $f_\lambda(z_i) = x_i$. Clearly $f_\lambda(M_\lambda) \subseteq L$. Using 1.5 we show that there exists a homomorphism $e_\lambda \colon A_\lambda \longrightarrow F_\lambda$ with $e_\beta \subseteq e_\lambda$ and $e_\lambda(x_i) = (z_i \wedge e_\beta(b_i)) \vee e_\beta(a_i)$

$(i = 1, \dots, m)$, where $a_i = (x_i)_{A_\beta}$, $b_i = (x_i)^{A_\beta}$. Suppose that Y is a finite subset of $A_\beta \cup \{x_1, \dots, x_m, x'_1, \dots, x'_m\}$ with $\bigwedge Y = 0$. We have to verify that $d = \bigwedge\{e_\lambda(y) \mid y \in Y \setminus \{x'_1, \dots, x'_m\}\} \wedge \bigwedge\{e_\lambda(y')' \mid y \in Y \cap \{x'_1, \dots, x'_m\}\} = 0$. This is trivial if $\{x_k, x'_k\} \subseteq Y$ for some k. If there is no such k, by 1.6 and 1.7 we obtain that $0 = (\bigwedge Y)^{A_\beta} = \bigwedge\{y^{A_\beta} \mid y \in Y\}$. Since e_β is an homomorphism, we have $0 = e_\beta(\bigwedge\{y^{A_\beta} \mid y \in Y\}) = \bigwedge\{e_\beta(y^{A_\beta}) \mid y \in Y\}$. For $y \in A_\beta$ we have $e_\beta(y^{A_\beta}) = e_\beta(y) = e_\lambda(y)$. If $y = x_i$ then $e_\beta(y^{A_\beta}) = e_\beta(b_i) \geq (z_i \wedge e_\beta(b_i)) \vee e_\beta(a_i) = e_\lambda(y)$. Similarly, if $y = x'_i$ then $e_\lambda(y')' = e_\lambda(x_i)' = (z'_i \vee e_\beta(b_i)') \wedge e_\beta(a_i)' \leq e_\beta(a_i)' = e_\beta(a'_i) = e_\beta(((x_i)_{A_\beta})') = e_\beta((x'_i)^{A_\beta}) = e_\beta(y^{A_\beta})$. We obtain that $d \leq \bigwedge\{e_\beta(y^{A_\beta}) \mid y \in Y\} = 0$. Thus, there is a homomorphism e_λ fulfilling the above conditions. From $a_i, b_i \in f_\beta(D_\beta)$ and $e_\beta(f_\beta(D_\beta)) \subseteq D_\beta \subseteq D_\lambda$ we deduce that $e_\lambda(f_\lambda(z_i)) = (z_i \wedge e_\beta(b_i)) \vee e_\beta(a_i) \in D_\lambda$, hence $e_\lambda(f_\lambda(D_\lambda)) \subseteq D_\lambda$. Further, $f_\lambda(e_\lambda(x_i)) = f_\lambda((z_i \wedge e_\beta(b_i)) \vee e_\beta(a_i)) = (f_\lambda(z_i) \wedge b_i) \vee a_i = x_i$, hence $f_\lambda e_\lambda$ is the identity on a generating set, which implies that $f_\lambda e_\lambda = \mathrm{id}(A_\lambda)$.

II. Let λ be a limit ordinal. Let us set $M_\lambda = \bigcup\{M_\alpha \mid \alpha < \lambda\}$, $F_\lambda = \bigcup\{F_\alpha \mid \alpha < \lambda\}$, $f_\lambda = \bigcup\{f_\alpha \mid \alpha < \lambda\}$, $e_\lambda = \bigcup\{e_\alpha \mid \alpha < \lambda\}$. Validity of (i) and (ii) is evident.

Finally, set $D = \bigcup\{f(D_\alpha) \mid \alpha < \tau\}$, $f = \bigcup\{f_\alpha \mid \alpha < \tau\}$, $e = \bigcup\{e_\alpha \mid \alpha < \tau\}$. It is clear that $D \subseteq L$ and $\langle D \rangle = A$. Moreover, D is a retract of the free distributive lattice $D_\tau = \bigcup\{D_\alpha \mid \alpha < \tau\}$ via $e \upharpoonright D$ and $f \upharpoonright D_\tau$. □

In particular, every projective Boolean algebra is generated by some of its projective distributive sublattices.

We can also formulate the consequence of 2.7 for ordered topological spaces, using the Priestley duality (see [**9**]). By this duality, projective Boolean algebras are associated with injective Boolean spaces (also called Dugundji spaces), i.e. retracts of powers of a two element discrete space. Duals of projective distributive lattices are injective Priestley spaces (with respect to the class of all embeddings), i.e. retracts of powers of a two element chain.

2.8. Corollary. *If the topology of a Priestley space P is injective, then we can extend the ordering on P in such a way that we get an injective Priestley space.*

Finally, let us present some problems. First, every free distributive lattice is a free product of three element lattices (i.e. free distributive lattices with one generator). Projective distributive lattices are just retracts of such free products. Free products of arbitrary finite (or countable) distributive lattices need not be projective, but they still generate projective Boolean algebras. The question now arises, whether the converse of this is true.

2.9. Problem. *Let a distributive lattice D generate a projective Boolean algebra $B(D)$. Is D a retract of the free product of some finite (or countable) distributive lattices?*

Another kind of problems is connected with a possible generalization of what we have proved for Boolean algebras and distributive lattices. Suppose that $\mathcal{V}$ is any variety (equational class) of algebras and $\mathcal{W}$ its subvariety. (Or, more generally, let $\mathcal{V}$ be a category and $\mathcal{W}$ its full epireflective subcategory.) For every algebra $A \in \mathcal{V}$ there exists the least congruence θ on A such that the algebra A/θ (the reflection of A) belongs to $\mathcal{W}$. It is easy to prove that if A is projective in $\mathcal{V}$ then A/θ is projective in $\mathcal{W}$. The question is whether all algebras projective in $\mathcal{W}$ are of this form.

2.10. Problem. *Given a variety $\mathcal{V}$ and its subvariety $\mathcal{W}$, decide whether every algebra projective in $\mathcal{W}$ is the reflection of some algebra projective in $\mathcal{V}$.*

References

1. Balbes R., *Projective and injective distributive lattices*, Pacific J. Math. **21** (1967), 405–420.
2. Balbes R. and Horn A., *Projective distributive lattices*, Pacific J. Math. **33** (1970), 273–279.
3. Freese R. and Nation J. B., *Projective lattices*, Pacific J. Math. **75** (1978), 93–106.
4. Grätzer G., *General lattice theory*, Akademie-Verlag Berlin, 1978.
5. Halmos P. R., *Injective and projective Boolean algebras*, *Proc. Symp. Pure Math. II*, 1961, pp. 114–122.
6. ______, *Lectures on Boolean algebras*, Toronto-New York-London, 1963.
7. Haydon R., *On a problem of Pelczynski: Milutin spaces, Dugundji spaces and AE*(0-dim), Studia Math. **52** (1974), 23–31.
8. Koppelberg S., *Projective Boolean algebras*, *Handbook on Boolean algebras*, North-Holland, 1989.
9. Priestley H. A., *Ordered sets and duality for distributive lattices*, *Orders: description and roles* (*L'Arbresle* 1982), North-Holland, Amsterdam-New York, 1984, pp. 39–60.

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