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ON JOINTLY SOMEWHAT NEARLY CONTINUOUS FUNCTIONS

Ö. VANCSÓ

Abstract. This article answers three questions of Piotrowski (see [**1**]) namely whether or not the separately somewhat, separately nearly or separately somewhat nearly continuity implies the jointly somewhat nearly continuity.
Further we investigate the weakening of the conditions of Piotrowski on the spaces X, Y, Z.

1. Introduction

This work is based on an article of Piotrowski [**1**] which gave a survay of several generalizations of the notion of continuity. In the section 4 of [**1**] three questions were raised. In the first part of our paper we shall answer these questions, in the second part we make some remarks about the conditions of the above mentioned statements.

First we summarize the basic definitions.

A space means a topological space, a function means a map from one topological space to another.

A space is M_1 (M_2) if every point has a countable base for its neighborhood system (has a countable base for the topology).

A subset A of a space X is said to be **Semi-open** if there exists an open set U in X such that $U \subset A \subset \mathrm{cl}(U)$ ($\mathrm{cl}(U)$ denotes the closure of the set U in the space X).

A subset B of a space X is said to be **Nearly open** if $B \subset \mathrm{int}(\mathrm{cl}(B))$ ($\mathrm{int}(B)$ denotes the interior of the set B in the space X).

A subset C of a space X is **Somewhat Nearly open**, if $\mathrm{int}(\mathrm{cl}(C)) \neq \emptyset$.

Let $f\colon X \to Y$ be a function. We say that f is

(i) **Quasi-Continuous**, shortly QC, if the inverse image of every open set is Semi-open.
(ii) **Somewhat Continuous**, shortly SC, if the inverse image of every open set, if it is not empty, has a non-empty interior.

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(iii) **Nearly Continuous**, shortly NC, if the inverse image of every open set is Nearly open. (This notion is called NC by Piotrowski or earlier by Pták [**2**] but almost continuous is also used by Hussain [**3**] or Long-McGehee [**4**]).

(iv) **Somewhat Nearly Continuous**, shortly SNC, if the inverse image of every open set is empty or Somewhat Nearly open.

The continuous functions will be denoted by C-functions. The diagram below illustrates the relations between the above defined classes of functions ($\longrightarrow$ denotes inclusion).

$$\begin{tikzcd}
 & & \mathrm{C} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{QC} \arrow[dl] & & \mathrm{NC} \arrow[dr] & \\
\mathrm{SC} & & & & \mathrm{SNC}
\end{tikzcd}$$

The sections of a map $f\colon X \times Y \to Z$ are defined by

$$f_x(y) = f(x,y), \quad f^y(x) = f(x,y) \quad \text{for all } x \in X,\ y \in Y.$$

The sections of a set $H \subset X \times Y$ are defined by

$$H_x = \{y : (x,y) \in H\}, \quad H^y = \{x : (x,y) \in H\}.$$

Let A be a class of functions, and $f\colon X \times Y \to Z$ a function. We say this function is **separately** of class A (e.g. separately C) if all x-sections f_x and all y-sections f^y are of class A.

To express that a function $f\colon X \times Y \to Z$ is of class A as a function of two variables we will say that f is **jointly** of class A (e.g. jointly C).

II. The Answers

In this paragraph we suppose that X is Baire, Y is M_2, Z is metric. The next theorem gives a positive answer to one of Piotrowski's questions.

Theorem. *If $f\colon X \times Y \to Z$ is separately* SC *then f is jointly* SNC.

Proof. Suppose that f is separately SC, let G be an open set in Z, and $(x_0, y_0) \in f^{-1}(G)$. Since f^{y_0} is SC, there exists a nonempty open set H in X, such that $f^{y_0}(x) \in G$ for every $x \in H$. Because f_x is SC for all $x \in H$, there is an index n (depending on x) with $f_x(V_n) \in G$, where V_n is a countable base of Y.

Let $H_n = \{x : x \in H \text{ and } f_x(V_n) \in G\}$. It is clear that $\cup_{n=1}^{\infty} H_n = H$. Since X is a Baire space, there is an index k, so that H_k is dense in some nonempty open subset U of H. Then $H_k \times V_k$ is dense in $U \times V_k$ and $f(H_k \times V_k) \subset G$ which implies that f is jointly SNC. $\square$

In what follows we show an example, which gives a negative answer to the other two questions of Piotrowski. We construct a function from I^2 to $\{0,1\}$ which is separately NC but not jointly SNC (I denotes the $[0,1]$ interval endowed with the usual topology). In this case the spaces are very nice, $X = Y$ and all spaces involved are compact, metrizable spaces.

We remark that the essence of our construction is really a subset of the unit square which is nowhere dense, but all points of this set are contained in a "cross" in which our set is sectionwise dense. The basic idea and this version of the construction was suggested by M. Laczkovich.

Example 1. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base for the unit square I^2, and let $\{(x_n, y_n) : n \in \mathbb{N}\}$ be an enumeration of the rational points of I^2. Let

$$K_{n,\delta} = [((x_n - \delta, x_n + \delta) \cap \mathbb{Q}) \times \{y_n\}] \cup [\{x_n\} \times ((y_n - \delta, y_n + \delta) \cap \mathbb{Q})],$$

where $\mathbb{Q}$ is the set of rationals. Let $k_1 = 1$, $\delta_1 = 1$, and let B_1 be a nonempty open subset of U_1 such that $\mathrm{cl}\,(B_1) \cap K_{1,\delta_1} = \emptyset$.

Let $n > 1$ and suppose that the indexes k_i, positive numbers δ_i and open sets B_i have been defined for $i < n$ such that $\emptyset \neq B_i \subset U_i$ $(i < n)$, and $(\cup_{i=1}^{n-1} \mathrm{cl}\,(B_i)) \cap K_{n-1} = \emptyset$ where $K_{n-1} = \cup_{i=1}^{n-1} K_{k_i,\delta_i}$. Then let $k_n = \min\{j : j \geq n, (x_j, y_j) \notin \cup_{i=1}^{n-1} \mathrm{cl}\,(B_i)\}$, and let $\delta_n > 0$ be so small that $K_{k_n,\delta_n} \cap \cup_{i=1}^{n-1} \mathrm{cl}\,(B_i) = \emptyset$. The set $K_n = K_{n-1} \cup K_{k_n,\delta_n}$ is nowhere dense, hence there is an open set B_n such that $\emptyset \neq B_n \subset U_n$ and $K_n \cap \mathrm{cl}\,(B_n) = \emptyset$. In this way we define k_n, δ_n, B_n for every n.

Let $K = \cup_{n=1}^{\infty} K_{k_n,\delta_n}$; then K is nowhere dense, since $K \cap B_n = \emptyset$ and $\emptyset \neq B_n \subset U_n$ for every n.

If $(x_n, y_n) \in K$ then there is $\delta > 0$ such that $K_{n,\delta} \subset K$. Indeed, the construction gives $k_n = n$ and hence $K_{n,\delta_n} = K_{k_n,\delta_n} \subset K$.

Finally, let f be the characteristic function of K. Then f is not jointly SNC, because $f^{-1}(\{1\})$ is nowhere dense although $\{1\}$ is open in $Z \overset{\text{def}}{=} \{0,1\}$.

Nevertheless it is easy to see that f is separately NC (and of course SNC as well).

This shows that separately NC or SNC does not imply jointly SNC. These results complete Table 3 of [**1**].

III. Generalization

In this paragraph we examine the question whether or not the conditions on the spaces X, Y, Z in Piotrowski's paper can be weakened. It is clear that if there is negative answer somewhere in Table 3 of [**1**], it remains negative under weakened conditions. Those positive answers that come directly from the definitions remain valid. We have to examine the positive not obvious statements only. We can not omit the Baire property of X, an example is the $\mathbb{Q} \times \mathbb{Q}$ space, where the set $\mathbb{Q}$ of rational numbers is endowed with the usual topology. In this case none of the nontrivial positive answers remain valid, as the following example shows:

Let $X = Y = \mathbb{Q}$, $Z = \{0, 1\}$. Let K be the set constructed in Example 1, and let g be the characteristic function of K in the space $\mathbb{Q} \times \mathbb{Q}$. Then g is separately C but not jointly SNC.

In the rest of this section we investigate the validity of the nontrivial positive statements of Table 3 of [**1**], supposing that Y is M_1 (instead of M_2), and Z is regular (instead of metric).

We shall examine eight positive non-trivial statements, four in the second row, three in the fourth row and one in the seventh row of Table 3 of [**1**].

The following theorem can be found in a survey article by T. Neubrunn about quasi-continuity (see p. 275, 4.1.2 Theorem in [**5**]):
Let X be Baire, Y be M_1 and Z be regular space, then the separately QC implies the jointly QC. It guarantees six positive answers, where (4.7) is used. The appropriate guaranty of (4.9) is Theorem 1, p. 350 in [**6**].

Only one question remains (in seventh row) whether separately SC implies the jointly SNC or not.

The following counter-example is due to M. Bognar:

Example 2. Let $X = Y$ be the topological sum of continuum many copies of $\mathbb{R}$ (the real numbers) with the usual topology. We may take $X = Y = \mathbb{R}^2$ endowed with the topology given by the base: $\{(a, b) \times \{c\} : a, b, c \in \mathbb{R}\}$. This space is Baire and M_1 (even metrizable). Let Z be again a discret metric space having two points. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a bijection. We define now:

$$\begin{aligned} H = \{(x, y) : x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2,\ y_2 = F(x_1, x_2)\} \\ \cup \{(x, y) : x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2,\ x_2 = F(y_1, y_2)\}. \end{aligned}$$

It is easy to check that each section H_x, H^y contains a non-empty open subset of X, resp. Y.

One can see that H is nowhere dense in $X \times Y$. Indeed, every non-empty open set in $X \times Y$ contains a non-empty open set of the form $((\alpha, \beta) \times \{c\}) \times ((\gamma, \delta) \times \{d\})$ with intervals (α, β), (γ, δ) so short that $F^{-1}(d) \notin (\alpha, \beta) \times c$ and $F^{-1}(c) \notin (\gamma, \delta) \times d$, hence $H \cap [(\alpha, \beta) \times c) \times ((\gamma, \delta) \times d)] = \emptyset$.

Finally, let f be the characteristic function of H. Then f is separately SC but not jointly SNC. So the eight statement is negative.

Remark. Y is metrizable, consequently the separately SC does not imply the jointly SNC if yet Y is metrizable.

Problem. *Does separately* SC *of* $f\colon X \times Y \to Z$ *imply jointly* SNC *if* X *is Baire,* Z *is regular and* Y *is* M_1 *and separable?*
(Our conjecture is negative.)

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Ö. Vancsó, Eötvös Loránd University, Rákóczi út 5, Budapest, Hungary