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ON SWELL–COLORED COMPLETE GRAPHS

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Abstract. An edge-colored graph is said to be **swell-colored** if each triangle contains exactly 1 or 3 colors but never 2 colors and if the graph contains more than one color. It is shown that a swell-colored complete graph with n vertices contains at least $\sqrt{n}+1$ colors. The complete graph with n^2 vertices has a swell coloring using $n+1$ colors if and only if there exists a finite affine plane of order n.

A graph with its edges colored is said to be well-colored if each triangle contains exactly 1 or 3 colors but never 2 colors. Since all graphs can be well-colored using exactly one color, those graphs which are well-colored with more than one color will be referred to as **swell-colored** graphs or **swell** graphs for short.

We shall investigate the number of colors with which a complete graph can be swell-colored. The complete graph on n vertices (generically denoted K_n) can never be swell-colored with exactly two colors. A simple investigation shows that K_3 and K_4 are the only K_n swell-colorable with exactly 3 colors; the other K_n require more colors since they are more highly connected.

For a particular value of n, what is the fewest number of colors that can give a swell K_n? This minimum completely characterizes the possible number of colors found in other swell-colorings of K_n:

Proposition 1. *If the complete graph on n vertices can be swell-colored using exactly ρ colors, $\rho < \binom{n}{2}$, then it can be swell-colored using exactly $\rho + 1$ colors.*

Before we prove this, we shall specify some terms and notation.

Definition. A **color-component** of edge-colored graph G is a maximally connected subgraph (with edge colorings inherited from G) whose edges are all of the same color. If a color-component of G has edges of color c, then we call it a c-**component** of G. If G is complete, then every two vertices v_1, v_2 are contained in a color-component, which we denote $\overleftrightarrow{v_1v_2}$. This is to be distinguished from $\overline{v_1v_2}$ which denotes the edge connecting v_1 and v_2.

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First we shall make some simple observations.

(1) Two color-components of the same color cannot share a common vertex.
(2) If G is complete and well-colored, then distinct color-components of G can share at most one vertex.
(3) A complete graph G is well-colored if and only if all its color-components are complete.
(4) Suppose G is swell and complete, and H is a c-component of G containing vertices v_1 and v_2, and P is a vertex of G not in H. Then the colors of $\overline{Pv_1}$ and $\overline{Pv_2}$ are distinct and different from c.
(5) If color-component H of G is re-colored so that each edge of H becomes a color-component, then the newly colored G is swell.

Proof of Propostion 1. Let G denote a K_n which is swell-colored with ρ colors. Suppose G has two distinct c-components of the same color, c. According to observation 1, these c-components are disjoint; re-color one of them with a new color not occurring in G. In view of observation 3, the resulting graph is swell-colored with $\rho + 1$ colors. Thus, we shall suppose, without loss of generality, that G is the union of ρ distinctly colored color-components. It is sufficient to show that we can re-color G so that the resulting graph, G', still has ρ colors in all, still is swell but has two color-components of the same color.

Since $\rho < \binom{n}{2}$, there must be some color-component, say H, containing $m \geq 3$ vertices. According to observation 3, H is a complete subgraph with vertices, say $v_1, \ldots, v_m$. Since $\rho > 1$, H is a proper subgraph of G, thus there exists a vertex of G not contained in H, say P. Denote the colors of H and $\overline{Pv_k}$ by c_0 and c_k respectively. Since G is swell, $c_0, c_1, \ldots, c_m$ are $m + 1$ distinct colors. For every two vertices of H, v_i and v_j, re-color edge $\overline{v_i v_j}$ with color c_k where $k \equiv i + j \pmod{m+1}$. We shall refer to the newly colored G and H as G' and H' respectively.

Every color of G' occurs in G, and every color of G is found in G' ($\overline{v_1 v_m}$ still has color c_0.) Thus, G' uses exactly ρ colors.

We now show that G' is swell. Consider some edge, $\overline{v_i v_j}$, of H' with color c_k where $k \equiv i + j \pmod{m+1}$. In light of the formula defining the edge-coloring of H', we see that no edge of H' adjacent to $\overline{v_i v_j}$ shares color c_k. In addition, k is neither i nor j. Thus, $\overline{Pv_i}$ and $\overline{Pv_j}$ are colored differently than $\overline{v_i v_j}$. Now, consider any vertex $Q \neq P$ of G' which is outside of H'. We claim that $\overline{Qv_i}$ cannot have color c_k. Certainly, if $k = 0$, $\overline{Qv_i}$ can't have color c_k since H is a c_0-component of G. Suppose $k \neq 0$. Then, $\overline{Pv_k}$ has color c_k, and $\overline{v_i v_k}$ has color c_0 in G. By assumption, G can't have two distinct c_k-components, so the swellness of G implies that $\overline{Qv_i}$ can't have color c_k. An identical argument shows that $\overline{Qv_j}$ can't have color c_k. We have shown that no edge adjacent to $\overline{v_i v_j}$ shares its color and so $\overline{v_i v_j} = \overleftrightarrow{v_i v_j}$. According to observation 5, G' is swell.

G' has several color-components of the same color. In particular, $\overleftrightarrow{Pv_1}$ and $\overleftrightarrow{v_2v_m}$ are disjoint c_1-components. This is sufficient to complete the proof. □

A pigeonhole argument gives us a lower bound on the number of colors in a swell K_n. We will use $\lceil m \rceil$ to denote the smallest integer not less than m.

Proposition 2. *The complete graph on n vertices cannot be swell-colored with fewer than* $\lceil \sqrt{n} \rceil + 1$ *colors.*

Proof. Let K_n be swell-colored with exactly ρ distinct colors. Let $N(v, c)$ denote the number of edges incident to vertex v which have color c. Let $\alpha = N(v_0, c_0) = max N(v, c)$, where the maximum is taken over all vertex-color combinations (v, c).

The $n - 1$ edges incident to any particular vertex can be sorted into ρ color classes, each with α or fewer members, and therefore,

$$\alpha\rho \geq n - 1. \tag{1}$$

Let $v_1, v_2, \ldots, v_\alpha$ be the vertices connected to v_0 by the α edges of color c_0. Let G denote the subgraph of K_n induced by the vertex subset $\{v_0, v_1, \ldots, v_\alpha\}$. The well-coloredness of K_n is inherited by G and so all edges of G have color c_0. Since K_n is assumed to be properly well-colored, there must be some vertex of K_n not in subgraph G, call it v_*. It will be shown that the $\alpha + 1$ edges connecting v_* to G are all distinctly colored with colors other than c_0. A consequence is that

$$\rho \geq \alpha + 2. \tag{2}$$

If an edge connecting v_* to G, say $\overline{v_*v_j}$, $0 \leq j \leq \alpha$, has color c_0 then by the well-coloredness of G, $\overline{v_*v_0}$ would have color c_0, contrary to the definition of v_*. Furthermore, if any two edges connecting v_* to G, say $\overline{v_*v_j}$ and $\overline{v_*v_k}$, $0 \leq j, k \leq \alpha$, $k \neq j$, have the same color, then the well-coloredness of K_n implies that $\overline{v_jv_k}$ shares this same color. But $\overline{v_jv_k}$ belongs to G, hence has color c_0 and so $\overline{v_*v_j}$ would have color c_0 which we have seen is impossible. Thus, inequality (2) has been established.

Inequalities (1) and (2) imply that $\rho^2 \geq 2\rho + n - 1$. It is easy to see that $\rho \geq \lceil \sqrt{n} \rceil + 1$. □

Is the lower bound of Proposition 2 ever obtained? In the next Theorem we show that it is when n is an even power of a prime. We do this by algebraically constructing the desired well-coloring. There are several approaches that work; one is to use difference quotients. Assume n is any power of a prime. Label each vertex of K_n with an element of the Galois field $F = GF(n)$. Also, associate each element of F with a unique color. Fix an arbitrary function $\Phi \colon F \to F$. The **difference quotient of** Φ, defined for distinct elements x, y of F, is

$$\Delta(x, y) = (\Phi(x) - \Phi(y))(x - y)^{-1}.$$

Color each edge of K_n using color $\Delta(x, y)$ for the edge with vertices labelled x and y. It is easy to show that this gives a well-coloring which we call **the well-coloring generated by the difference quotient of** Φ.

Theorem 3. *Suppose $n = p^{2k}$ where p is prime. The complete graph on n vertices can be swell-colored with $p^k + 1$ colors but not with fewer.*

Proof. Let F be the Galois field of order p^{2k}. Define $\Phi\colon F \to F$ by $\Phi(x) = x^{p^k}$. Consider the well-coloring generated by the difference quotient of Φ,

$$\begin{aligned}\Delta(x,y) &= (\Phi(x) - \Phi(y))(x-y)^{-1} \\ &= (x^{p^k} - y^{p^k})(x-y)^{-1} \\ &= (x-y)^{p^k}(x-y)^{-1} \\ &= (x-y)^{p^k-1}.\end{aligned}$$

Note that all values of Δ must be non-zero perfect $p^k - 1$ -th powers. There are exactly $p^k + 1$ such elements of F. Therefore, the well-coloring generated by the difference quotient of Φ uses exactly $p^k + 1$ colors. This equals the bound of Proposition 2 and so no fewer colors can suffice. □

Rédei [**1**, Theorem 24], shows when $F = GF(p^k)$, p prime, then the number of distinct values of the difference quotient of any non-linear $\Phi\colon F \to F$ must lie in one of the intervals

$$\left[1 + \frac{p^k - 1}{p^e + 1}, \frac{p^k - 1}{p^e - 1}\right],\ e = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor;\ \left[\frac{p^k + 1}{2}, p^k\right].$$

When $k = 2m$ is even then the smallest of Rédei's intervals becomes $[p^m, p^m + 1]$. According to Proposition 1, the value p^m is never attained, whereas Theorem 3 — as well as Rédei — shows that value $p^m + 1$ is attained. It would be interesting to find other simple characteristics of swell K_n, $n = p^k$, that guarantee that they are not generated by difference quotients.

What can be said about the K_{j^2} where j is not a prime power? A geometric view provides a partial answer:

A **finite affine plane** is a finite system of points and lines and an incidence relation satisfying the following three Axioms:

(1) Every two distinct points, P and Q, determine a unique line, l, such that both P and Q lie on l. We say that two lines are parallel if they are identical or if they share no common points.
(2) Given a point, P, and a line, l, there exists a unique line, parallel to l that contains P.
(3) There exist three points which do not all lie on the same line.

Any finite affine plane contains exactly j^2 points for some integer j, called its **order**. A plane of order j contains exactly $j + 1$ pencils of parallel lines, each containing j lines. Each line contains j points and each point lies on $j + 1$ lines.

Theorem 4. *The graph K_{j^2} $(j > 1)$ can be swell-colored with exactly $j+1$ colors if and only if there exists a finite affine plane of order j.*

Proof. Suppose that the graph K_{j^2} is swell-colored with exactly $j+1$ colors. Let the points of the putative affine plane consist of the vertices of the graph.

We define the lines be the color-components of K_{j^2}. We take point P to be incident to line l if and only if vertex P belongs to color-component l. The color components of a certain color form a pencil of parallel lines. From inequalities (1) and (2) of Proposition 2, it follows that $\alpha = j-1$. This means that every vertex belongs to exactly $j+1$ color-components, exactly one of each color. Thus, Axiom 2 is satisfied; the other two Axioms are obviously satisfied.

To show the converse, suppose that we have an finite affine plane, Π, of order j. Choose some correspondence betweeen the vertices of K_{j^2} and the points of Π and associate some unique color with each of the $j+1$ pencils of parallel lines of Π. In order to define a swell-coloring, consider two distinct vertices, v_1 and v_2, of K_{j^2}. Let l be the line in Π determined by the points associated with v_1 and v_2. Line l belongs to exactly one of the $j+1$ pencils of Π. Color $\overline{v_1 v_2}$ with the color associated with this pencil.

The graph, K_{j^2}, is now swell-colored by virtue of the fact that the parallel relation is an equivalence relation in Π. $\square$

For certain j, it is unknown whether there exists a finite affine plane of order j. The orders of the known affine planes are all powers of a prime. In 1949, Bruck and Ryser [**2**] proved that if j is congruent to 1 or 2 $\pmod 4$ and if j cannot be written as the sum of two squares, then there is no finite affine plane of order j. In 1988, it was shown [**3**] (by extensive computer calculation) that no affine plane of order 10 exists. At present, order 12 is the smallest order for which the issue is undecided. We view this in terms of the minimum number of colors in a swell K_{j^2}; let $\Psi(n)$ denote the smallest number of colors that could give a swell K_n.

Corollary 5. *We know that $\Psi(36) = 8$ and $\Psi(100) = 12$. Also, $\Psi(144)$ is either 13 or 14 — the exact value being unknown.*

Proof. There is no affine plane of order 6, thus no 7 color swell K_{36} exists. To produce one with 8 colors, one can just swell color K_{49} with 8 colors and remove any 13 vertices. All 8 colors must remain in the subgraph and so $\Psi(36) = 8$. The arguments for K_{100} and K_{144} are similar. $\square$

Many interesting questions remain open. It is unknown whether the only increases in Ψ occur between $n = m^2$ and $n = m^2 + 1$. In particular, where does $\Psi(n)$ first exceed 6? Does Ψ ever equal 7? It is also unknown whether there is an m for which $\Psi(m^2) > m + 2$.

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