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# SUBLATTICES OF TOPOLOGICALLY REPRESENTED LATTICES

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## 1. Introduction

In [**5**], a representation of bounded lattices within so-called standard topological contexts has been developed. Based on the theory of formal concept analysis [**14**] it includes Stone's representation of Boolean algebras by totally disconnected compact spaces [**10**], Priestley's representation of bounded distributive lattices by totally order disconnected compact spaces [**7**] as well as Urquhardt's representation of bounded lattices by so-called $L$-spaces [**11**].

In the present paper we characterize the 0-1-sublattices of an arbitrary bounded lattice within its standard topological context. To do so, the concept of a closed relation of a formal context [**16**] is generalized to the concept of a topological relation of a topological context. This is then used to describe finite subdirect products of bounded lattices. Finally, the idea of a subdirect product construction for complete lattices [**13, 16**] motivates an approach to the fusion of standard topological contexts.

Several examples illustrate the theoretical results.

## 2. Preliminaries

We briefly sketch the duality between bounded lattices and standard topological contexts worked out in [**5**]. For basic notions of the theory of formal concept analysis see [**14**]. By $(X, \tau)$ we denote a topological space where $X$ is the underlying set and $\tau$ is the family of all closed sets of the space.

We start with a triple $\mathbb{K}^\tau := ((G,\rho),(M,\sigma),I)$ consisting of two topological spaces $(G,\rho)$, $(M,\sigma)$ and a binary relation $I \subseteq G \times M$. For $A \subseteq G$ and $B \subseteq M$ we define

$$A' := \{m \in M \mid (g,m) \in I \text{ for all } g \in A\};$$
$$B' := \{g \in G \mid (g,m) \in I \text{ for all } m \in B\}.$$

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This establishes a Galois-connection between $G$ and $M$ and we obtain a complete lattice by setting

$$\underline{\mathfrak{B}}(G,M,I) := \{(A,B) \mid A \subseteq G,\ B \subseteq M,\ A' = B,\ B' = A\}$$

where $(A,B) \leq (C,D) :\Leftrightarrow A \subseteq C (\Leftrightarrow B \supseteq D)$. The lattice $\underline{\mathfrak{B}}(G,M,I)$ is called the **concept lattice** of the **context** $(G,M,I)$. Its elements are called **concepts** of $(G,M,I)$. A set $A \subseteq G$ is said to be an **extent** of $(G,M,I)$ if $(A,A')$ is a concept of $(G,M,I)$. We call a set $B \subseteq M$ an **intent** of $(G,M,I)$ if, analogously, $(B',B)$ is a concept of $(G,M,I)$. Subsequently, we write $\underline{\mathfrak{B}}(\mathbb{K}^\tau)$ instead of $\underline{\mathfrak{B}}(G,M,I)$. A **closed concept** of $\mathbb{K}^\tau$ is a concept in each component consisting of a closed set with respect to the given topologies $\rho$ and $\sigma$. The ordered set of all closed concepts is denoted by $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$.

The structure $\mathbb{K}^\tau := ((G,\rho),(M,\sigma),I)$ is called a **topological context** if the following conditions are satisfied:

(i) $A \in \rho \ \Rightarrow\ A'' \in \rho$; $\quad B \in \sigma \ \Rightarrow\ B'' \in \sigma$;

(ii) $\mathfrak{S}_\rho := \{A \subseteq G \mid (A,A') \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)\}$ is a subbasis of $\rho$ and $\mathfrak{S}_\sigma := \{B \subseteq M \mid (B',B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)\}$ is a subbasis of $\sigma$.

If $\mathbb{K}^\tau$ is a topological context the lattice $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ is bounded but not necessarily complete. In fact, it is a 0-1-sublattice of $\underline{\mathfrak{B}}(\mathbb{K}^\tau)$. A topological context is called a **standard topological context** if, in addition, the following hold:

(R) $\mathbb{K}^\tau$ is reduced, i.e., the map $g \mapsto (g'',g')$ is a bijection between $G$ and the completely join-irreducible elements of $\underline{\mathfrak{B}}(\mathbb{K}^\tau)$ and $m \mapsto (m',m'')$ is a bijection between $M$ and the completely meet-irreducible elements of $\underline{\mathfrak{B}}(\mathbb{K}^\tau)$;

(S) For every $(g,m) \in I$ there exists some $(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ such that $g \in A$ and $m \in B$;

(Q) $\left(I^c, (\rho \times \sigma)_{|I^c}\right)$ is a quasicompact space where $I^c := (G \times M) \setminus I$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

For every bounded lattice $L$ a standard topological context $\mathbb{K}^\tau(L)$ can be constructed as follows: A nonempty lattice filter $F$ of $L$ is called an ***I*-maximal filter** [**11**] if there exists a nonempty lattice ideal $I$ of $L$ such that $F \cap I = \emptyset$ and every proper superfilter $E \supset F$ already contains an element of $I$. We denote the set of all $I$-maximal filters of $L$ by $\mathfrak{F}_0(L)$. Dually, the set $\mathfrak{I}_0(L)$ of all $F$-maximal ideals of $L$ is introduced. The standard topological context of $L$ is then defined by

$$\mathbb{K}^\tau(L) := ((\mathfrak{F}_0(L),\rho_0),(\mathfrak{I}_0(L),\sigma_0),\Delta)$$

where $(F,I) \in \Delta :\Leftrightarrow F \cap I \neq \emptyset$ and $\rho_0$ and $\sigma_0$ are given by the subbasis $\mathfrak{S}_{\rho_0} := \{\{F \in \mathfrak{F}_0(L) \mid a \in F\} \mid a \in L\}$ and $\mathfrak{S}_{\sigma_0} := \{\{I \in \mathfrak{I}_0(L) \mid a \in I\} \mid a \in L\}$, respectively. For every bounded lattice $L$ the mapping

$$\iota_L : L \longrightarrow \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(L)) \qquad \iota_L(a) = (\mathfrak{F}_a, \mathfrak{I}_a)$$

where $\mathfrak{F}_a := \{F \in \mathfrak{F}_0(L) \mid a \in F\}$ and $\mathfrak{I}_a := \{I \in \mathfrak{I}_0(L) \mid a \in I\}$) is an isomorphism. Moreover, every standard topological context $\mathbb{K}^\tau$ is isomorphic to $\mathbb{K}^\tau(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau))$ via the following pair of homeomorphisms:

$$\begin{aligned} \alpha : G &\longrightarrow \mathfrak{F}_0(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)) \qquad & \alpha(g) &= \{(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau) \mid g \in A\}; \\ \beta : M &\longrightarrow \mathfrak{I}_0(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)) \qquad & \beta(m) &= \{(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau) \mid m \in B\}. \end{aligned}$$

This establishes a dual equivalence between the category of bounded lattices with onto-lattice-homomorphisms and the category of standard topological contexts with so-called standard embeddings. In [**6**], an extended version of this duality is presented keeping the objects in both categories and taking arbitrary 0-1-lattice-homomorphisms as morphisms between bounded lattices and so-called multivalued standard morphisms between standard topological contexts.

## 3. 0-1-Sublattices

Using the duality described in the previous section properties of lattices can be reformulated in the language of topological contexts. This idea has already been successfully used for complete lattices. These are investigated in terms of their formal contexts (see e.g. [**9, 8**]) which can be viewed as a kind of spectral representation. This gave rise to efficient algorithms calculating properties by computer [**3, 4, 17, 18**]. Understanding data-sets as formal contexts this yields meanings and interpretations of such properties for reality.

In this section we give a characterization of 0-1-sublattices of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ where $\mathbb{K}^\tau$ is a standard topological context. We start by recalling the description of complete sublattices of concept lattices [**16**]. Let $(G,M,I)$ be a context. A relation $J \subseteq I$ is called a **closed relation** of $(G,M,I)$ if every concept of $(G,M,J)$ is already a concept of $(G,M,I)$. There is a bijection from the set of all complete sublattices of $\underline{\mathfrak{B}}(G,M,I)$ onto the set of closed relations of $(G,M,I)$. In particular, for every complete sublattice $\mathfrak{S}$ of $\underline{\mathfrak{B}}(G,M,I)$, the relation $J_\mathfrak{S} := \bigcup_{(A,B)\in\mathfrak{S}} A \times B$ is closed and $\underline{\mathfrak{B}}(G,M,J_\mathfrak{S}) = \mathfrak{S}$. The following lemma [**16**] gives a useful characterization for closed relations.

**Lemma 1.** *A relation $J$ is a closed relation of $(G,M,I)$ if and only if $J$ is a subset of $I$ and satisfies the following conditions:*

$$\begin{aligned} (g,m) \in I \setminus J \textit{ implies } (h,m) \notin I \textit{ for some } h \in G \textit{ with } g^J \subseteq h^J \textit{ and} \\ (g,n) \notin I \textit{ for some } n \in M \textit{ with } m^J \subseteq n^J. \end{aligned}$$

The generalization of closed relations for topological contexts are topological relations.

| | 0' | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|---|
| 1' | | x | x | x | x | $\cdots$ | x |
| 1 | x | | x | x | x | $\cdots$ | x |
| 2 | x | | | x | x | $\cdots$ | x |
| 3 | x | | | | x | $\cdots$ | x |
| $\vdots$ | $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | | | | | . |
| $\omega+1$ | x | | | | | | |

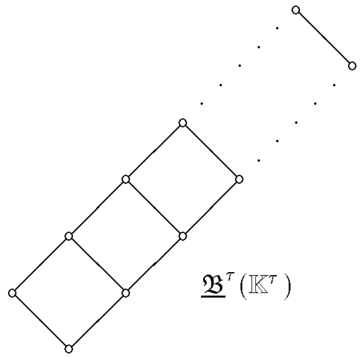


**Figure 1.** A topological context and its closed concepts.

**Definition 1.** Let $\mathbb{K}^{\tau} := ((G,\rho),(M,\sigma),I)$ be a topological context. A triple $R := (\rho_R, \sigma_R, I_R)$ is called a **topological relation** of $\mathbb{K}^{\tau}$ if the following conditions are satisfied:

(i) $\rho_R \subseteq \rho$ and $\sigma_R \subseteq \sigma$;
(ii) $I_R$ is a closed relation of $(G,M,I)$;
(iii) $\mathbb{K}^{\tau}(R) := ((G,\rho_R),(M,\sigma_R),I_R)$ is a topological context.

Note that $\underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}(R))$ is a 0-1-sublattice of $\underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau})$. But, unlike to the complete case, we can not hope to get a bijection between 0-1-sublattices and topological relations. We may have several choices to describe a given sublattice. To see this, consider the context $\mathbb{K}^{\tau}$ in Fig. 1 which is equipped with the topologies $\rho$ and $\sigma$ generated by the subbases

$$\mathfrak{S}_{\rho} := \{\{1,2,3,\dots,n\} \mid n \in \mathbb{N}\} \cup \{\{1',1,2,\dots,n\} \mid n \in \mathbb{N}_0\} \cup \{1,2,3,\dots,\omega+1\},$$
$$\mathfrak{S}_{\sigma} := \{\{0',n,n+1,\dots,\omega\} \mid n \in \mathbb{N}\} \cup \{n,n+1,n+2,\dots,\omega\} \mid n \in \mathbb{N}_0\} \cup \{0'\}.$$

This yields a topological context, which is already standard, and its closed concepts form the lattice also shown in Fig. 1. We define two topological relations $R_1 := (\rho_1,\sigma_1,I_{R_1})$ and $R_2 := (\rho_2,\sigma_2,I_{R_2})$ of $\mathbb{K}^{\tau}$ by the closed relations $I_{R_1}$ and $I_{R_2}$ shown in Fig. 2 and Fig. 3 and the topologies $\rho_1,\sigma_1,\rho_2$ and $\sigma_2$ generated by the subbases

$$\mathfrak{S}_{\rho_1} = \mathfrak{S}_{\rho_2} := \mathfrak{S}_{\rho} \setminus \{\{1'\},\{1',1\},\{1',1,2\}\};$$
$$\mathfrak{S}_{\sigma_1} = \mathfrak{S}_{\sigma_2} := \mathfrak{S}_{\sigma} \setminus \{\{0,1,2,\dots,\omega\},\{1,2,3,\dots,\omega\},\{2,3,4,\dots,\omega\}\}.$$

But though $R_1$ and $R_2$ are different they establish the same 0-1-sublattice.

In the following we investigate the case of standard topological contexts. This still includes the general situation in bounded lattices because of the duality described in Section 1. Moreover, for every topological relation, quasicompactness is available.

| | 0' | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|---|
| 1' | | | | | x | $\cdots$ | x |
| 1 | x | | x | x | x | $\cdots$ | x |
| 2 | x | | | x | x | $\cdots$ | x |
| 3 | x | | | | x | $\cdots$ | x |
| $\vdots$ | $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | | | | | . |
| $\omega+1$ | x | | | | | | |

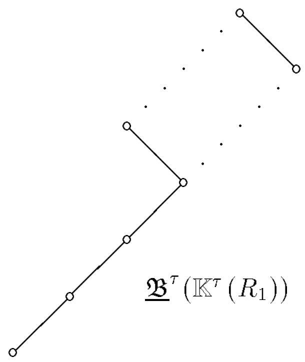


**Figure 2.** The topological relation $R_1$ and its corresponding 0-1-sublattice.

| | 0' | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|---|
| 1' | | | x | x | x | $\cdots$ | x |
| 1 | x | | x | x | x | $\cdots$ | x |
| 2 | x | | | x | x | $\cdots$ | x |
| 3 | x | | | | x | $\cdots$ | x |
| $\vdots$ | $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | | | | | . |
| $\omega+1$ | x | | | | | | |

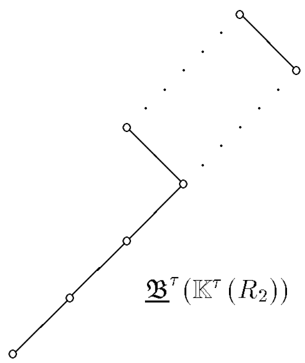


**Figure 3.** The topological relation $R_2$ and its corresponding 0-1-sublattice.

**Proposition 1.** *Let $\mathbb{K}^{\tau}$ be a standard topological context and $R := (\rho_R, \sigma_R, I_R)$ be a topological relation of $\mathbb{K}^{\tau}$. Then $\mathbb{K}^{\tau}(R)$ fulfils (Q).*

*Proof.* A subbasis of $(\rho_R \times \sigma_R)_{|I_R^c}$ is given by

$$\begin{aligned}\mathfrak{S} = &\{\{(g,m) \in I_R^c \mid g \in A\} \mid (A, A') \in \underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}(R))\}\\ &\cup \{\{(g,m) \in I_R^c \mid m \in B\} \mid (B', B) \in \underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}(R))\}\,.\end{aligned}$$

Now, let $\mathfrak{A}$ be a subset of $\mathfrak{S}$ having the finite intersection property. We define

$$\begin{aligned}\mathfrak{A}_1 &:= \{(A, A') \in \underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}(R)) \mid \{(g,m) \in I_R^c \mid g \in A\} \in \mathfrak{A}\}\,,\\ \mathfrak{A}_2 &:= \{(B', B) \in \underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}(R)) \mid \{(g,m) \in I_R^c \mid m \in B\} \in \mathfrak{A}\}\,.\end{aligned}$$

For any finite collection $(A_1, A_1'), \ldots, (A_n, A_n') \in \mathfrak{A}_1$ and $(B_1', B_1), \ldots, (B_l', B_l) \in \mathfrak{A}_2$ there is a pair $(g, m) \in I_R^c$ such that $\bigwedge_{i=1}^{n}(A_i, A_i') \in \alpha(g)$ and $\bigvee_{j=1}^{l}(B_j', B_j) \in \beta(m)$, i.e., the filter $F$ generated by $\mathfrak{A}_1$ in $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$ and the ideal $I$ generated by $\mathfrak{A}_2$ in $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$ are disjoint. Hence there exists a pair $(\tilde{g}, \tilde{m}) \in I^c$ such that $(\alpha(\tilde{g}), \beta(\tilde{m}))$ is a maximal pair of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$ with $\alpha(\tilde{g}) \supseteq F$ and $\beta(\tilde{m}) \supseteq I$ an so $(\tilde{g}, \tilde{m}) \in \bigcap \mathfrak{A}$. □

If $\mathfrak{S}$ is a 0-1-sublattice of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ then there are two canonical topologies coming along with $\mathfrak{S}$, namely the topology $\rho_{\mathfrak{S}}$ on $G$ generated by the subbasis $\{A \subseteq G \mid (A, A') \in \mathfrak{S}\}$ and the topology $\sigma_{\mathfrak{S}}$ generated by the subbasis $\{B \subseteq M \mid (B', B) \in \mathfrak{S}\}$. Furthermore, $\mathfrak{S}$ yields the canonical relation $I_{\mathfrak{S}} := \bigcup_{(A,B)\in\mathfrak{S}}(A \times B)$.

**Proposition 2.** *Let $\mathbb{K}^\tau$ be a standard topological context and $\mathfrak{S}$ be a 0-1-sublattice of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$. Then the following are equivalent:*

(i) *$R := (\rho_R, \sigma_R, I_R)$ is a topological relation of $\mathbb{K}^\tau$ and $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R)) = \mathfrak{S}$.*
(ii) *$\rho_R = \rho_{\mathfrak{S}}$, $\sigma_R = \sigma_{\mathfrak{S}}$ and $I_R$ is a closed relation with $I_{\mathfrak{S}} \subseteq I_R$.*

*Proof.* The implication (i) ⇒ (ii) is obvious. Conversely, let $R := (\rho_{\mathfrak{S}}, \sigma_{\mathfrak{S}}, J)$ be a triple where $J$ is a closed relation with $I_{\mathfrak{S}} \subseteq J$. First we prove that $R$ is a topological relation. The crucial point is to identify $\mathbb{K}^\tau(R)$ as a topological context. To see this, let $A \in \rho_{\mathfrak{S}}$, i.e.,

$$A = \bigcap_{t\in T} \bigcup_{r\in R_t} A_{tr} \quad \text{where} \quad (A_{tr}, A_{tr}^J) \in \mathfrak{S} \quad \text{and} \quad R_t = \{1, \ldots, n_t\}.$$

We claim and prove further below that $A^J = \bigcup_{\hat{A}\in\mathfrak{B}_T} \hat{A}^J$ $(*)$ where

$$\hat{A} \in \mathfrak{B}_T \;:\Longleftrightarrow\; \hat{A} = \left( \bigcup_{\varphi\in \times_{t\in E} R_t} \left( \bigcap_{t\in E} A_{t\varphi(t)} \right) \right)^{JJ} \quad \text{for some finite } E \subseteq T.$$

Since $J$ is a closed relation $(\hat{A}, \hat{A}') \in \mathfrak{S}$ for every $\hat{A} \in \mathfrak{B}_T$, i.e., $\mathfrak{B}_T \subseteq \rho_{\mathfrak{S}}$. Then $(*)$ implies $A^{JJ} = \bigcap_{\hat{A}\in\mathfrak{B}_T} \hat{A}^{JJ} = \bigcap_{\hat{A}\in\mathfrak{B}_T} \hat{A} \in \rho_{\mathfrak{S}}$. Similar arguments show $B^{JJ} \in \sigma_{\mathfrak{S}}$ for every $B \in \sigma_{\mathfrak{S}}$. Thus, $\mathbb{K}^\tau(R)$ is a topological relation.

Now we prove $(*)$. Let $m \in \hat{A}^J$ for some $\hat{A} \in \mathfrak{B}_T$ and $g \in A$. For every $t \in T$ there exists $\hat{r} \in R_t$ such that $g \in A_{t\hat{r}}$. Let $\hat{E}$ be the finite subset of $T$ corresponding to $\hat{A}$. Then there exists $\hat{\varphi} \in \times_{t\in\hat{E}} R_t$ such that $A_{t\hat{\varphi}(t)} = A_{t\hat{r}}$ for all $t \in \hat{E}$ and so $g \in \bigcap_{t\in\hat{E}} A_{t\hat{\varphi}(t)} \subseteq \hat{A}$. Hence $(g, m) \in J$ and therefore $m \in A^J$.

Now, let $m \notin \hat{A}^J$ for all $\hat{A} \in \mathfrak{B}_T$. Then, for every finite $E \subseteq T$, there is some $\hat{\varphi} \in \times_{t\in E} R_t$ with $m \notin \left(\bigcap_{t\in E} A_{t\hat{\varphi}(t)}\right)^J$, i.e., for every finite $E \subseteq T$, there exists a function

$$f_E \;:\; E \longrightarrow \bigcup_{t\in E} \{A_{tr} \mid r \in R_t\}$$

such that $f_E(t) \in \{A_{tr} \mid r \in R_t\}$ for all $t \in E$ and $m \notin \left(\bigcap_{t\in E} f_E(t)\right)^J$. Using Rado's Selection Theorem [**1**] we get the existence of a global function

$$f \,:\, T \longrightarrow \bigcup_{t\in T}\{A_{tr} \mid r \in R_t\}$$

such that $f(t) \in \{A_{tr} \mid r \in R_t\}$ for all $t \in T$. Moreover, for every finite $E \subseteq T$, there is some finite $F \subseteq T$ such that $E \subseteq F$ and $f_{|E} = f_{F|E}$. Let $\hat{F}$ be the filter of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ generated by $\{(f(t), f(t)') \mid t \in T\}$. For $A_{t_1r_1}, \ldots, A_{t_nr_n} \in f(T)$ and $E := \{t_1, \ldots, t_n\}$ there is some finite $F \supseteq E$ such that $\bigcap_{i=1}^n A_{t_ir_i} = \bigcap_{t\in E} f(t) = \bigcap_{t\in E} f_F(t) \supseteq \bigcap_{t\in F} f_F(t)$. Since $m \notin \left(\bigcap_{t\in F} f_F(t)\right)^J$ we conclude $m \notin (\bigcap_{i=1}^n A_{t_nr_n})^J$. Hence $\hat{F}$ and $\beta(m) := \{(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau) \mid m \in B\}$ form a disjoint filter-ideal pair and, by [**5**, Lemma 2.1.5], there exists a maximal filter-ideal pair $(\tilde{F}, \tilde{I})$ such that $\tilde{F} \supseteq \hat{F}$ and $\tilde{I} \supseteq \beta(m)$. By [**5**, Theorem 2.2.4], $\tilde{F} = \alpha(\tilde{g})$ for some $\tilde{g} \in G$ and $\tilde{I} = \beta(\tilde{m})$ for some $\tilde{m} \in M$. Then $\tilde{g} \in A$ and $\tilde{m} \notin A^J$ imply $m \notin A^J$. Thus, $(*)$ is proved.

It remains to show $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R)) = \mathfrak{S}$. Clearly, $\mathfrak{S} \subseteq \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$. If $(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$ with $A \neq \emptyset$ then $A = \bigcap_{\hat{A}\in\mathfrak{B}} \hat{A}$ for some suitable $\mathfrak{B} \subseteq \{A \subseteq G \mid (A, A') \in \mathfrak{S}\}$. Suppose that, for every finite $E \subseteq \mathfrak{B}$, the extent $A$ is a proper subset of $\bigcap_{\hat{A}\in E} \hat{A}$. We define a nonempty family of closed sets by $\mathfrak{N} := (N_E)_{E\in\mathfrak{E}_T}$ where $\mathfrak{E}_T := \{E \subseteq \mathfrak{B} \mid E \text{ is finite }\}$ and

$$N_E := \{(g,m) \in I_R^c \mid g \in \bigcap_{\hat{A}\in E} \hat{A} \text{ and } m \in B\}.$$

Since $N_E \cap N_F = N_{E\cup F} \neq \emptyset$ the family $\mathfrak{N}$ has the finite intersection property. By quasicompactness, there exists some $(\hat{g}, \hat{m}) \in \bigcap \mathfrak{N}$, i.e., $\hat{g} \in A$ and $\hat{m} \in B$. This is a contradiction. Hence there exists some finite $E \subseteq \mathfrak{B}$ such that $A = \bigcap_{\hat{A}\in E} \hat{A}$ which proves $(A,B) = \bigwedge_{\hat{A}\in E}(\hat{A}, \hat{A}') \in \mathfrak{S}$. □

Of course, $I$ is a closed relation and therefore $F_\mathfrak{S} := (\rho_\mathfrak{S}, \sigma_\mathfrak{S}, I)$ is the greatest topological relation among all topological relations describing a given 0-1-sublattice $\mathfrak{S}$. On the other hand, given a topological relation $R := (\rho_R, \sigma_R, I)$, we immediately conclude $\rho_R = \rho_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$ and $\sigma_R = \sigma_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$ since in both cases the generating subbases coincide. Let us call a topological relation $R := (\rho_R, \sigma_R, I_R)$ **full** if $I_R = I$. We proved the following theorem.

**Theorem 1.** *Let $\mathbb{K}^\tau$ be a standard topological context. Then there is a bijection from the set of all* 0-1*-sublattices of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ onto the set of all full topological relations. In particular, for a* 0-1*-sublattice $\mathfrak{S}$, the relation $F_\mathfrak{S} := (\rho_\mathfrak{S}, \sigma_\mathfrak{S}, I)$ is a full topological relation with $\mathfrak{S} = \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(F_\mathfrak{S}))$.*

In fact, there is also a smallest topological relation among all topological relations describing a given 0-1-sublattice $\mathfrak{S}$.

**Proposition 3.** *Let $\mathfrak{S}$ be a* 0-1-*sublattice of* $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ *where* $\mathbb{K}^\tau$ *is a standard topological context. Then* $I_{\mathfrak{S}}$ *is a closed relation.*

*Proof.* We use the characterization given in Lemma 1. Let $(g,m) \in I \setminus I_{\mathfrak{S}}$ and define

$$\begin{aligned}\mathfrak{N}_{(g,m)} := &\{\{(h,n) \in I^c \mid h \in A\} \mid g \in A \text{ and } (A,A') \in \mathfrak{S}\}\\ &\cup \{\{(h,n) \in I^c \mid n \in B\} \mid m \in B \text{ and } (B',B) \in \mathfrak{S}\}.\end{aligned}$$

Then $\mathfrak{N}_{(g,m)}$ is a nonempty family of closed sets. Moreover, $\mathfrak{N}_{(g,m)}$ has the finite intersection property because otherwise

$$\{(h,n) \in I^c \mid h \in A\} \cap \{(h,n) \in I^c \mid n \in B\} = \emptyset$$

for some $A$ fulfilling $g \in A$ and $(A,A') \in \mathfrak{S}$ and for some $B$ fulfilling $m \in B$ and $(B',B) \in \mathfrak{S}$. This implies $(A,A') \leq (B',B)$ which is a contradiction to $(g,m) \notin I_{\mathfrak{S}}$. By quasicompactness, there exists some $(\tilde{h},\tilde{n}) \in \bigcap \mathfrak{N}_{(g,m)}$, i.e., $\alpha(\tilde{h}) \supseteq \alpha(g) \cap \mathfrak{S}$ and $\beta(\tilde{n}) \supseteq \beta(m) \cap \mathfrak{S}$ which means $\tilde{h}^{I_{\mathfrak{S}}} \supseteq g^{I_{\mathfrak{S}}}$ and $\tilde{n}^{I_{\mathfrak{S}}} \supseteq m^{I_{\mathfrak{S}}}$. Thus, by Lemma 1, $I_{\mathfrak{S}}$ is a closed relation. □

We call a topological relation $R := (\rho_R, \sigma_R, I_R)$ **separating** if $\mathbb{K}^\tau(R)$ satisfies condition (S). Clearly, given a 0-1-sublattice $\mathfrak{S}$, the topological context $\mathbb{K}^\tau(S_{\mathfrak{S}})$ is separating where $S_{\mathfrak{S}} := (\rho_{\mathfrak{S}}, \sigma_{\mathfrak{S}}, I_{\mathfrak{S}})$. Conversely, separating topological relations are characterized by this construction.

| | 0' | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|---|
| 1' | | | | | x | $\cdots$ | x |
| 1 | x | | | x | x | $\cdots$ | x |
| 2 | x | | | x | x | $\cdots$ | x |
| 3 | x | | | | x | $\cdots$ | x |
| $\vdots$ | $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | | | | | | . |
| $\omega+1$ | x | | | | | | |

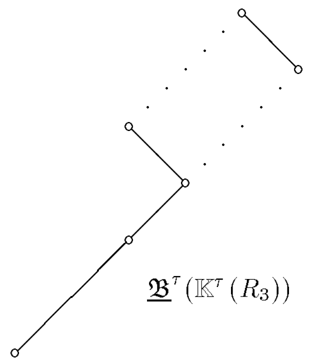


**Figure 4.** The topological relation $R_3$ and its corresponding 0-1-sublattice.

**Proposition 4.** *Let* $\mathbb{K}^\tau$ *be a standard topological context and* $R := (\rho_R, \sigma_R, I_R)$ *be a separating topological relation. Then* $R = S_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$.

*Proof.* The family $\{A \subseteq G \mid (A,A') \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))\}$ is a subbasis for both, $\rho_R$ and $\rho_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$, and $\{B \subseteq M \mid (B',B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))\}$ is a subbasis for $\sigma_R$ and $\sigma_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$.

Clearly, $I_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))} \subseteq I_R$. Since $R$ is separating, for every $(g,m) \in I_R$, there is a concept $(A,B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))$ such that $g \in A$ and $m \in B$. Hence $(g,m) \in A \times B \subseteq I_{\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R))}$. □

**Theorem 2.** *Let $\mathbb{K}^\tau$ be a standard topological context. Then there is a bijection from the set of all 0-1-sublattices of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ onto the set of all separating topological relations. In particular, for a 0-1-sublattice $\mathfrak{S}$, the relation $S_\mathfrak{S} := (\rho_\mathfrak{S}, \sigma_\mathfrak{S}, I_\mathfrak{S})$ is a separating topological relation with $\mathfrak{S} = \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(S_\mathfrak{S}))$.*

For every closed relation $J$ of a standard topological context $\mathbb{K}^\tau$ we obtain a 0-1-sublattice of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ by $\mathfrak{S}_J := \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau) \cap \underline{\mathfrak{B}}(G,M,J)$. Hence $R_J := (\rho_J, \sigma_J, J)$ is a topological relation where $\rho_J$ is generated by $\{A \subseteq G \mid (A,A') \in \mathfrak{S}_J\}$ and $\sigma_J$ is generated by $\{B \subseteq M \mid (B',B) \in \mathfrak{S}_J\}$. We call a closed relation $J$ of a standard topological context **separating** if $R_J$ is separating.

**Corollary 1.** *Let $\mathbb{K}^\tau$ be a standard topological context. Then there is a bijection from the set of all 0-1-sublattices of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ onto the set of all separating closed relations. In particular, for a 0-1-sublattice $\mathfrak{S}$, the relation $I_\mathfrak{S}$ is a separating closed relation with $\mathfrak{S} = \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_{I_\mathfrak{S}}))$.*

| | 0' | 0 | 1 | 2 | 3 | ··· | $\omega$ |
|---|---|---|---|---|---|---|---|
| 1' | | | | x | x | ··· | x |
| 1 | x | | | x | x | ··· | x |
| 2 | x | | | x | x | ··· | x |
| 3 | x | | | | x | ··· | x |
| ⋮ | ⋮ | | | | | ⋱ | ⋮ |
| ⋮ | ⋮ | | | | | | . |
| $\omega+1$ | x | | | | | | |

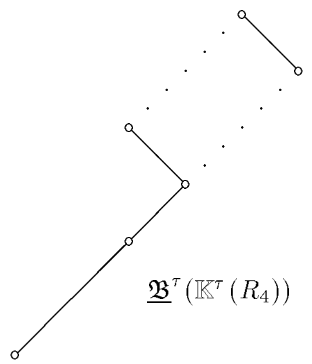


**Figure 5.** The topological relation $R_4$ and its corresponding 0-1-sublattice.

The only full and separating topological relation is $(\rho, \sigma, I)$. Of course, every full topological relation satisfies (R). Separating topological relations may or may not satisfy (R). There are topological relations being neither full nor separating. Among these some fulfil (R) and others do not. For illustration we consider some examples. The topological relation $R_1$ in Fig. 2 is separating and reduced. The relation $R_2$ in Fig. 3 is neither full nor separating but reduced. The topological relation $R_3$ shown in Fig. 4 is separating but not reduced whereas $R_4$ in Fig. 5 is neither full nor separating and, in addition, not reduced. $(R_3 := (\rho_3, \sigma_3, I_{R_3})$

and $R_4 := (\rho_4, \sigma_4, I_{R_4})$ are topological relations of the context $\mathbb{K}^\tau$ in Fig. 1. Their topologies $\rho_3, \sigma_3, \rho_4$ and $\sigma_4$ are generated by

$$\mathfrak{S}_{\rho_3} = \mathfrak{S}_{\rho_4} := \mathfrak{S}_{\rho_1} \setminus \{1\},$$
$$\mathfrak{S}_{\sigma_3} = \mathfrak{S}_{\sigma_4} := \mathfrak{S}_{\sigma_1} \setminus \{0', 1, 2, \dots, \omega\}.)$$

## 4. Direct Products

Before we can start to investigate subdirect products it is necessary to characterize direct products of bounded lattices within the corresponding standard topological contexts. For an arbitrary family of contexts $(\mathbb{K}_t)_{t\in T}$ it is well-known (see e.g. [**16**]) that

$$\bigtimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t) \cong \underline{\mathfrak{B}}\left(\sum_{t\in T} \mathbb{K}_t\right)$$

where

$$\sum_{t\in T} \mathbb{K}_t = \sum_{t\in T} (G_t, M_t, I_t) := \left(\dot{\bigcup}_{t\in T} G_t, \dot{\bigcup}_{t\in T} M_t, \dot{\bigcup}_{t\in T} I_t \dot{\cup} \dot{\bigcup}_{\substack{s,t\in T\\ s\neq t}} (G_s \times M_t)\right)$$

is called the **sum of the contexts** $(\mathbb{K}_t)_{t\in T}$. An isomorphism is given by

$$\iota_c\colon \bigtimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t) \longrightarrow \underline{\mathfrak{B}}\left(\sum_{t\in T} \mathbb{K}_t\right) \qquad \iota_c\left(((A_t, B_t))_{t\in T}\right) := \left(\dot{\bigcup}_{t\in T} A_t, \dot{\bigcup}_{t\in T} B_t\right).$$

Let us define the **sum of the topological contexts** $(\mathbb{K}_t^\tau)_{t\in T}$ by

$$\begin{aligned}\sum_{t\in T} \mathbb{K}_t^\tau &= \sum_{t\in T} ((G_t, \rho_t), (M_t, \sigma_t), I_t) \\ &:= \left(\left(\dot{\bigcup}_{t\in T} G_t, \rho\right), \left(\dot{\bigcup}_{t\in T} M_t, \sigma\right), \dot{\bigcup}_{t\in T} I_t \dot{\cup} \dot{\bigcup}_{\substack{s,t\in T\\ s\neq t}} (G_s \times M_t)\right)\end{aligned}$$

where $A \in \rho :\Leftrightarrow A \cap G_t \in \rho_t$ for all $t \in T$ and $B \in \sigma :\Leftrightarrow B \cap M_t \in \sigma_t$ for all $t \in T$. It is straightforward to see that this definition yields again a topological context. Moreover, we get a description for the direct product.

**Proposition 5.** *Let $(\mathbb{K}_t^\tau)_{t\in T}$ be a family of topological contexts. Then*

$$\bigtimes_{t\in T} \underline{\mathfrak{B}}^\tau(\mathbb{K}_t^\tau) \cong \underline{\mathfrak{B}}^\tau\left(\sum_{t\in T} \mathbb{K}_t^\tau\right).$$

*Proof.* An isomorphism is given by

$$\iota_b\colon \bigtimes_{t\in T} \underline{\mathfrak{B}}^\tau(\mathbb{K}_t^\tau) \to \underline{\mathfrak{B}}^\tau\left(\sum_{t\in T} \mathbb{K}_t^\tau\right) \qquad \iota_b\left(((A_t, B_t))_{t\in T}\right) := \left(\dot{\bigcup}_{t\in T} A_t, \dot{\bigcup}_{t\in T} B_t\right).$$

$\square$

Taking a family of standard topological contexts their sum satisfies (R) and (S). But unfortunately, if the given family has infinite cardinality (Q) is no longer valid. Therefore, the infinite sum is not isomorphic to the standard topological context of the direct product. As an example we take the countable direct product $2^{\mathbb{N}}$ where 2 is the two-element lattice. The set $G$ of the standard topological context $\mathbb{K}^{\tau}(2)$ consists of exactly one element and therefore the set $G$ of the sum contains countably many elements. On the other hand, every element of the set $G$ of the standard topological context of $2^{\mathbb{N}}$ is an $I$-maximal filter of $2^{\mathbb{N}}$. But this lattice has uncountably many $I$-maximal filters since those are exactly the ultrafilters. However, if we restrict ourselves to finite families of standard topological contexts the sum stays standard.

**Proposition 6.** *Let $(\mathbb{K}_i^{\tau})_{i=1,\dots,n}$ be a family of standard topological contexts. Then*

$$\sum_{i=1}^{n} \mathbb{K}_i^{\tau} \cong \mathbb{K}^{\tau}\left( \bigtimes_{i=1}^{n} \underline{\mathfrak{B}}^{\tau}(\mathbb{K}_i^{\tau}) \right).$$

*Proof.* The set $\left(\dot{\bigcup}_{i=1}^{n} I_i \dot{\cup} \dot{\bigcup}_{\substack{j,i\in\{1,\dots,n\}\\ j\neq i}} (G_j \times M_i)\right)^c = \dot{\bigcup}_{i=1}^{n} (I_i)^c$ is a finite disjoint union of quasicompact spaces and therefore quasicompact. □

## 5. Subdirect Products

The last two sections suggest a method how to find finite subdirect products of bounded lattices within the sum of the corresponding standard topological contexts. We have to look for certain topological relations. For two reasons we concentrate on separating topological relations. Firstly, they give a minimal description of 0-1-sublattices providing the existence-property (S). Secondly, they fit in with the theory of bonds [**16**] which we briefly review in the following.

A **bond** from a context $(G_i, M_i, I_i)$ to a context $(G_j, M_j, I_j)$ is a subset $J_{ij}$ of $G_i \times M_j$ such that for every $g \in G_i$ the set $g^j := \{m \in M_j \mid (g,m) \in J_{ij}\}$ is an intent of $(G_j, M_j, I_j)$ and for every $m \in M_j$ the set $m^i := \{g \in G_i \mid (g,m) \in J_{ij}\}$ is an extent of $(G_i, M_i, I_i)$. If $J_{ij}$ is a bond from $(G_i, M_i, I_i)$ to $(G_j, M_j, I_j)$ and $J_{jk}$ is a bond from $(G_j, M_j, I_j)$ to $(G_k, M_k, I_k)$ then $J_{ij} \circ J_{jk} := \{(g,m) \in G_i \times M_k \mid g^{jj} \subseteq m^j\}$ is a bond from $(G_i, M_i, I_i)$ to $(G_k, M_k, I_k)$.

Now, let $(\mathbb{K}_t)_{t\in T}$ be a family of contexts and $\iota_c$ be the isomorphism from $\bigtimes_{t\in T} \underline{\mathfrak{B}}(\mathbb{K}_t)$ onto $\underline{\mathfrak{B}}\left(\sum_{t\in T} \mathbb{K}_t\right)$. Furthermore, let $J$ be a subset of $\dot{\bigcup}_{t\in T} G_t \times \dot{\bigcup}_{t\in T} M_t$ and let $J_{st} := J \cap (G_s \times M_t)$ for $s,t \in T$. Then the following conditions are equivalent (see [**16**, Theorem 6]):

(i) $\iota_c^{-1}\left(\underline{\mathfrak{B}}\left(\dot{\bigcup}_{t\in T} G_t, \dot{\bigcup}_{t\in T} M_t, J\right)\right)$ is a complete subdirect product of the $(\underline{\mathfrak{B}}(\mathbb{K}_t))_{t\in T}$.

(ii) $J$ is a closed relation of $\sum_{t\in T} \mathbb{K}_t$ with $J_{tt} = I_t$ for all $t \in T$.

(iii) The $J_{st}$ are bonds from $(G_s, M_s, I_s)$ to $(G_t, M_t, I_t)$ with $J_{tt} = I_t$ and $J_{rt} \subseteq J_{rs} \circ J_{st}$ for all $r, s, t \in T$.

| | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|
| 1 | | x | x | x | $\cdots$ | x |
| 2 | | | x | x | $\cdots$ | x |
| 3 | | | | x | $\cdots$ | x |
| $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | | | | | | . |
| $\omega+1$ | | | | | | |



$\underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau})$

**Figure 6.**

If we consider the finite sum of standard topological contexts closed relations fulfilling those conditions may occur which are not separating. Fig. 6 presents a standard topological context $\mathbb{K}^{\tau}$ and its lattice of all closed concepts where the topologies $\rho$ and $\sigma$ are given by the subbases

$$\mathfrak{S}_{\rho} := \{\{1\}, \{1,2\}, \{1,2,3\} \ldots\},$$
$$\mathfrak{S}_{\sigma} := \{\{1,2,3,\ldots,\omega\}, \{2,3,4,\ldots,\omega\}, \{3,4,5,\ldots,\omega\}, \ldots\}.$$

Now, there is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K}^{\tau} + \mathbb{K}^{\tau})$, boldface in the line diagram in Fig. 7, corresponding to a complete subdirect product of $(\underline{\mathfrak{B}}(\mathbb{K}^{\tau}))^2$ which does not induce a subdirect product of $(\underline{\mathfrak{B}}^{\tau}(\mathbb{K}^{\tau}))^2$. But then this yields a closed relation which is not separating (Fig. 8).

Let us call a bond $J_{ij}$ from a topological context $\mathbb{K}_i^{\tau}$ to a topological context $\mathbb{K}_j^{\tau}$ **topological** if $(g,m) \in J_{ij}$ always implies $\overline{g} \times \overline{m} \subseteq J_{ij}$ where $\overline{g}$ and $\overline{m}$ are the topological closures of $g$ and $m$ in $(G_i, \rho_i)$ and $(M_j, \sigma_j)$, respectively.

**Proposition 7.** *Let $J_{ij}$ be a topological bond from $\mathbb{K}_i^{\tau}$ to $\mathbb{K}_j^{\tau}$ and $J_{jk}$ be a topological bond from $\mathbb{K}_j^{\tau}$ to $\mathbb{K}_k^{\tau}$. Then $J_{ij} \circ J_{jk}$ is a topological bond from $\mathbb{K}_i^{\tau}$ to $\mathbb{K}_k^{\tau}$.*

*Proof.* Let $(g,m) \in J_{ij} \circ J_{jk}$. For $h \in \overline{g}$ we conclude $g^j \subseteq h^j$ since $J_{ij}$ is a topological bond and so $g^{jjk} \subseteq h^{jjk}$. On the other hand, $g^{jj} \subseteq m^j$ is equivalent to $m \in g^{jjk}$. Since $J_{jk}$ is a topological bond we obtain $n \in g^{jjk}$ for every $n \in \overline{m}$ showing $n \in h^{jjk}$ which is equivalent to $h^{jj} \subseteq n^j$. Hence $\overline{g} \times \overline{m} \subseteq J_{ij} \circ J_{jk}$. This proves that $J_{ij} \circ J_{jk}$ is a topological bond. $\square$

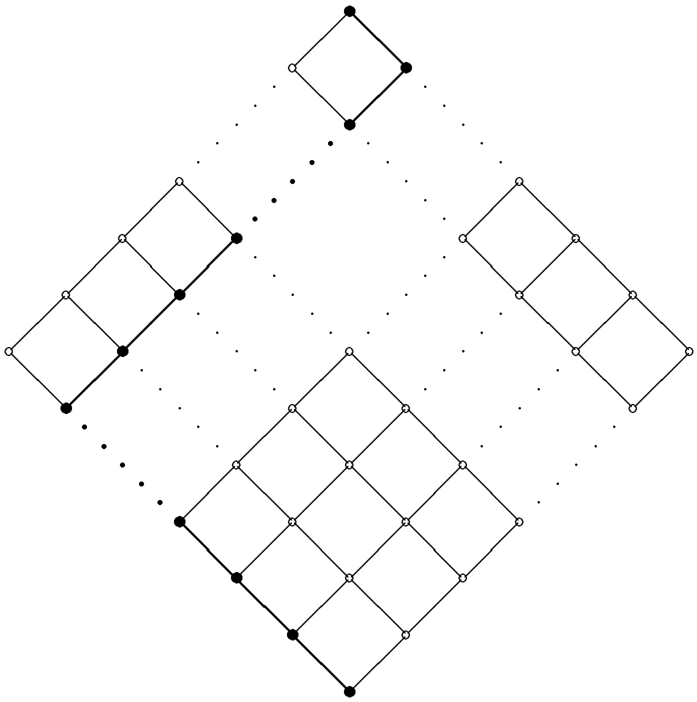

**Figure 7.** $\underline{\mathfrak{B}}(\mathbb{K}^\tau + \mathbb{K}^\tau)$.

| | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ | 0 | 1 | 2 | 3 | $\cdots$ | $\omega$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | x | x | x | $\cdots$ | x | x | x | x | x | $\cdots$ | x |
| 2 | | | x | x | $\cdots$ | x | x | x | x | x | $\cdots$ | x |
| 3 | | | | x | $\cdots$ | x | x | x | x | x | $\cdots$ | x |
| $\vdots$ | | | | | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | | | | | . | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $\omega+1$ | | | | | | | x | x | x | x | $\cdots$ | x |
| 1 | | | | | | x | | x | x | x | $\cdots$ | x |
| 2 | | | | | | x | | | x | x | $\cdots$ | x |
| 3 | | | | | | x | | | | x | $\cdots$ | x |
| $\vdots$ | | | | | | $\vdots$ | | | | | $\ddots$ | $\vdots$ |
| $\vdots$ | | | | | | $\vdots$ | | | | | | . |
| $\omega+1$ | | | | | | x | | | | | | |

**Figure 8.** A closed relation of $\mathbb{K}^\tau + \mathbb{K}^\tau$ which is not separating.

Before we can give the characterization of the separating closed relations of $\sum_{i=1}^{n}(\mathbb{K}_i^\tau)$ corresponding to subdirect products of the $(\underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau))_{i\in\{1,\dots,n\}}$ we prove a result about standard topological contexts which needs similar arguments as we have already used in the proof of Proposition 2.

**Lemma 2.** *Let $\mathbb{K}^\tau$ be a standard topological context, let $\mathfrak{S}$ be a proper 0-1-sublattice of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ and let $(A,B)\in\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)\setminus\mathfrak{S}$. Then $A\notin\rho_{\mathfrak{S}}$ and $B\notin\sigma_{\mathfrak{S}}$.*

*Proof.* Suppose $A\in\rho_{\mathfrak{S}}$, i.e.,

$$A=\bigcap_{t\in T}\bigcup_{r\in R_t}A_{tr}\quad\text{where}\quad (A_{tr},A'_{tr})\in\mathfrak{S}\quad\text{and}\quad R_t=\{1,\dots,n_t\}.$$

For every finite $E\subseteq T$ we find $A\subset\bigcap_{t\in E}\bigcup_{r\in R_t}A_{tr}$ because otherwise

$$A=\bigcap_{t\in E}\bigcup_{r\in R_t}A_{tr}=\left(\bigcup_{\varphi\in\times_{t\in E}R_t}\left(\bigcap_{t\in E}A_{t\varphi(t)}\right)\right)''$$

would be an extent belonging to a concept in $\mathfrak{S}$ which is contrary to our assumption. Hence, for every finite $E\subseteq T$, there is a function

$$f_E\ :\ E\longrightarrow\bigcup_{t\in E}\{A_{tr}\mid r\in R_t\}$$

such that $f_E(t)\in\{A_{tr}\mid r\in R_t\}$ for all $t\in E$ and $A\subset\bigcap_{t\in E}f_E(t)$. By Rado's Selection Theorem [**1**] we get the existence of a global function

$$f:T\longrightarrow\bigcup_{t\in T}\{A_{tr}\mid r\in R_t\}$$

such that $f(t)\in\{A_{tr}\mid r\in R_t\}$ for all $t\in T$. Moreover, for every finite $E\subseteq T$, there is some finite $F\subseteq T$ such that $E\subseteq F$ and $f_{|E}=f_{F|E}$. Let $\hat{F}$ be the filter of $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ generated by $\{(f(t),f(t)')\mid t\in T\}$. For $A_{t_1r_1},\dots,A_{t_nr_n}\in f(T)$ and $E:=\{t_1,\dots,t_n\}$ there is some finite $F\supseteq E$ such that $\bigcap_{i=1}^{n}A_{t_ir_i}=\bigcap_{t\in E}f(t)=\bigcap_{t\in E}f_F(t)\supseteq\bigcap_{t\in F}f_F(t)\supset A$. Then $(A,B)\notin\hat{F}$. By [**5**, Lemma 2.1.5], there is some $\tilde{F}\in\mathfrak{F}_0(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau))$ such that $\hat{F}\subseteq\tilde{F}$ and $(A,B)\notin\tilde{F}$. By [**5**, Theorem 2.2.4], $\tilde{F}=\alpha(g)$ for some $g\in G$. But then, $g\in\left(\bigcap_{t\in T}\bigcup_{r\in R_t}A_{tr}\right)$ and $g\notin A$ which is a contradiction. Analogous arguments show $B\notin\sigma_{\mathfrak{S}}$. □

**Theorem 3.** *Let $(\mathbb{K}_i^\tau)_{i=1,\dots,n}$ be a family of standard topological contexts and let $\iota_b$ be the isomorphism from $\times_{i=1}^{n}\underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau)$ onto $\underline{\mathfrak{B}}^\tau(\sum_{i=1}^{n}\mathbb{K}_i^\tau)$. Furthermore,* *let $J$ be a subset of $\dot{\bigcup}_{i=1}^{n} G_i \times \dot{\bigcup}_{i=1}^{n} M_i$ and let $J_{ij} := J \cap (G_i \times M_j)$ for $i, j \in \{1, \ldots, n\}$. Then the following conditions are equivalent:*

(i) *$J$ is a separating closed relation and $\iota_b^{-1}(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)))$ is a subdirect product of the $(\underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau))_{i=1,\ldots,n}$.*

(ii) *$J$ is a separating closed relation of $\sum_{i=1}^{n} \mathbb{K}_i^\tau$ with $J_{ii} = I_i$ and $\rho_{J|G_i} = \rho_i$ and $\sigma_{J|M_i} = \sigma_i$ for all $i \in \{1, \ldots, n\}$.*

(iii) *The $J_{ij}$ are topological bonds from $\mathbb{K}_i^\tau$ to $\mathbb{K}_j^\tau$ with $J_{ii} = I_i$, $J_{ik} \subseteq J_{ij} \circ J_{jk}$ and $\rho_{J|G_i} = \rho_i$ and $\sigma_{J|M_i} = \sigma_i$ for all $i, j, k \in \{1, \ldots, n\}$.*

*Proof.* (i) $\Rightarrow$ (ii): For a closed relation $J$ of $\sum_{i=1}^{n} \mathbb{K}_i^\tau$ we always have $\rho_{J|G_i} \subseteq \rho_i$ and $\sigma_{J|M_i} \subseteq \sigma_i$ for all $i \in \{1, \ldots, n\}$. Now, $\iota_b^{-1}(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)))$ is a subdirect product of the $(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau{}_i))_{i\in\{1,\ldots,n\}}$ if and only if, for every $i \in \{1, \ldots, n\}$ and for every $(A, B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau)$, there is some $(C, D) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))$ such that $C \cap G_i = A$ and $D \cap M_i = B$. This implies $\rho_{J|G_i} \supseteq \rho_i$ and $\sigma_{J|M_i} \supseteq \sigma_i$ for all $i \in \{1, \ldots, n\}$ and since $\mathbb{K}_i^\tau$ satisfies (S) we get $J_{ii} = I_i$ for all $i \in \{1, \ldots, n\}$.

(ii) $\Rightarrow$ (iii): Since $J$ is closed the $J_{ij}$ are bonds from $\mathbb{K}_i^\tau$ to $\mathbb{K}_j^\tau$ satisfying $J_{ik} \subseteq J_{ij} \circ J_{jk}$ for all $i, j, k \in \{1, \ldots, n\}$. Let $i, j \in \{1, \ldots, n\}$ and $(g, m) \in J_{ij}$. Since $J$ is separating there is some $(C, D) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))$ such that $g \in C$ and $m \in D$. Then $\overline{g} \times \overline{m} \subseteq (C \cap G_i) \times (D \cap M_j) \subseteq J_{ij}$ and $J_{ij}$ is a topological bond.

(iii) $\Rightarrow$ (i): Certainly, $J$ is a closed relation. First we show that $\iota_b^{-1}(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)))$ is a subdirect product of the $(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau{}_i))_{i=1,\ldots,n}$. Let $(A, B) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau)$ for some $i \in \{1, \ldots, n\}$. Then $A \in \rho_{J|G_i}$. This topology is generated by the subbasis

$$\mathfrak{S}_i := \{C \cap G_i \mid (C, C^J) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))\}.$$

Then $J_{ii} = I_i$ yields that

$$\mathfrak{T}_i := \{(C \cap G_i, C^J \cap M_i) \mid (C, C^J) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))\}$$

is a 0-1-sublattice of $\underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau)$ and therefore, by Lemma 2, $(A, B) \in \mathfrak{T}_i$. Hence there is some $(C, C^J) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))\}$ such that $(A, B) = (C \cap G_i, C^J \cap M_i)$ and the lattice $\iota_b^{-1}(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)))$ is a subdirect product.

Finally, we prove that $J$ is separating. To see this, let $(g, m) \in J_{ij}$ for some $i, j \in \{1, \ldots, n\}$ and suppose that there is no $(C, D) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J))$ such that $g \in C$ and $m \in D$. We get a nonempty family of nonempty closed sets by

$$\begin{aligned}\mathfrak{N}_{(g,m)} :=& \{\{(h, n) \in J \mid h \in C\} \mid (C, C') \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)) \text{ and } g \in C\} \\ &\cup \{\{(h, n) \in J \mid n \in D\} \mid (D', D) \in \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(R_J)) \text{ and } m \in D\}.\end{aligned}$$

Quasicompactness yields the existence of a pair $(\hat{h}, \hat{n}) \in \bigcap \mathfrak{N}_{(g,m)}$, i.e., $\hat{h} \in \overline{g}$ and $\hat{n} \in \overline{m}$. This contradicts the fact that $J_{ij}$ is a topological bond. $\square$

## 6. Fusion of Standard Topological Contexts

Subdirect products are of course not uniquely determined by their factors. Nevertheless, uniqueness can be obtained if some additional conditions about the linkage of the factors are required. This has been studied extensively (e.g. [**12, 13, 16**]).

We follow this idea and give a finite subdirect product construction for bounded lattices in terms of standard topological contexts. We introduce some notions which are similar to those in [**16**]: For a set $P$ the pair $(L,\alpha)$ is called a **bounded** $P$**-lattice** if $L$ is a bounded lattice and $\alpha$ maps $P$ onto a generating subset of $L$. Given bounded $P$-lattices $(L_1,\alpha_1),\dots,(L_n,\alpha_n)$ their (finite) $P$**-product** is defined by $(L,\alpha)$ where $\alpha(p) := (\alpha_1(p),\dots,\alpha_n(p))$ for all $p\in P$ and $L$ is the 0-1-sublattice of $\bigtimes_{i=1}^n L_i$ generated by $\alpha(P)$.

For a given topological context $\mathbb{K}^\tau$ let us call the pair $(\mathbb{K}^\tau,\alpha)$ a **topological** $P$**-context** if $(\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau),\alpha)$ is a bounded $P$-lattice. In the following, $(A_i^p,B_i^p)$ denotes the concept $\alpha_i(p)$. Now, let $(\mathbb{K}_1^\tau,\alpha_1),\dots,(\mathbb{K}_n^\tau,\alpha_n)$ be standard topological $P$-contexts. We define their **standard topological** $P$**-fusion** to be

$$\left(\left(\left(\dot{\bigcup}_{i=1}^{n} G_i,\rho_J\right),\left(\dot{\bigcup}_{i=1}^{n} M_i,\sigma_J\right),J\right),\alpha\right)$$

where $J$ and $\alpha$ are determined by the following conditions:

(i) $J_{ii} = I_i$ for all $i\in\{1,\dots,n\}$;

(ii) For all $i\neq j\in\{1,\dots,n\}$, the relation $J_{ij}$ is the smallest topological bond from $\mathbb{K}_i^\tau$ to $\mathbb{K}_j^\tau$ containing the set $(A_i^p\times B_j^p)$ for every $p\in P$;

(iii) $\alpha(p) := (\dot{\bigcup}_{i=1}^n A_i^p, \dot{\bigcup}_{i=1}^n B_i^p)$ for all $p\in P$.

The relations $J_{ij}$ in (ii) are well-defined since the intersection of topological bonds is again a topological bond.

**Theorem 4.** *Let* $((\mathbb{K}_i^\tau,\alpha_i))_{i=1,\dots,n}$ *be standard topological* $P$*-contexts. Then*

$$\left(\left(\left(\dot{\bigcup}_{i=1}^{n} G_i,\rho_J\right),\left(\dot{\bigcup}_{i=1}^{n} M_i,\sigma_J\right),J\right),\alpha\right)$$

*is a topological* $P$*-context,* $J$ *is a separating closed relation of the context* $\sum_{i=1}^n \mathbb{K}_i^\tau$ *and* $\iota_b^{-1}\left(\underline{\mathfrak{B}}^\tau\left((\dot{\bigcup}_{i=1}^n G_i,\rho_J),(\dot{\bigcup}_{i=1}^n M_i,\sigma_J),J\right)\right)$ *is the* $P$*-product of the bounded* $P$*-lattices* $((\underline{\mathfrak{B}}^\tau(\mathbb{K}_i^\tau),\alpha_i))_{i=1,\dots,n}$.

*Proof.* We check (iii) of Theorem 3. Let $i,j,k\in\{1,\dots,n\}$ and $A_i^p\times B_k^p\subseteq J_{ik}$. Then, for any $(g,m)\in A_i^p\times B_k^p)$, we have $g^{jj}\subseteq A_j^p\subseteq m^j$. Hence $(g,m)\in J_{ij}\circ J_{jk}$. Proposition 7 yields $J_{ik}\subseteq J_{ij}\circ J_{jk}$. In particular, $J$ is a closed relation of $\sum_{i=1}^n \mathbb{K}_i^\tau$ and $\underline{\mathfrak{B}}\left((\dot{\bigcup}_{i=1}^n G_i,\rho_J),(\dot{\bigcup}_{i=1}^n M_i,\sigma_J),J\right)$ is a complete sublattice of

$\underline{\mathfrak{B}}(\sum_{i=1}^{n}\mathbb{K}_i^{\tau})$ containing $\alpha(P)$. If $(A,B)\in\underline{\mathfrak{B}}^{\tau}(\mathbb{K}_i^{\tau})$ for some $i\in\{1,\dots,n\}$ there is some $(C,D)\in\underline{\mathfrak{B}}^{\tau}\left((\dot{\bigcup}_{i=1}^{n}G_i,\rho_J),(\dot{\bigcup}_{i=1}^{n}M_i,\sigma_J),J\right)$ such that $A=C\cap G_i$ and $B=D\cap M_i$ because $\alpha_i(P)$ is a generating set of $\underline{\mathfrak{B}}^{\tau}(\mathbb{K}_i^{\tau})$. This shows $\rho_{J|G_i}=\rho_i$ and $\sigma_{J|M_i}=\sigma_i$. Theorem 3 yields that $J$ is a separating closed relation such that

$$\iota_b^{-1}\left(\underline{\mathfrak{B}}^{\tau}\left(\left(\dot{\bigcup}_{i=1}^{n}G_i,\rho_J\right),\left(\dot{\bigcup}_{i=1}^{n}M_i,\sigma_J\right),J\right)\right)$$

is a subdirect product of the $(\underline{\mathfrak{B}}^{\tau}(\mathbb{K}_i^{\tau}))_{i=1,\dots,n}$ containing their $P$-product $(\mathfrak{L},\alpha)$. Since $J\subseteq I_{\iota_b(\mathfrak{L})}$ we obtain equality. $\square$

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