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A NOTE ON TOPOLOGICAL D–POSETS OF FUZZY SETS

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Abstract. Kôpka and Chovanec in [**KCH**] defined the difference poset (D-poset) as a partially ordered set with a partial difference operation. We show in this paper that every difference operation on a dense subset of $\langle 0, 1\rangle$ is continuous with respect to the usual topology of the real line. We prove also some consequences for the continuity of the difference operation on D-posets of fuzzy sets.

Difference posets were defined by Kôpka and Chovanec in [**KCH**] and they are investigated in many recent papers (see for example [**DR**], [**NP**], [**P**] and [**RB**]).

Definition 1. Difference poset (briefly D-poset) is a couple $(D, \ominus)$, where D is a partially ordered set with the largest element 1 and the difference $\ominus$ is the partial operation, which defines for every $a, b \in D$, $a \leq b$, an element $b \ominus a$ in such a way that the following conditions are satisfied:
i) $b \ominus a \leq b$
ii) $b \ominus (b \ominus a) = a$
iii) if $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Special cases of D-posets are orthomodular posets and another example are D-posets of fuzzy sets defined in [**K**]. D-poset $(F, \ominus)$ is a **D-poset of fuzzy sets**, if elements of F are functions defined on a nonempty set X with values in $\langle 0, 1\rangle$ and the largest (smallest) element of F is the function identically equal to 1 (0). Moreover, the partial ordering of F is given via: for $f, g \in F$, $f \leq g$, if $f(t) \leq g(t)$ for every $t \in X$.

The continuity of $\ominus$ with respect to various topologies was studied in [**P**]. There was also introduced the notion of a topological D-poset. If a D-poset D with a topology $\mathcal{T}$ forms a topological space $(D, \mathcal{T})$ and $\mathcal{T} \times \mathcal{T}$ is the usual product topology, let $\mathcal{T}_0$ be the relative topology on the set $G = \{(a, b) \in D \times D;\ a \leq b\}$ induced by $\mathcal{T} \times \mathcal{T}$.

Definition 2. $(D, \ominus, \mathcal{T})$ is called a **topological D-poset**, if $\ominus\colon (G, \mathcal{T}_0) \to (D, \mathcal{T})$ is a continuous mapping.

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We use the following notation. For x, $x_n \in R$, $x_n \to x$ denotes $\lim\limits_{n\to\infty} x_n = x$ and $x_n \nearrow x$ ($x_n \searrow x$) means that $x_n \to x$ and x_n is increasing (decreasing). Let F be a dense subset of $\langle 0,1\rangle$, containing 0 and 1. Let F be ordered by the standard order of real numbers and let $(F,\ominus)$ be a D-poset.

Lemma 3. *Let x, y, $x_n \in F$. Then*
a) $x_n \nearrow x \leq y$ implies $y \ominus x_n \searrow y \ominus x$,
b) $x_n \searrow x$, $x \leq x_n \leq y$ implies $y \ominus x_n \nearrow y \ominus x$.

Proof. a) Obviously, $y \ominus x_n$ is decreasing. If $y \ominus x_n$ would not converge to $y \ominus x$, then there exists $p \in F$ such that $y \ominus x_n > p > y \ominus x$. This implies $y \ominus (y \ominus x_n) = x_n < y \ominus p < y \ominus (y \ominus x) = x$, a contradiction.

b) Clearly, $y \ominus x_n$ is increasing. If it does not converge to $y \ominus x$, then there exists $p \in F$, $y \ominus x_n < p < y \ominus x$, and this implies $x_n > y \ominus p > x$, a contradiction. □

A simple consequence of this lemma is

Lemma 4. *If x_n, x, $y \in F$, $x_n \to x$, $x_n \leq y$, $x \leq y$, then $y \ominus x_n \to y \ominus x$.*

Lemma 5. *Let x, y, $y_n \in F$. Then*
a) $x \leq y_n \nearrow y$ implies $y_n \ominus x \nearrow y \ominus x$,
b) $y_n \searrow y \geq x$ implies $y_n \ominus x \searrow y \ominus x$.

Proof. a) We have $1 \ominus y \leq 1 \ominus y_n \leq 1 \ominus x$ and, by Lemma 3a), $1 \ominus y_n \searrow 1 \ominus y$. Then by Lemma 3b), $(1 \ominus x) \ominus (1 \ominus y_n) = y_n \ominus x \nearrow (1 \ominus x) \ominus (1 \ominus y) = y \ominus x$.

b) We have $1 \ominus y_n \leq 1 \ominus y \leq 1 \ominus x$ and, by Lemma 3, $(1 \ominus x) \ominus (1 \ominus y_n) = y_n \ominus x \searrow (1 \ominus x) \ominus (1 \ominus y) = y \ominus x$. □

An immediate consequence is

Lemma 6. *If x, y, $y_n \in F$, $y_n \to y$, $x \leq y$, $x \leq y_n$, then $y_n \ominus x \to y \ominus x$.*

Lemma 7. *Let x, y, x_n, $y_n \in F$, $x_n \leq y_n$, $x \leq y$. Let arbitrary of the following conditions be satisfied:*
a) $x_n \nearrow x$, $y_n \searrow y$,
b) $x_n \searrow x$, $y_n \nearrow y$,
c) $x_n \nearrow x$, $y_n \nearrow y$,
d) $x_n \searrow x$, $y_n \searrow y$.
Then $y_n \ominus x_n \to y \ominus x$.

Proof. a) In this case $y_n \ominus x_n$ is decreasing and $y_n \ominus x_n \geq y \ominus x$. If it does not converge to $y \ominus x$, there exist p, $q \in F$ such that $y_n \ominus x_n > p > q > y \ominus x$. Then $y_n \ominus (y_n \ominus x_n) = x_n < y_n \ominus p < y_n \ominus q < y_n \ominus (y \ominus x)$. Since $x_n \to x$ and, by Lemma 6, $y_n \ominus p \to y \ominus p$, $y_n \ominus q \to y \ominus q$, $y_n \ominus (y \ominus x) \to y \ominus (y \ominus x) = x$, we obtain $x \leq y \ominus p < y \ominus q \leq x$, a contradiction.

b) The case $x = y$ is trivial. Let us assume $x < y$. Then $y_n \geq y \ominus x$ for $n \geq n_0$. Obviously, $y_n \ominus x_n$ is increasing, $y_n \ominus x_n \leq y \ominus x$. If it does not converge to $y \ominus x$, then there again exist p, $q \in F$ such that $y_n \ominus x_n < p < q < y \ominus x$. Then for $n \geq n_0$, $y_n \ominus (y_n \ominus x_n) = x_n > y_n \ominus p > y_n \ominus q > y_n \ominus (y \ominus x)$. Hence, by Lemma 6, $x \geq y \ominus p > y \ominus q \geq y \ominus (y \ominus x) = x$, a contradiction.

c) Let us assume the case $x = y$. Then $0 \leq y_n \ominus x_n \leq x \ominus x_n$. Since $x \ominus x_n \to x \ominus x = 0$, we obtain $y_n \ominus x_n \to 0 = y \ominus x$.
If $x < y$, we can assume $x < y_n$. Then we have $x_n \leq x < y_n \leq y$. This implies $y \ominus x_n \geq y_n \ominus x_n \geq y_n \ominus x$. Since both of $y \ominus x_n$ and $y_n \ominus x$ converge to $y \ominus x$, we obtain $y_n \ominus x_n \to y \ominus x$.

d) If $x = y$, then $y_n \ominus x \geq y_n \ominus x_n \geq 0$ and $y_n \ominus x \to 0$ implies $y_n \ominus x_n \to 0$.
If $x < y$, we can assume $x \leq x_n < y \leq y_n$. Then $y_n \ominus x \geq y_n \ominus x_n \geq y \ominus x_n$. Immediately, $y_n \ominus x_n \to y \ominus x$. Lemma is proved. □

Theorem 8. *If $\ominus$ is an arbitrary difference operation on F and x_n, y_n, x, $y \in F$, then $x_n \to x$, $y_n \to y$, $x \leq y$, $x_n \leq y_n$ implies $y_n \ominus x_n \to y \ominus x$, i.e. $(F, \ominus, \mathcal{T})$, where $\mathcal{T}$ is the topology induced by the standard topology on the real line, is a topological D-poset.*

Proof. If $y_n \ominus x_n$ would not converge to $y \ominus x$, then there would exist subsequences x_{n_k}, y_{n_k}, each of them increasing or decreasing, such that $x_{n_k} \to x$, $y_{n_k} \to y$ and $y_{n_k} \ominus x_{n_k} \nrightarrow y \ominus x$. This is a contradiction with Lemma 7. □

Previous result gives some simple consequences for the continuity of difference operations on some D-posets of fuzzy sets.

In the following, $(F, \ominus)$ denotes a D-poset of fuzzy sets and X is the domain of elements of F.

Definition 9. We say that the difference operation $\ominus$ on F is **coordinate dependent**, if for every f_1, f_2, g_1, $g_2 \in F$ and $t \in X$, $f_1(t) = f_2(t)$, $g_1(t) = g_2(t)$, $f_1 \leq g_1$, $f_2 \leq g_2$ implies $(g_1 \ominus f_1)(t) = (g_2 \ominus f_2)(t)$.

Example. Let $F = \langle 0, 1\rangle^X$ and let for every $t \in X$ a continuous strictly increasing function $u_t : \langle 0, 1\rangle \to R$, $u_t(0) = 0$, be given. Let us define the difference operation $\ominus$ on F in the following way: for f, $g \in F$, $f \leq g$, $(g \ominus f)(t) = u_t^{-1}(u_t(f(t)) - u_t(g(t)))$, $t \in X$. Then $\ominus$ is coordinate dependent. □

Let $\mathcal{T}_{pc}$ be the topology of pointwise convergence on F, i.e. a net f_α of elements of F converges to $f \in F$ iff $f_\alpha(t)$ converges to $f(t)$ for every $t \in X$.

For every f, $g \in F$, let us define function $f \vee g$ as follows: $(f \vee g)(t) = \max\{f(t), g(t)\}$.

Theorem 10. *Let $f \vee g \in F$ for every f, $g \in F$. Let for every $t \in X$ the set $\{f(t);\ f \in F\}$ be dense in $\langle 0, 1\rangle$. Then for every coordinate dependent difference operation $\ominus$, $(F, \ominus, \mathcal{T}_{pc})$ is a topological D-poset.*

Proof. Let us prove the continuity of $\ominus$ by contradiction. If $\ominus$ is not continuous in $(f_0, g_0) \in G$, then there exist $\varepsilon > 0$ and $t_0 \in X$ such that for every $n \in N$ there exist f_n, $g_n \in F$, $f_n \le g_n$ such that, $|f_n(t_0) - f_0(t_0)| < \frac{1}{n}$, $|g_n(t_0) - g_0(t_0)| < \frac{1}{n}$ and $|(g_n \ominus f_n)(t_0) - (g_0 \ominus f_0)(t_0)| \ge \varepsilon$.

Let us denote $F_0 = \{f(t_0); f \in F\}$. If $x, y \in F_0$, $x \le y$ and $x = f(t_0)$, $y = g(t_0)$, where $f, g \in F$, then $g(t_0) = (f \vee g)(t_0)$ and $f \le f \vee g$. So, if $x = f(t_0)$, then we can choose $g \in F$, $g \ge f$ such that $y = g(t_0)$.

Let us define the difference operation $\ominus_{t_0}$ on the set $F_0 = \{f(t_0);\ f \in F\}$ in the following way. For $x, y \in F_0$, $x \le y$, let us define $y \ominus_{t_0} x = (g \ominus f)(t_0)$, where $x = f(t_0)$, $y = g(t_0)$, $f, g \in F$, $f \le g$. Since $\ominus$ is coordinate dependent, $\ominus_{t_0}$ is well defined. The verification of the difference properties of $\ominus_{t_0}$ is a routine.

Then we have $f_n(t_0) \to f_0(t_0)$, $g_n(t_0) \to g_0(t_0)$ and $g_n(t_0) \ominus_{t_0} f_n(t_0) \nrightarrow g_0(t_0) \ominus_{t_0} f_0(t_0)$, what is a contradiction to the Lemma 7. Theorem is proved. □

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