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# POSITIVE SOLUTIONS OF VOLTERRA INTEGRO–DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for existence of positive solutions of integro-differential equations of Volterra type are given and existence of solutions with zero crossing $(0, +\infty)$ of integro-differential equations is investigated.

## Introduction

In this paper, we investigate existence of positive solutions and existence of zero points of solutions on $(0, \infty)$ of the Volterra integro-differential equations

$$\dot{x}(t) + \int_0^t P(t,s)x(g(s))\,ds = 0\,, \quad t \geq 0\,. \tag{1}$$

The functions $P \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ and $g \in C(\mathbb{R}^+, \mathbb{R}^+)$. The function $g$ satisfies the following conditions:

$$\begin{gathered} g \text{ is nondecreasing}, \quad g(t) < t \;\text{ for }\; t \in (0,\infty) \quad \text{and} \\ \lim_{t\to\infty} g(t) = \lim_{t\to\infty}(t - g(t)) = +\infty\,. \end{gathered} \tag{2}$$

We present some sufficient conditions such that Eq. (1) only has solutions with zero points in $(0, \infty)$. Moreover, we also obtain some conditions such that Eq. (1) has a positive solution on $[0, +\infty)$.

The motivation of this work comes from the work of Ladas, Philos and Sficas [**5**]. They discussed the oscillation behavior of Eq. (1) when $P(t,s) = P(t-s)$ and $g(t) = t$. They obtained a necessary and sufficient condition under which every solution of the equation is positive on $[0, +\infty)$. Note that Eq. (1) is not a generalization of the equation in [**5**] because of the condition $g(t) < t$, which we require here.

From (2), we see that the function $g$ is nondecreasing and $g(0) = 0$, so, Eq. (1) has a lag with a finite fixed point $t = 0$. Karakostas [**4**] has studied linear delay differential equations with delays having fixed point and obtained that solutions

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of such equations are well defined by giving an initial point instead of an initial function as for general delay differential equations.

By a solution of Eq. (1), we mean that $x \in C^1(\mathbb{R}^+, \mathbb{R})$ and satisfies Eq. (1). For the fundamental theory of integro-differential equations, we refer to [**1**], and for some related work, we refer to [**3**].

## Main Results

Before giving the main results, we present some lemmas which will be used in the proofs of theorems.

**Lemma 1.** *The function g has the properties*

$$g(g(t)) \le g(t), \quad t > 0\,,$$

*and*

$$\lim_{t\to+\infty} g(g(t)) = \lim_{t\to+\infty}(t - g(g(t)) = +\infty\,.$$

*Proof.* By assumption (2), $g$ is nondecreasing, and

$$g(t) < t \ \text{ for } \ t > 0\,.$$

So we have

$$g(g(t)) \le g(t), \quad \text{for } t > 0\,.$$

Moreover, by this inequality, we can see easily that

$$t - g(t) \le t - g(g(t)), \quad t > 0\,,$$

taking limit on both sides, we obtain

$$+\infty = \lim_{t\to+\infty}(t - g(t)) \le \lim_{t\to+\infty}(t - g(g(t)))\,.$$

By $g(t) \to \infty$ as $t \to +\infty$, it is obvious that $g(g(t)) \to +\infty$ as $t \to +\infty$. □

**Lemma 2.** *Assume that*

$$\liminf_{t\to+\infty} \int_0^t P(t,s)\,ds \ne 0\,.$$

*Then we have*

$$\lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds = \lim_{t\to+\infty} \int_{g(g(t))}^t \int_0^s P(s,u)\,du\,ds = +\infty\,.$$

*Proof.* Since $P(t,s) \geq 0$, for $t \in \mathbb{R}^+$, $s \in \mathbb{R}^+$, by assumption, we have

$$\liminf_{t\to+\infty} \int_0^t P(t,s)\,ds > 0\,.$$

On the other hand, by mean value theorem, we have

$$\int_{g(t)}^t \int_0^s P(s,u)\,du\,ds = (t-g(t)) \int_0^{\bar{t}} P(\bar{t},s)\,ds, \quad t>0\,,$$

where $\bar{t} \in [g(t),t]$. Thus $\bar{t} \to +\infty$ as $t \to +\infty$. Then it is clear that

$$\lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds = +\infty\,.$$

Since

$$\int_{g(t)}^t \int_0^s P(s,u)\,du\,ds \leq \int_{g(g(t))}^t \int_0^s P(s,u)\,du\,ds\,,$$

we have

$$\lim_{t\to+\infty} \int_{g(g(t))}^t \int_0^s P(s,u)\,du\,ds = +\infty\,.$$

□

Let us see the main theorem.

**Theorem 1.** *Assume that*

$$\liminf_{t\to+\infty} \int_0^t P(t,s)\,ds \neq 0\,.$$

*Then every solution of Eq.* (1) *has, at least, one zero point on* $(0,+\infty)$.

*Proof.* For the sake of contradiction, assume that there exists a positive solution $x$ on $(0,+\infty)$. For the case that there is a negative solution $y$, we simply let $x=-y$. So here we only consider the case $x(t)>0$, for $t\in(0,+\infty)$. Then we see that $\dot{x}(t)\leq 0$, $t\geq 0$, so $x$ is a nonincreasing function on $[0,+\infty)$. Thus we have

$$0 < x(t) \leq x(g(t)), \quad \text{for} \quad t>0\,.$$

Dividing both sides of Eq. (1) by $x(t)$, we obtain

$$\frac{\dot{x}(t)}{x(t)} + \int_0^t P(t,s)\frac{x(g(s))}{x(t)}\,ds = 0\,, \quad t>0\,.$$

Hence, by using the facts that $x$ is noincreasing an $g$ is nondecreasing, we have

$$\frac{\dot{x}(t)}{x(t)} + \frac{x(g(t))}{x(t)} \int_0^t P(t,s)\,ds \le 0\,, \quad t > 0\,.$$

Integrating both sides of this inequality from $g(t)$ to $t$, we have

$$\ln \frac{x(t)}{x(g(t))} + \int_{g(t)}^t \frac{x(g(s))}{x(s)} \int_0^s P(s,u)\,du\,ds \le 0\,, \quad t > 0\,.$$

a Setting $W(t) := \frac{x(g(t))}{x(t)}$ , it is clear that $W(t) \ge 1$, $t > 0$.

So by the last inequality, we have

$$\int_{g(t)}^t W(s) \int_0^s P(s,u)\,du\,ds \le \ln W(t), \quad t > 0\,.$$

Let $\ell := \liminf_{t\to+\infty} W(t)$, then $1 \le \ell \le +\infty$. Now we divide our discussion into the following two cases: $\alpha)$ $\ell \ne +\infty$, $\beta)$ $\ell = +\infty$.

$\alpha)$ $\ell$ is finite.

There exists a sequence $(t_n)$ such that

$$\lim_{n\to+\infty} t_n = +\infty, \quad \text{and} \quad \liminf_{t\to+\infty} W(t) = \lim_{n\to+\infty} W(t_n) = \ell\,.$$

Thus

$$\begin{aligned} \ell \cdot \liminf_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds &\le \liminf_{t\to+\infty} \int_{g(t)}^t W(s) \int_0^s P(s,u)\,du\,ds \\ &\le \liminf_{t\to+\infty} \ln W(t) = \ln \ell\,. \end{aligned}$$

On the other hand, since $g(t)$ is nondecreasing and $P(s,u)$ is nonnegative, so it follows

$$\liminf_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds = \lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds\,.$$

Therefore we have

$$\lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\,du\,ds \le \frac{\ln \ell}{\ell} \le \frac{1}{e}\,.$$

By Lemma 2, we see that it is a contradiction.

$\beta)$ $\ell = +\infty$.

Thus

$$\lim_{t\to+\infty} \frac{x(g(t))}{x(t)} = +\infty\,. \tag{3}$$

Integrating (1) on both sides from $g(g(t))$ to $g(t)$, we have

$$x(g(t)) - x(g(g(t))) + x(g(g(t))) \int_{g(g(t))}^{g(t)} \int_0^s P(s,u)\,du\,ds \le 0, \quad t > 0\,.$$

Dividing both sides of this inequality by $x(g(g(t)))$, we have

$$\frac{x(g(t))}{x(g(g(t)))} - 1 + \int_{g(g(t))}^{g(t)} \int_0^s P(s,u)\,du\,ds \le 0, \quad t > 0\,. \tag{4}$$

And by (3), we know

$$\lim_{t\to+\infty} \frac{x(g(t))}{x(g(g(t)))} = \lim_{t\to+\infty} \frac{x(t)}{x(g(t))} = 0\,.$$

Taking limit on both sides of inequality (4), in view of Lemmas 1 and 2, we have a contradiction.

The proof is complete. □

**Example 1.** Consider the integro-differential equation

$$\dot{x}(t) + \int_0^t \frac{-2s}{\alpha t^2} x(\alpha s)\,ds = 0\,, \quad t > 0\,,$$

where $\alpha \in (0,1)$. Thus, we see that $g(t) = \alpha t$, $g(g(t)) = \alpha^2 t$, $g$ satisfies all conditions in Theorem 1. It is easy to check that $x(t) = t$ is a solution of the equation, and $x(t)$ has no zero point in the interval $(0,+\infty)$. It is clear that the function $P(t,s)$ is negative. Thus $P$ does not satisfy the conditions in Theorem 1. We can also see that Eq. (1) could have positive solution when the kernel $P(t,s)$ is negative no matter what the function $g$ is. In above example, even if $\alpha$ takes value in the interval $[1,+\infty)$, $x(t) = t$ is always a solution of the equation.

**Example 2.** Consider the following integro-differential equation

$$\dot{x}(t) + \int_0^t P(s) x\left(\frac{s}{2}\right) ds = 0\,, \quad t > 0\,, \tag{5}$$

where $P \in C(\mathbb{R}^+, \mathbb{R}^+)$.

As we can see, this integral equation is equivalent to the following second order functional differential equation

$$\ddot{x}(t) + p(t) x\left(\frac{t}{2}\right) = 0\,, \quad t > 0 \tag{6}$$

if we only consider the solutions which belong to $C^2(\mathbb{R}^+, \mathbb{R})$ and satisfy the initial condition $\dot{x}(0) = 0$. The oscillation of this equation has been studied in [**2**] where sufficient conditions have been established. Thus if we have (see Corollary 2.4 in [**2**]),

$$\int^{\infty} t^{\alpha} P(t)\, dt = +\infty, \quad \text{for some } \alpha \in (0,1)\,,$$

then every solution of Eq. (6) with the condition $\dot{x}(0) = 0$ is oscillatory. So for Eq. (5), if the function $P(t)$ is nonnegative and no identically zero on $[0,+\infty)$, then all the conditions in Theorem 1 hold. Hence, every solution of Eq. (5) has, at least, zero point on $(0,+\infty)$.

From the proof of Theorem 1, we can have the following results without giving further proof.

**Corollary 1.** *Assume that*

$$\liminf_{t\to+\infty} \int_0^t P(t,s)\, ds \neq 0\,.$$

*Then the integro-differential inequality*

$$\dot{x}(t) + \int_0^t P(t,s)x(g(s))\, ds \le 0 \quad (\textit{or } \ \ge 0), \quad \textit{for } \ t > 0\,, \tag{7}$$

*does not have positive (or negative) solutions on* $[0,+\infty)$.

**Corollary 2.** *Assume that*

$$\lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\, du\, ds > 1\,. \tag{8}$$

*Then every solution of Eq.* (1) *has, at least, one zero in* $(0,+\infty)$ *and every solution of inequality* (7) *is not positive (or negative) on* $[0,+\infty)$.

*Proof.* For Corollary 2, we can see that in the proof of Theorem 1, if $x(t) > 0$ on $(0,+\infty)$, when $\ell$ is finite, then we have

$$\lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\, du\, ds \le \frac{1}{e}$$

which contradicts (8). When $\ell = +\infty$, in view of (4), we have

$$\lim_{t\to+\infty} \int_{g(g(t))}^{g(t)} \int_0^s P(s,u)\, du\, ds \le 1\,.$$

Since $g(t) \to +\infty$ as $t \to +\infty$, it follows

$$\lim_{t\to+\infty} \int_{g(g(t))}^{g(t)} \int_0^s P(s,u)\, du\, ds = \lim_{t\to+\infty} \int_{g(t)}^t \int_0^s P(s,u)\, du\, ds \le 1\,,$$

which contradicts (8). Thus the result of Corollary 2 holds. $\square$

Note that the condition (8) is much weaker than the condition in Theorem 1. We can see this from Lemma 2.

Consider the following Volterra integro-differential equation

$$\dot{x}(t) + \int_0^t f(t, s, x(g(s)))\, ds = 0\,, \quad t > 0 \tag{9}$$

and the inequality

$$\dot{x}(t) + \int_0^t f(t, s, x(g(s)))\, ds \le 0 \quad (\text{or } \ \ge 0), \quad t > 0\,. \tag{10}$$

The function $f \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions: $f(t, s, v)v > 0$ for $s \le t$, $v \in \mathbb{R}$, $v \ne 0$ and

$$f(t, s, 0) = 0, \quad |f(t, s, v)| \ge p(t, s)|v|, \quad v \in \mathbb{R},\ t, s \in \mathbb{R}^+,$$

where $P(t, s)$ is as the function appeared in Eq. (1) and satisfies all the conditions mentioned at the beginning of this paper.

It follows a similar way to prove the following results.

**Theorem 2.** *Assume that*

$$\liminf_{t \to +\infty} \int_0^t P(t, s)\, ds \ne 0\,.$$

*Then every solution of Eq.* (9) *has, at least, one zero point on* $(0, +\infty)$ *and no solution of inequality* (10) *is positive (or negative) on* $(0, +\infty)$.

As a matter of fact, if there exists a positive solution $x$ of Eq. (9), then by Eq. (9), we have

$$\dot{x}(t) + \int_0^t P(t, s) x(g(s))\, ds \le \dot{x}(t) + \int_0^t f(t, s, x(g(s)))\, ds = 0\,, \quad t > 0\,.$$

Then the rest proof can follow the one that we have done in the proof of Theorem 1. It has similar steps if we have a negative solution $x$ to Eq. (9). Indeed, if $x(t) < 0$, $t \in (0, +\infty)$, we have

$$\dot{x}(t) + \int_0^t P(t, s) x(g(s))\, ds \ge \dot{x}(t) + \int_0^t f(t, s, x(g(s)))\, ds = 0$$

for $t > 0$. Let $x(t) = -y(t)$, then $y(t) > 0$, $t > 0$, it follows

$$\dot{y}(t) + \int_0^t P(t, s) y(g(s))\, ds \le 0\,, \quad t > 0\,.$$

In the following, we investigate existence of positive solutions of Eq. (1) and Eq. (9).

**Theorem 3.** *Assume that*

$$\int_0^{+\infty} \int_0^s P(s,u)\,du\,ds \le \frac{1}{e}\,.$$

*Then Eq.* (1) *has a positive solution on* $[0,+\infty)$.

*Proof.* For the convenience, we set

$$x(t) = \exp\Big(\int_0^t \lambda(u)\,du\Big), \quad t \ge 0\,, \tag{11}$$

where $x$ is a solution of Eq. (1). By this form, from Eq. (1), we have the following integral equation

$$\lambda(t) = -\int_0^t P(t,s)\exp\Big(-\int_{g(s)}^t \lambda(u)\,du\Big)\,ds, \quad t > 0\,. \tag{12}$$

If we can prove that Eq. (12) has a solution $\lambda(t)$, then by the form of $x(t)$ in (11), we see that Eq. (1) has a positive solution on $[0,+\infty)$.
Construct a sequence as follows

$$\begin{aligned}
\lambda_0(t) &= -e\int_0^t P(t,s)\,ds,\\
\lambda_1(t) &= -\int_0^t P(t,s)\exp\Big[\int_{g(s)}^t -\lambda_0(u)\,du\Big]\,ds,\\
&\dots\\
\lambda_n(t) &= -\int_0^t P(t,s)\exp\Big[\int_{g(s)}^t -\lambda_{n-1}(u)\,du\Big]\,ds.
\end{aligned}$$

Using the induction, we can prove that $\lambda_n(t)$ is a nondecreasing sequence, namely

$$\lambda_n(t) \ge \lambda_{n-1}(t), \quad n = 1,2,\dots$$

and we also have

$$-e\int_0^t P(t,s)\,ds \le \lambda_n(t) \le 0\,, \quad t \in [0,+\infty),$$

for $n = 1,2,\dots$.

Using the monotone convergence theorem, we know that there exists a function $\lambda(t)$ such that $\lambda_n(t) \to \lambda(t)$ as $n \to +\infty$, and

$$\lim_{n\to+\infty}\int_{g(s)}^t \lambda_n(u)\,du = \int_{g(s)}^t \lambda(u)\,du, \quad s \le t.$$

Hence

$$\lim_{n\to+\infty}\int_0^t P(t,s)\exp\Big[\int_{g(s)}^t -\lambda_n(u)\,du\Big]\,ds = \int_0^t P(t,s)\exp\Big[\int_{g(s)}^t -\lambda(u)\,du\Big]\,ds,\ t > 0\,.$$

It concludes that $\lambda(t)$ is a solution of Eq. (12). $\square$

**Theorem 4.** *Assume that the function $f(t,s,v)$ is nonincreasing in $v$ and $f(t,s,v)v > 0$, $v \neq 0$, and*

$$\int_0^{+\infty} \int_0^t f\Big(t, s, \frac{1}{e}\Big)\, ds\, dt \leq \frac{1}{e}\,.$$

*Then Eq.* (9) *has a positive solution on* $[0, +\infty)$.

*Proof.* We can prove this result by a similar way as we have done in the proof of Theorem 3. Set

$$x(t) = \exp\Big(\int_0^t \lambda(s)\, ds\Big), \quad t \geq 0\,,$$

where $x$ is a solution of Eq. (9). Then by Eq. (9) and the form of $x$, we have the integral equation

$$\lambda(t) = -\int_0^t \frac{f(t, s, \exp[\int_0^{g(s)} \lambda(u)\, du])}{\exp[\int_0^t \lambda(u)\, du]}\, ds, \quad t \geq 0\,. \tag{13}$$

If Eq. (13) has a solution $\lambda(t)$ on $[0, +\infty)$, then it follows that Eq. (9) has a positive solution on $[0, +\infty)$. Construct a sequence as follows

$$\lambda_0(t) = -e \int_0^t f\Big(t, s, \frac{1}{e}\Big)\, ds,$$
$$\lambda_n(t) = -\int_0^t \frac{f(t, s, \exp[\int_0^{g(s)} \lambda_{n-1}(u)\, du])}{\exp[\int_0^t \lambda_{n-1}(u)\, du]}\, ds,$$

for $t \geq 0$, $n = 1, 2, \ldots$.

In view of the assumption, we see that $\lambda_n(t) \leq 0$, for $t \geq 0$, $n = 1, 2, 3, \ldots$. Furthermore by using the induction, we can prove that

$$-e \int_0^t f\Big(t, s, \frac{1}{e}\Big)\, ds \leq \lambda_{n-1}(t) \leq \lambda_n(t), \quad n = 1, 2, 3, \ldots,\ t \geq 0\,.$$

Indeed,

$$\exp\Big[\int_0^t \lambda_0(u)\, du\Big] = \exp\Big[\int_0^t -e \int_0^s f\Big(s, u, \frac{1}{e}\Big)\, du\, ds\Big] \geq \frac{1}{e}\,,$$

for $t \geq 0$, and since $f$ is nonincreasing in $v$, we have

$$f\Big(t, s, \exp\Big[\int_0^{g(s)} \lambda_0(u)\, du\Big]\Big) \leq f\Big(t, s, \frac{1}{e}\Big), \quad t \geq s \geq 0\,.$$

Thus
$$\lambda_1(t) \geq -e \int_0^t f\Big(t, s, \frac{1}{e}\Big)\, ds = \lambda_0(t), \quad t \geq 0\,.$$
Now assume that $\lambda_{n-1}(t) \geq \lambda_{n-2}(t)$, $t \geq 0$. Then
$$0 < \exp\Big[\int_0^t \lambda_{n-2}(u)\, du\Big] \leq \exp\Big[\int_0^t \lambda_{n-1}(u)\, du\Big],$$
and
$$f\Big(t, s, \exp\Big[\int_0^{g(s)} \lambda_{n-2}(u)\, du\Big]\Big) \leq f\Big(t, s, \exp\Big[\int_0^{g(s)} \lambda_{n-1}(u)\, du\Big]\Big) > 0\,.$$
Thus, it is clear that $\lambda_n(t) \geq \lambda_{n-1}(t)$.

By the monotone convergence theorem, there exists a function $\lambda(t)$ such that $\lambda_n(t) \to \lambda(t)$ as $n \to +\infty$. So there exists a solution $\lambda(t)$ of Eq. (13).

The proof is complete. □

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## References

1. Balachandran K. and Ilamaran S., *An existence theorem for Volterra integral equation with deviating arguments*, J. Appl. Math. Stoch. Anal. **3** (1990), 155–162.
2. Erbe L., *Oscillation criteria for second order nonlinear delay equations*, Canad. Math. Bull **16**(1) (1973), 49–56.
3. Gopalsamy K., *Oscillations in integro-differential equations of arbitrary order*, J. Math. Anal. Appl. **126** (1987), 100–109.
4. Karakostas G., *Expression of solutions of linear differential equations of retarded type*, Bolletino U.M.I. (5)17-A (1980), 428–435.
5. Ladas G., Philos Ch. G. and Sficas Y. G., *Oscillations of integro-differential equations*, Differential and Integral Equations **4** (1991), 1113–1120.

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