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NOTE ON AN INEQUALITY INVOLVING $(n!)^{1/n}$

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Abstract. We prove: If $G(n) = (n!)^{1/n}$ denotes the geometric mean of the first n positive integers, then

$$\frac{1}{e^2} < (G(n))^2 - G(n-1)G(n+1)$$

holds for all $n \geq 2$. The lower bound $\frac{1}{e^2}$ is best possible.

In 1964 H. Minc and L. Sathre [**2**] published several remarkable inequalities involving the geometric mean of the first n positive integers. Their main result states:
If $G(n) = (n!)^{1/n}$, then

$$1 < n\frac{G(n+1)}{G(n)} - (n-1)\frac{G(n)}{G(n-1)} \tag{1}$$

holds for all integers $n \geq 2$. The lower bound 1 is best possible.

Recently, the author [**1**] proved the following refinement of (1):
If $n \geq 2$, then

$$1 < 1 + \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} < n\frac{G(n+1)}{G(n)} - (n-1)\frac{G(n)}{G(n-1)}\,. \tag{2}$$

The left-hand inequality of (2), written as

$$0 < (G(n))^2 - G(n-1)G(n+1) \qquad (n \geq 2), \tag{3}$$

leads to the question: What is the greatest real number c (which is independent of n) such that

$$c < (G(n))^2 - G(n-1)G(n+1)$$

holds for all $n \geq 2$? It is the aim of this paper to answer this question.

We note that inequalities of the type

$$W_n(x) = (y_n(x))^2 - y_{n-1}(x)y_{n+1}(x) \geq 0 \quad \text{and} \quad W_n(x) \leq 0\,,$$

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where $y_n(x)$ $(n = 1, 2, \dots)$ are particular sequences of functions defined on a real interval, have found much attention; see the paper of D. K. Ross [**3**] which contains not only new results but also interesting historical remarks and many references on this subject.

We prove the following refinement of inequality (3).

Theorem. *If* $n \geq 2$, *then*

$$\frac{1}{e^2} < (G(n))^2 - G(n-1)G(n+1)\,. \tag{4}$$

The lower bound $\frac{1}{e^2}$ *is best possible.*

Proof. In the first part of the proof we establish the double-inequality

$$[G(n+1) - G(n)]^2 < (G(n))^2 - G(n-1)G(n+1) < \left(\frac{G(n)}{n}\right)^2 \quad (n \geq 2)\,. \tag{5}$$

Thereafter we show that the sequence $n \mapsto G(n+1) - G(n)$ is strictly decreasing and converges to $\frac{1}{e}$ as n tends to ∞. This implies (4). Since $\lim_{n\to\infty} \frac{G(n)}{n} = \frac{1}{e}$ we conclude $\lim_{n\to\infty} \left[(G(n))^2 - G(n-1)G(n+1)\right] = \frac{1}{e^2}$. Hence, the constant $\frac{1}{e^2}$ cannot be replaced by a greater number (which is independent of n.)

The function $x \mapsto (\Gamma(x+1))^{1/x}$ $(0 < x \in \mathbb{R})$ is strictly concave on $[7, \infty)$ (see [**4**]). From Jensen's inequality we obtain for all integers $n \geq 8$:

$$\frac{1}{2}(G(n-1) + G(n+1)) < G(n)\,, \tag{6}$$

which is equivalent to the first inequality of (5). For $2 \leq n \leq 7$ we get (6) by direct computation. The approximate values of $\frac{(G(n-1)+G(n+1))}{2}$ and $G(n)$ are given in the following table.

n	$\dfrac{G(n-1)+G(n+1)}{2}$	$G(n)$
2	1.4085	1.4142
3	1.8137	1.8171
4	2.2111	2.2133
5	2.6035	2.6051
6	2.9925	2.9937
7	3.3790	3.3800

Thus, the left-hand inequality of (5) holds for all $n \geq 2$.

To prove the second inequality of (5) we establish that the function $f(x) = \log \frac{(\Gamma(x+1))^{1/x}}{x}$ $(0 < x \in \mathbb{R})$ is strictly convex on $[1, \infty)$. Differentiation yields

$$f'(x) = \frac{\Psi(x+1) - 1}{x} - \frac{\log \Gamma(x+1)}{x^2},$$

where $\Psi = \frac{\Gamma'}{\Gamma}$ denotes the logarithmic derivative of the gamma function, and

$$x^3 f''(x) = x^2 \Psi'(x+1) - 2x\Psi(x+1) + x + 2\log\Gamma(x+1).$$

Applying the inequalities

$$\Psi'(x) > \frac{1}{x} \qquad (x > 1),$$

$$\Psi(x) < \log x - \frac{1}{2x} \qquad (x > 1),$$

$$\log\Gamma(x) > \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) \qquad (x > 1)$$

(see [**2**]), we obtain for $x \geq 1$:

$$x^3 f''(x) \log(x+1) + \log(2\pi) - 2 \geq \log(4\pi) - 2 > 0.$$

Therefore, f is strictly convex on $[1, \infty)$. This implies

$$f(n) < \frac{1}{2}\left(f(n-1) + f(n+1)\right) \qquad (n \geq 2),$$

which is equivalent to the second inequality of (5). Inequality (6) can be written as

$$G(n+1) - G(n) < G(n) - G(n-1) \qquad (n \geq 2).$$

Hence, $n \mapsto G(n+1) - G(n)$ is strictly decreasing. A simple calculation reveals the validity of

$$G(n+1) - G(n) = \frac{v_n - 1}{\log v_n} \frac{G(n)}{n} \frac{\log n + 1}{G(n+1)}$$

with $v_n = \frac{G(n+1)}{G(n)}$. Since $\lim_{n\to\infty} \frac{G(n)}{n} = \frac{1}{e}$ and $\lim_{n\to\infty} \frac{(v_n - 1)}{\log v_n} = 1$ we conclude

$$\lim_{n\to\infty} (G(n+1) - G(n)) = \frac{1}{e}.$$

The proof of the Theorem is complete. □

References

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