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THE FRACTAL DIMENSION OF INVARIANT SUBSETS FOR PIECEWISE MONOTONIC MAPS ON THE INTERVAL

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Abstract. We consider completely invariant subsets A of weakly expanding piecewise monotonic transformations T on $[0,1]$. It is shown that the upper box dimension of A is bounded by the minimum t_A of all parameters t for which a t-conformal measure with support A exists. In particular, this implies equality of box dimension and Hausdorff dimension of A.

1. Introduction

During the last years the fractal dimension of invariant subsets in dynamical systems has attracted much interest. Different notions of dimension have been considered. The best known are box dimension, Hausdorff dimension and packing dimension. We need here only the definition of box dimension of a subset X of $[0,1]$. Let $N_r(X)$ be the number of closed intervals of length r required to cover X. The lower and upper box dimension of X are defined by

$$\mathrm{BD}^-(X) = \liminf_{r\to 0} \frac{\log N_r(X)}{-\log r} \quad \text{and} \quad \mathrm{BD}^+(X) = \limsup_{r\to 0} \frac{\log N_r(X)}{-\log r}$$

If $\mathrm{BD}^+(X) = \mathrm{BD}^-(X)$ this number is called the box dimension $\mathrm{BD}(X)$ of X. The definitions of Hausdorff dimension $\mathrm{HD}(X)$ and of packing dimension $\mathrm{PD}(X)$ of a set X can be found in [**1**] or in [**3**]. It is well known that $\mathrm{HD}(X) \le \mathrm{PD}(X) \le \mathrm{BD}^+(X)$ and that $\mathrm{HD}(X) \le \mathrm{BD}^-(X) \le \mathrm{BD}^+(X)$.

In this paper we investigate the fractal dimension of invariant subsets of piecewise monotonic transformations on the interval. A map $T\colon [0,1] \to [0,1]$ is called piecewise monotonic, if there are $c_i \in [0,1]$ for $0 \le i \le N$ with $0 = c_0 < c_1 < \cdots < c_N = 1$ such that $T|(c_{i-1}, c_i)$ is monotone and continuous for $1 \le i \le N$. Since T is allowed to be discontinuous at the points in $P\colon = \{c_0, c_1, \dots, c_N\}$, we call a closed subset of $[0,1]$ invariant, if $T(A \setminus P) \subset A$, and completely invariant, if $x \in A$ is equivalent to $T(x) \in A$ for all $x \in [0,1] \setminus P$. For equivalent definitions of completely invariant subsets see Lemma 4 in [**7**]. One goal of this paper will be to find conditions under which $\mathrm{BD}^+(A) \le \mathrm{HD}(A)$, which implies equality of notions of dimension introduced above.

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The investigation of the dimension of an invariant subset A usually involves the derivative of T. In this paper a measurable function $T'\colon [0,1] \to \mathbb{R}$ is called a derivative of T, if $T(b)-T(a) = \int_a^b T'dx$ for all a and b satisfying $c_{i-1} < a < b < c_i$ for some i. A function $f\colon [0,1] \to \mathbb{R}$ is called regular, if $f(x+)\colon = \lim_{y\downarrow x} f(y)$ for $x \in [0,1)$ and $f(x-)\colon = \lim_{y\uparrow x} f(y)$ for $x \in (0,1]$ exist. We shall always assume that T has a derivative, which is regular.

For an invariant subset A of a piecewise monotonic transformation T various quantities associated with the dynamical system $(A, T|A)$ have been introduced in order to prove results about dimension. We give a short review.

The essential Hausdorff dimension was introduced in [**2**] (see also [**10**]). For an invariant subset A let $M_T(A)$ be the set of all T-invariant probability measures μ with $\mu(A) = 1$, and let $E_T(A)$ be the set of all $\mu \in M_T(A)$ which are ergodic. For a probability measure μ define $\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(B) : \mu(B) = 1\}$. Then one defines the essential Hausdorff dimension of an invariant subset A by

$$\mathrm{HD}_{\mathrm{ess}}(A) = \sup\{\mathrm{HD}(\mu) : \mu \in E_T(A), h_\mu > 0\}$$

where h_μ denotes the entropy of μ. It is clear from the definition that $\mathrm{HD}_{\mathrm{ess}}(A) \leq \mathrm{HD}(A)$.

Now let A be a completely invariant subset which is topologically transitive. For a measurable function $f\colon [0,1] \to \mathbb{R}$ we define the pressure $p(T|A, f)\colon = \sup\{h_\mu + \int f d\mu : \mu \in E_T(A)\}$. We fix a regular derivative T' of T and set $\pi(t) = p(T|A, t\varphi)$, where $\varphi = -\log|T'|$. Then π is a convex function on $\mathbb{R}^+$ with $p(0) \geq 0$. It is shown in [**6**] that $z_A := \inf\{t \geq 0 : \pi(t) = 0\}$ exists if $|T'| \geq 1$ or if T is continuous, and that $z_A \leq \mathrm{HD}_{\mathrm{ess}}(A)$ if $|T'|$ is of bounded variation or if $\inf|T'| > 0$. It is not known whether z_A exists in general. In the general case one can define a modified pressure $q(T|A, f)$ exhausting $(A, T|A)$ by Markov maps (see [**9**]). Again we set $\tilde\pi(t) = q(T|A, t\varphi)$. Theorem 1 in [**9**] implies that $\tilde z_A := \inf\{t \geq 0 : \tilde\pi(t) = 0\}$ exists under seme weak assumptions on T. Furthermore, if $|T'|$ is of bounded variation, then $\tilde z_A = \mathrm{HD}_{\mathrm{ess}}(A)$ (Theorem 5 in [**9**]) and $\tilde z_A = z_A$ whenever z_A exists (remark after Theorem 5 in [**9**]).

Now we consider conformal measures. A probability measure m is called t-conformal, if

$$(1.1) \qquad m(TB) = \int_B |T'|^t \, dm \text{ for all } B \text{ contained in } (c_{i-1}, c_i) \text{ for some } i$$

Lemma 5 in [**7**] says that the support of a conformal measure is a completely invariant subset. For a completely invariant subset A let t_A be the infimum of all $t \geq 0$ for which a t-conformal measure with support A exists. Theorem 2 in [**9**] implies that $t_A \leq \tilde z_A$ and hence $t_A \leq z_A$ whenever z_A exists, if $h_{top}(T|A) > 0$ and $(A, T|A)$ is topologically transitive.

Therefore, under the assumptions on T used in [**9**], for a completely invariant topologically transitive subset A with $h_{top}(T|A) > 0$ we have that $t_A \le \tilde{z}_A \le \mathrm{HD}_{\mathrm{ess}}(A) \le \mathrm{HD}(A) \le \mathrm{BD}^+(A)$. The question arises under which conditions we have $\mathrm{BD}^+(A) \le t_A$ holds. We consider this for weakly expanding piecewise monotonic transformations. Let F^+ be the set of all $x \in [0,1)$ with $T(x+) = x$ and $T'(x+) = 1$ and let F^- be the set of all $x \in (0,1]$ with $T(x-) = x$ and $T'(x-) = 1$. These sets need not be disjoint. Set $F = F^+ \cup F^-$. We say that a piecewise monotonic map T is weakly expanding, if the following properties are satisfied.

(a) F is finite
(b) there is $\delta > 0$, such that $T'|(p-\delta,p)$ is decreasing, if $p \in F^-$, and $T'|(p,p+\delta)$ is increasing, if $p \in F^+$
(c) $\inf\left\{|T'(y)| : y \notin P \cup \bigcup_{p\in F^-}(p-\delta,p] \cup \bigcup_{p\in F^+}[p,p+\delta)\right\} > 1$ for each $\delta > 0$.

We shall prove the following theorem.

Theorem. *Let A be an invariant subset of a weakly expanding piecewise monotonic transformation T with regular derivative. Suppose that there is a t-conformal measure with support A. Then* $\mathrm{BD}^+(A) \le t$.

This theorem implies that for a weakly expanding transformation T, such that T' is equicontinuous on $f|(c_{i-1},c_i)$ for all i (then the assumptions of [**9**] are satisfied), and a completely invariant topologically transitive subset A with $h_{top}(T|A) > 0$ we have $t_A = z_A = \mathrm{HD}(A) = \mathrm{PD}(A) = \mathrm{BD}(A)$.

Under the assumption, that T is expanding, which means that $F = \emptyset$, the above theorem is already proved in [**8**]. In this paper also an example of a transformation T and a set A is given, for which all assumptions of the above theorem are satisfied except (b) in the definition of a weakly expanding transformation, but for which $\mathrm{HD}(A) < \mathrm{BD}^+(A)$. Therefore it cannot expected that the above theorem holds under weaker assumptions.

For the proof of the the above theorem we have to construct suitable covers of A by intervals. In Section 2 we define a directed graph, called Markov diagram, whose paths can be used to define such covers of A. In Section 3 we deal with indifferent fixed points. Estimates of the lengths of halfneighbourhoods of the points in F are given. Together with estimates of the cardinality of certain sets of paths in the Markov diagram, which are given in Section 4, this gives upper bounds of $N_r(A)$ used in the definition of $\mathrm{BD}^+(A)$.

2. Intervals Defined By Paths of a Graph

In order to estimate box dimension, we have to construct covers by intervals. To this end we construct a directed graph, called Markov diagram, whose finite paths correspond to certain intervals.

In this paper a finite collection of open intervals, which cover A up to a finite set, is called a cover of A. A cover of $[0,1]$, which consists of open disjoint intervals, is called a partition.

Set $\mathcal{W} = \{(p-\delta, p) : p \in F^-\} \cup \{(p, p+\delta) : p \in F^+\}$, where $\delta > 0$ is chosen so small that the intervals in $\mathcal{W}$ are disjoint and that (b) holds. For each $W \in \mathcal{W}$ let $V_0(W) = W \supset V_1(W) \supset V_2(W) \supset \dots$ be the uniquely determined open intervals with an indifferent fixed point as common endpoint, such that $T(V_i(W)) = V_{i-1}(W)$ for $i \geq 1$. Furthermore, for $i \geq 0$ set $U_i(W) = V_i(W) \setminus V_{i+1}(W)$ and $V_i = \bigcup_{W \in \mathcal{W}} V_i(W)$.

We fix a regular derivative T' of T and set $\varphi = \log |T'| \geq 0$. Set $\Gamma = \sup \varphi$ and $\gamma = \inf_{x \notin \overline{V_1}} \varphi > 0$. We fix $\varepsilon \in (0, \gamma)$ and $\theta = \theta(\varepsilon) \in \mathbb{N}$, such that $\sup_{x \in V_\theta} \varphi < \frac{\varepsilon}{4}$. We fix a partition $\mathcal{Z}$ such that

(2.1) $\quad T|Z$ is monotone and continuous for each $Z \in \mathcal{Z}$

(2.2) $\quad V_\theta(W) \in \mathcal{Z}$ for each $W \in \mathcal{W}$

(2.3) $\quad$ if $Z \in \mathcal{Z}$, $W \in \mathcal{W}$ and $0 \leq i < \theta$ then $Z \cap U_i(W) = \emptyset$ or $Z \subset U_i(W)$

(2.4) $\quad \sup_Z \varphi - \inf_Z \varphi < \dfrac{\varepsilon}{4\theta}$ for all $Z \in \mathcal{Z} \setminus \{V_\theta(W) : W \in \mathcal{W}\}$

We define the Markov diagram of $([0,1], T)$ with respect to the partition $\mathcal{Z}$. If D is an open interval contained in an element of $\mathcal{Z}$, the nonempty sets among $T(D) \cap Z$ for $Z \in \mathcal{Z}$ are called the successors of D. These successors are again open intervals contained in elements of $\mathcal{Z}$, so that one can iterate the formation of successors. We write $D \to C$ if C is a successor of D. Set $\mathcal{D}_0 = \mathcal{Z}$. For $n \geq 1$ let $\mathcal{D}_n$ be the union of $\mathcal{D}_{n-1}$ and the set of all successors of elements of $\mathcal{D}_{n-1}$. Since the number of successors of an interval is always bounded by card $\mathcal{Z}$, the sets $\mathcal{D}_n$ for $n \geq 0$ are finite. Set $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$. The directed graph $(\mathcal{D}, \to)$ is called the Markov diagram of $([0,1], T)$ with respect to the partition $\mathcal{Z}$.

If $D_0 D_1 \dots D_{k-1}$ is a path in $(\mathcal{D}, \to)$, then $\bigcap_{j=0}^{k-1} T^{-j} D_j$ is a nonempty open interval by the definition of a successor. We shall use intervals of this kind to define covers of an invariant subset A. We begin with the definition

$$h(D_0 D_1 \dots D_{k-1}) = \sum_{i=0}^{k-1} \inf_{Q_i} \varphi \quad \text{where} \quad Q_i = \bigcap_{j=i}^{k-1} T^{i-j} D_j \tag{2.5}$$

Observe that $Q_0 = \bigcap_{j=0}^{k-1} T^{-j} D_j$ and that $Q_i = T^i(Q_0)$ for $1 \leq i \leq k-1$ by the definition of a successor. We define also $\tilde{h}(D_0 D_1 \dots D_{k-1}) = \sum_{i=0}^{k-1} \sup_{Q_i} \varphi$. We have

Lemma 1. *For a path $D_0 D_1 \dots D_{k-1}$ in $(\mathcal{D}, \to)$ we have*

(i) $h(D_0 D_1 \dots D_{k-1}) \geq h(D_0 D_1 \dots D_{k-2})$

(ii) $\tilde{h}(D_0 D_1 \dots D_{k-1}) \leq \tilde{h}(D_0 D_1 \dots D_{k-2}) + \Gamma$

(iii) $h(D_0 D_1 \dots D_{k-1}) \geq h(D_0 D_1 \dots D_{l-1}) + h(D_l D_{l+1} \dots D_{k-1})$

(iv) $\tilde{h}(D_0 D_1 \dots D_{k-1}) \leq \tilde{h}(D_0 D_1 \dots D_{l-1}) + \tilde{h}(D_l D_{l+1} \dots D_{k-1})$

Proof. This follows easily from the definitions using $0 \le \varphi \le \Gamma$. $\square$

Set $\mathcal{G} = \{D \in \mathcal{D} : D \cap V_1 = \emptyset\}$ and let $\mathcal{P}_n$ be the set of all paths $D_0 D_1 \dots D_{k-1}$ in $(\mathcal{D}, \to)$ with $k \ge 1$ satisfying

$$h(D_0 D_1 \dots D_{k-2}) < \gamma n \le h(D_0 D_1 \dots D_{k-1}) \tag{2.6}$$

$$D_0 \in \mathcal{Z} = \mathcal{D}_0 \text{ and } D_{k-1} \in \mathcal{G} \tag{2.7}$$

$$A \cap \bigcap_{i=0}^{k-1} T^{-i} D_i \ne \emptyset \tag{2.8}$$

If $k = 1$ we set $h(D_0 D_1 \dots D_{k-2}) = 0$. By Lemma 1 for each infinite path $D_0 D_1 \dots$ in $(\mathcal{D}, \to)$ there is at most one k such that (2.6) holds.

Now we can estimate length and measure of the intervals associated with paths in $\mathcal{P}_n$. Let $|I|$ denote the length of the interval I.

Lemma 2. *For $D_0 D_1 \dots D_{k-1} \in \mathcal{P}_n$ we have $\tilde{h}(D_0 D_1 \dots D_{k-2}) - h(D_0 D_1 \dots D_{k-2}) \le \varepsilon n$. Furthermore, we have $|\bigcap_{i=0}^{k-1} T^{-i} D_i \cap T^{-k} J| \le e^{-n\gamma} |J|$ for any interval $J \subset [0,1]$ and $m(\bigcap_{i=0}^{k-1} T^{-i} D_i) \ge m(D_{k-1}) e^{-t(\gamma n + \varepsilon n)}$ for any t-conformal measure m.*

Proof. Set $Q_i = \bigcap_{j=i}^{k-2} T^{-(j-i)} D_j$ for $0 \le i \le k-2$. Let $i_1 < i_2 < \dots < i_r = k-1$ be all elements i of $\{0, 1, \dots, k-1\}$ with $D_i \in \mathcal{G}$. Consider some $s \ge 1$ with $i_{s-1} < i_s - 1$, where we set $i_0 = -1$. Then there is $W \in \mathcal{W}$ such that $D_j \subset V_1(W)$ for $i_{s-1} < j < i_s$ and $D_{i_s} \subset U_0(W)$. By (2.3) we have then $Q_{i_s - j} \subset D_{i_s - j} \subset U_j(W)$ for $0 \le j < \min(i_s - i_{s-1}, \theta)$. Since $T(Q_l) \subset Q_{l+1}$ this implies that $Q_j \subset U_{i_s - j}(W)$ for $i_{s-1} < j < i_s$. Set $\psi_l = \sup_{Q_l} \varphi - \inf_{Q_l} \varphi$ for $0 \le l \le k-2$ and $\psi_{k-1} = \sup_{D_{k-1}} \varphi - \inf_{D_{k-1}} \varphi$. The sets $U_l(W)$ are disjoint. Hence $\sum_{j=i_{s-1}+1}^{i_s - \theta} \psi_j < \frac{\varepsilon}{4}$ by the choice of θ, provided that $i_{s-1} < i_s - \theta$. Furthermore, $\psi_j < \frac{\varepsilon}{4\theta}$ for $\max(i_s - \theta, i_{s-1}) < j \le i_s$. Therefore $\sum_{j=i_{s-1}+1}^{i_s} \psi_j < \frac{\varepsilon}{2}$. If $i_s = i_{s-1} + 1$ then $\psi_{i_s} < \frac{\varepsilon}{4\theta} < \frac{\varepsilon}{2}$. We have shown that $\tilde{h}(D_0 D_1 \dots D_{k-2}) - h(D_0 D_1 \dots D_{k-2}) = \sum_{j=0}^{k-2} \psi_j < r\frac{\varepsilon}{2}$. Since $D_{i_s} \in \mathcal{G}$ and hence $\inf_{D_{i_s}} \varphi \ge \gamma$ for all s, we have $(r-1)\gamma \le h(D_0 D_1 \dots D_{k-2}) < \gamma n$ and hence $r < n+1$. This implies the first assertion.

Now set $R_i = \bigcap_{j=i}^{k-1} T^{-(j-i)} D_j$ for $0 \le i \le k-1$, which are intervals contained in elements of $\mathcal{Z}$. The sets $S_i := R_i \cap T^{-(k-i)} J$ satisfy $T(S_i) = S_{i+1}$. By the mean value theorem and (2.5) we get that $|S_0| \le |T(S_{k-1})| e^{-h(D_0 D_1 \dots D_{k-1})}$. As $T(S_{k-1}) = T(D_{k-1}) \cap J$ we get $|S_0| \le |J| e^{-\gamma n}$ by (2.6). This is the second assertion. Similarly we get for a t-conformal measure m that $m(R_0) \ge m(R_{k-1}) e^{-t\tilde{h}(D_0 D_1 \dots D_{k-2})}$. By (2.6) and the first assertion of this lemma we have $\tilde{h}(D_0 D_1 \dots D_{k-2}) \le \gamma n + \varepsilon n$ proving the last assertion. $\square$

3. Estimates Near Indifferent Fixed Points

In this section we use the existence of a t-conformal measure to estimate the length of halfneighbourhoods of indifferent fixed points. This leads to an estimate of $N_r(A)$ for $r = \gamma n$ and $n \in \mathbb{N}$ in terms of t and the cardinality of the sets $\mathcal{P}_l$. We begin with

Lemma 3. *Fix $W \in \mathcal{W}$ and set $\varphi_j(W) = \sup_{V_j(W)} \varphi$.*

(i) $\sum_{j=0}^{\infty} j = \infty$

(ii) $|V_k(W)| \leq \sum_{i=k}^{\infty} e^{-\sum_{j=1}^{i+1} \varphi_j(W)}$ *for* $k \geq 0$

(iii) *if m is a t-conformal measure and $m(W) > 0$ then $m(U_i(W)) > 0$ for all i and* $\sum_{i=1}^{\infty} e^{-t\sum_{j=1}^{i} \varphi_j(W)} < \infty$.

Proof. Let m be a t-conformal measure with $m(W) > 0$. By (1.1) and (b) we get

$$e^{t\varphi_{j+1}(W)} m(U_j(W)) \leq m(TU_j(W)) \leq e^{t\varphi_j(W)} m(U_j(W))$$

Since $T(U_j(W)) = U_{j-1}(W)$ for $j \geq 1$, we get $m(U_i(W)) = 0$ for all i or $m(U_i(W)) > 0$ for all i. Since $W = \bigcup_{i=0}^{\infty} U_i(W)$ the first assertion of (iii) follows. Furthermore,

$$e^{t\sum_{j=1}^{i+1} \varphi_j(W)} m(U_i(W)) \leq m(TU_0(W)) \quad \text{and} \quad m(U_0(W)) \leq e^{t\sum_{j=1}^{i} \varphi_j(W)} m(U_i(W))$$

The first inequlity gives $m(V_k(W)) \leq \sum_{i=k}^{\infty} e^{-t\sum_{j=1}^{i+1} \varphi_j(W)}$, since $m(TU_0(W)) \leq 1$. This shows (ii), since Lebesgue measure is a 1-conformal measure with $m(W) > 0$ for all $W \in \mathcal{W}$. The second inequlity gives $\sum_{i=1}^{\infty} e^{-t\sum_{j=1}^{i} \varphi_j(W)} \leq \frac{m(V_1(W))}{m(U_0(W))}$, which gives (iii). Taking for m again the Lebesgue measure, it gives $\sum_{i=1}^{\infty} e^{-\sum_{j=1}^{i} \varphi_j(W)} < \infty$ for all $W \in \mathcal{W}$, which implies (i). □

Lemma 4. *Set $\mathcal{W}_A = \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ and let m be a t-conformal measure with support A. There is $d > 0$ with $|V_k(W)| \leq de^{-(1-t)\sum_{i=1}^{k} \varphi_i(W)}$ for all $k \geq 1$ and all $W \in \mathcal{W}_A$, where $\varphi_i(W)$ is as in Lemma 3.*

Proof. Set $d = \sup_{W \in \mathcal{W}_A} \sum_{i=1}^{\infty} e^{-t\sum_{j=1}^{i} \varphi_j(W)}$ which is finite by Lemma 3 (iii) since m has support A. We have

$$\begin{aligned}
e^{\sum_{l=1}^{k} \varphi_l(W)} |V_k(W)| &\leq \sum_{i=k}^{\infty} e^{-\sum_{l=k+1}^{i+1} \varphi_l(W)} && \text{by Lemma 3 (ii)} \\
&\leq \sum_{i=k}^{\infty} e^{-t\sum_{l=k+1}^{i+1} \varphi_l(W)} && \text{as } t \leq 1 \text{ and } \varphi \geq 0 \\
&\leq de^{t\sum_{l=1}^{k} \varphi_l(W)} && \text{by definition of } d.
\end{aligned}$$

This gives the desired estimate. □

Lemma 5. *For each $u > 0$ there is $v \in \mathbb{N}$ such that each path $D_0 D_1 \dots D_{v-1}$ of length v in $(\mathcal{D}, \to)$ with $D_{v-1} \cap V_\theta = \emptyset$ satisfies $h(D_0 D_1 \dots D_{v-1}) \geq u$.*

Proof. For $W \in \mathcal{W}$ and $j \geq 0$ let $\varphi_j(W)$ be as in Lemma 3. This lemma says that $\sum_{j=0}^{\infty} \varphi_j(W) = \infty$. For fixed $u > 0$ choose $l > \theta$ such that $\sum_{j=\theta}^{l} \varphi_j(W) \geq u$ holds for each $W \in \mathcal{W}$. Then choose an integer $v > \frac{ul}{\gamma}$.

By (2.3) for a path $D_0 D_1 \dots D_{v-1}$ in $(\mathcal{D}, \to)$ with $D_{v-1} \cap V_\theta = \emptyset$ there are two cases. Either the number of D_i satisfying $D_i \cap V_1 = \emptyset$ is greater than $\frac{u}{\gamma}$ or there is $i < v - l$ such that $D_{i+j} \subset V_1$ for $0 \leq i < l$ and $D_{i+l} \cap V_\theta = \emptyset$. In the first case we get $h(D_0 D_1 \dots D_{v-1}) \geq \frac{u}{\gamma}\gamma = u$ by definition of γ. In the second case there is $W \in \mathcal{W}$ and s with $i \leq s < i + l$, such that $Q_{s+j} \subset U_{l-j-1}(W)$ for $0 \leq j \leq l - \theta$ by (2.3) and the definition of the sets $U_j(W)$, where Q_j is as in (2.5). By (b) we get $\inf_{Q_{s+j}} \varphi \geq \varphi_{l-j}(W)$ for $0 \leq j \leq l - \theta$. Therefore we have again that $h(D_0 D_1 \dots D_{k-1}) \geq \sum_{j=\theta}^{l} \varphi_j(W) \geq u$. □

Now we can give a first estimate of $N_r(A)$ for $r = e^{-\gamma n}$.

Proposition 1. *Let m be a t-conformal measure with support A. There is $c > 0$ such that $N_{e^{-\gamma n}}(A) \leq c e^{2\varepsilon n} \sum_{l=0}^{n} p_l e^{t\gamma(n-l)}$ for all n, where $p_0 = 1$ and $p_l = \operatorname{card} \mathcal{P}_l$ for $l \geq 1$.*

Proof. Let $\varphi_i(W)$ be as in Lemma 3. For $l < n$ and $W \in \mathcal{W}$ let $j(l, W)$ be the minimal j such that $\sum_{i=1}^{j} \varphi_i(W) \geq (n - l)\gamma - \varepsilon n - \Gamma$. The existence of $j(l, W)$ follows from Lemma 3 (i). We write $R_l(W)$ for $V_{j(l,W)}(W)$. Set $\mathcal{U}_n = \{\bigcap_{i=0}^{k-1} T^{-i} D_i : D_0 D_1 \dots D_{k-1} \in \mathcal{P}_n\}$. For $W \in \mathcal{W}_A := \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ set $\mathcal{U}_0(W) = \{R_0(W)\}$ and $\mathcal{U}_l(W) = \{\bigcap_{i=0}^{k-1} T^{-i} D_i \cap T^{-k} R_l(W) : D_0 D_1 \dots D_{k-1} \in \mathcal{P}_l\}$ for $1 \leq l \leq n-1$. We show first that $\mathcal{U} := \mathcal{U}_n \cup \bigcup_{l=0}^{n-1} \bigcup_{W \in \mathcal{W}_A} \mathcal{U}_l(W)$ covers A. To this end choose $q \geq \sup_{l<n} \sup_{W \in \mathcal{W}_A} j(l, W)$ such that $h(C_0 C_1 \dots C_{q-1}) > \gamma n$ for all paths $C_0 C_1 \dots C_{q-1}$ with $C_{q-1} \in \mathcal{G}$. This is possible by Lemma 5. FoLet k be maximal such that $C_{k-1} \in \mathcal{G}$ and $h(C_0 C_1 \dots C_{k-2}) < \gamma n$. If no such k exists set $k = 0$. If $k > 0$ and $h(C_0 C_1 \dots C_{k-1}) \geq \gamma n$ then $C_0 C_1 \dots C_{k-1} \in \mathcal{P}_n$ and Z is contained in an element of $\mathcal{U}_n$.

Therefore suppose that $k = 0$ or that $h(C_0 C_1 \dots C_{k-1}) < \gamma n$. If $k = 0$ set $l = 0$. If $k \geq 1$, there is $l \in \{1, 2, \dots, n-1\}$ such that $C_0 C_1 \dots C_{k-1} \in \mathcal{P}_l$, since $C_{k-1} \in \mathcal{G}$ and hence $\inf_{C_{k-1}} \varphi \geq \gamma$. We consider two cases.

Suppose first that there is no $i \geq k$ with $C_i \in \mathcal{G}$. Hence there is $W \in \mathcal{W}$ with $C_i \in V_1(W)$ for $k \leq i \leq s - 1$. By the choice of q and by Lemma 1 we have $k < q$. Since $s = 2q$ we get $\bigcap_{i=k}^{s-1} T^{k-i} C_i \subset V_{s-k}(W) \subset V_q(W)$ and hence $T^k(Z) \subset V_q(W)$. Since $T^k(Z) \cap A \neq \emptyset$, as A is invariant, $V_q(W)$ and hence also W has nonempty intersection with A. By the choice of q we get $V_q(W) \subset R_l(W)$. Thus $Z \subset \bigcap_{i=0}^{k-1} T^{-i} C_i \cap T^{-k} R_l(W)$ and we have found an element of $\mathcal{U}_l(W)$, which contains Z (empty intersections have to be considered as absent).

Now suppose that $u > k$ is minimal such that $C_{u-1} \in \mathcal{G}$. Because of $h(C_0 C_1 \dots C_{k-1}) < \gamma n$ and the choice of k we have $C_k \notin \mathcal{G}$. Hence $C_k \subset V_1(W)$

for some $W \in \mathcal{W}$. As above we get that $\bigcap_{i=k}^{u-2} T^{k-i} C_i \subset V_{u-k-1}(W)$ for some $W \in \mathcal{W}_A$. The choice of k implies that $h(C_0 C_1 \ldots C_{u-2}) \geq \gamma n$. Using Lemmas 1 and 2 we get

$$\begin{aligned}\sum_{i=1}^{u-k-1} \varphi_i(W) &\geq \tilde{h}(C_k C_{k+1} \ldots C_{u-2}) \\ &\geq \tilde{h}(C_0 C_1 \ldots C_{u-2}) - \tilde{h}(C_0 C_1 \ldots C_{k-2}) - \Gamma \\ &\geq h(C_0 C_1 \ldots C_{u-2}) - h(C_0 C_1 \ldots C_{k-2}) - \Gamma - \varepsilon n \\ &\geq \gamma n - \gamma l - \Gamma - \varepsilon n\end{aligned}$$

This says that $V_{u-k-1}(W) \subset R_l(W)$. As above we get that Z is contained in an element of $\mathcal{U}_l(W)$. Thus we have proved that $\mathcal{U}$ covers A.

We consider $I := \bigcap_{i=0}^{k-1} T^{-i} D_i \cap T^{-k} R_l(W) \in \mathcal{U}_l(W)$ and estimate the length $|I|$ of the interval I. If $l = n$ we have $[0,1]$ instead of $R_l(W)$, and if $l = 0$ then $k = 0$, which means that $I = R_0(W)$. By Lemma 2 we have $|I| \leq e^{-\gamma l} |R_l(W)|$. By Lemma 4 we get $|R_l(W)| \leq d e^{-(1-t)(\gamma(n-l) - \varepsilon n - \Gamma)}$ for $l < n$. Setting $b = d e^{\Gamma}$ we get $|I| \leq b e^{-\gamma n} e^{t\gamma(n-l)} e^{\varepsilon n}$. The number of intervals of length $e^{-\gamma n}$ necessary to cover I is bounded by $b e^{t\gamma(n-l)} e^{\varepsilon n}$. Since $p_l = \operatorname{card} \mathcal{U}_l(W)$ for $l \leq n-1$ and $p_n = \operatorname{card} \mathcal{U}_n$, the desired result follows with $c = b \operatorname{card} \mathcal{W}_A$. □

4. The Cardinality of Certain Sets of Paths

Proposition 1 leaves us with the problem of estimating the cardinality of the sets $\mathcal{P}_n$. For $E \in \mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$ let $\mathcal{Q}_n^E(\mathcal{B})$ be the set of all paths $D_0 D_1 \ldots D_{k-1}$ with $k \geq 1$ satisfying (2.6), such that $D_i \in \mathcal{B}$ for $1 \leq i \leq k-1$ and D_0 is a successor of E. We begin with

Lemma 6. *For each $\alpha > 0$ there is a finite subset $\mathcal{E}$ containing $\mathcal{Z} = \mathcal{D}_0$ such that* card $\mathcal{Q}_n^E(\mathcal{D} \setminus \mathcal{E}) \leq 4e^{\alpha n}$ *for all n and all $E \in \mathcal{D}$.*

Proof. Fix $u \geq \frac{\gamma}{\alpha} \log 2$ and let v be as in Lemma 5. Set $\mathcal{E} = \mathcal{D}_v$. Let $\mathcal{H}_1$ be the set of all $D \in \mathcal{D}$ which have a common endpoint with some $Z \in \mathcal{Z}$ and let $\mathcal{H}_2$ be the set of all $D \in \mathcal{H}_1$ which satisfy $D \cap V_\theta = \emptyset$. For $C \in \mathcal{D}$ we show the following.

(i) There are $j \geq 1$ and $C_0 = C, C_1, \ldots, C_{j-1}$ in $\mathcal{D}$ such that C_i is the only successor of C_{i-1} in $\mathcal{D} \setminus \mathcal{E}$ for $1 \leq i \leq j-1$ and C_{j-1} has at most two successors in $\mathcal{D} \setminus \mathcal{E}$, which are in $\mathcal{H}_1$. If $C \in \mathcal{H}_1$ then either $j \geq v$ or C_{j-1} has no successor in $\mathcal{D} \setminus \mathcal{E}$.

(ii) For each successor B of C_{j-1} in $\mathcal{D} \setminus \mathcal{E}$ there are $l \geq 1$ and $B_0 = B, B_1, \ldots, B_{l-1}$ in $\mathcal{D}$, such that B_i is the only successor of B_{i-1} in $\mathcal{D} \setminus \mathcal{E}$ for $1 \leq i \leq l-1$ and either $B_{l-1} \in \mathcal{H}_2$ or B_{l-1} has no successor in $\mathcal{D} \setminus \mathcal{E}$.

Assuming (i) and (ii) it is easy to prove the lemma. Each path $D_0 D_1 \ldots D_{k-1} \in \mathcal{Q}_n^E(\mathcal{D} \setminus \mathcal{E})$ is made up of segments $C_1 C_2 \ldots C_{j-1} B_0 B_1 \ldots B_{l-1}$, where $j \geq v$ except in the first segment, and $B_{l-1} \in \mathcal{H}_2$ except in the last segment, where the

B_i may be missing. For all segments except the first and the last one we have $h(C_1C_2 \dots C_{j-1}B_0B_1 \dots B_{l-1}) \geq u \geq \frac{\gamma}{\alpha}\log 2$ by Lemma 5, since $j \geq v$ and B_{l-1} has a successor in $\mathcal{D}\backslash\mathcal{E}$ and is therefore in $\mathcal{H}_2$ by (ii). Since $h(D_0D_1 \dots D_{k-2}) < \gamma n$, Lemma 1 implies that $D_0D_1 \dots D_{k-1}$ can consist of at most $2+\frac{\alpha n}{\log 2}$ such segments. Since all successors in these segments are uniquly determined except that of C_{j-1}, which can have two successors in $\mathcal{D} \setminus \mathcal{E}$, the number of paths in $\mathcal{Q}_n^E(\mathcal{D} \setminus \mathcal{E})$ is bounded by $2^{2+\frac{\alpha n}{\log 2}} = 4e^{\alpha n}$.

It remains to show (i) and (ii) for $C \in \mathcal{D}$. Lemmas 12 and 13 of [**4**] give the existence of j and C_i for $0 \leq i \leq j-1$ such that (i) holds. In order to show (ii) let $B \in \mathcal{H}_1$ be a successor of D_{j-1} in $\mathcal{D} \setminus \mathcal{E}$. If $B \cap V_\theta = \emptyset$, then $B \in \mathcal{H}_2$, and (ii) holds with $l = 1$ and $B_0 = B$. Hence using (2.3) we can assume that $B \subset V_\theta(W)$ for some $W \in \mathcal{W}$. Let p and y be the endpoints of $V_\theta(W)$, where $p \in F$. One of these points is also an endpoint of B.

Suppose first that p is an endpoint of B and denote the other endpoint of B by x. Choose s minimal such that $T^s(x) \notin V_\theta(W)$. Set $B_i = T^iB$ for $0 \leq i \leq s-1$. Then B_i is the only successor of B_{i-1} in $\mathcal{D}$ for $1 \leq i \leq s-1$. Furthermore, let B_s be that successor of B_{s-1}, which has $T^s(x)$ as endpoint. Since $T(B_{s-1})$ has endpoints p and $T^s(x)$ we have $B_s \in \mathcal{H}_2$ and all other successors of B_{s-1} are in $\mathcal{Z} \subset \mathcal{E}$. Either there is $l \leq s$ such that $B_i \in \mathcal{D} \setminus \mathcal{E}$ for $i < l$ and $B_l \in \mathcal{E}$ so that B_{l-1} has no successor in $\mathcal{D} \setminus \mathcal{E}$ or $B_i \in \mathcal{D} \setminus \mathcal{E}$ for all $i \leq s$. In the second we set $l = s+1$ and (ii) is shown.

Now suppose that p is not an endpoint of B. Then B has endpoint y. We denote its other endpoint by z. Choose s minimal such that $T^s(z) \notin V_\theta(W)$. Set $B_0 = B$ and $B_i = TB_{i-1} \cap V_\theta(W)$ for $1 \leq i \leq s-1$. The successors of B_{i-1} for $1 \leq i \leq s-1$ are then B_i and all $Z \in \mathcal{Z}$ contained in $T(V_\theta(W)) \setminus V_\theta(W)$. Let B_s be that successor of B_{s-1} which has $T^s(z)$ as endpoint. Since $T(B_{s-1})$ has endpoints $T(y)$ and $T^s(z)$, all successors of B_{s-1} are in $\mathcal{Z} \subset \mathcal{E}$ except B_s which is in $\mathcal{H}_2$. Either there is $l \leq s$ such that $B_i \in \mathcal{D}\setminus\mathcal{E}$ for $i < l$ and $B_l \in \mathcal{E}$ so that B_{l-1} has no successor in $\mathcal{D} \setminus \mathcal{E}$ or $B_i \in \mathcal{D} \setminus \mathcal{E}$ for $i \leq s$. In the second we set $l = s+1$ and again (ii) is shown. □

For $\mathcal{B} \subset \mathcal{D}$ let $\mathcal{P}_n(\mathcal{B})$ be the set of all $D_0D_1 \dots D_{k-1} \in \mathcal{P}_n$ with $D_{k-1} \in \mathcal{B}$. Then we have

Proposition 2. *For each $\alpha > 0$ there is a finite subset $\mathcal{F}$ of $\mathcal{D}$ and a constant a such that* card $\mathcal{P}_n \leq ae^{\alpha\varepsilon n/\gamma}\sum_{l=1}^n$ card $\mathcal{P}_l(\mathcal{F})e^{\alpha(n-l)}$ *for all n.*

Proof. For fixed $\alpha > 0$ let $\mathcal{E} \subset \mathcal{D}$ be as in Lemma 6. For each $E \in \mathcal{D}$ contained in some $W \in \mathcal{W}$ and each path $E_0E_1 \dots$ in $(\mathcal{D}, \to)$ with $E_0 = E$ one shows using similar arguements as in the second part of the proof of Lemma 6 that there is a minimal $s \geq 0$ such that either $E_s \subset V_0(W)$ and hence $E_s \in \mathcal{G}$ or $E_s = V_\theta(W) \in \mathcal{Z}$. For each $E \in \mathcal{E}$ contained in some $W \in \mathcal{W}$ and each path $E_0E_1 \dots$ with $E_0 = E$ we add E_i for $1 \leq i \leq s$ to $\mathcal{E}$ and denote the resulting set by $\mathcal{F}$. This set is still finite and contains $\mathcal{E}$ and hence also $\mathcal{Z}$, as $\mathcal{Z} \subset \mathcal{E}$ by Lemma 6. Let $\tilde{\mathcal{F}}$ be the set of all $D \in \mathcal{F}$, which have a successor outside $\mathcal{F}$. Then $\tilde{\mathcal{F}} \subset \mathcal{G}$ by the construction of $\mathcal{F}$, since each $V_\theta(W)$ has all its successors in $\mathcal{Z}$ by (2.3).

For each path $D_0D_1 \dots D_{k-1} \in \mathcal{P}_n$ let q be minimal, such that $D_i \notin \mathcal{F}$ for $q \le i \le k-1$. Since $D_0 \subset \mathcal{Z} \subset \mathcal{F}$ by (2.7), this q exists and satisfies $1 \le q \le k$. Set $\mathcal{R}_l = \{D_0D_1 \dots D_{k-1} \in \mathcal{P}_n : q < k, D_0D_1 \dots D_{q-1} \in \mathcal{P}_l(\mathcal{F})\}$ for $1 \le l \le n-1$. We show that $\mathcal{P}_n \subset \mathcal{P}_n(\mathcal{F}) \cup \bigcup_{l=1}^{n-1} \mathcal{R}_l$. If $D_0D_1 \dots D_{k-1} \in \mathcal{P}_n$ and $q = k$ then $D_0D_1 \dots D_{k-1} \in \mathcal{P}_n(\mathcal{F})$. If $q < k$ then $h(D_0D_1 \dots D_{q-1}) < \gamma n$ and, since D_{q-1} is in $\mathcal{F}$ and its successor D_q is not in $\mathcal{F}$, we get $D_{q-1} \in \tilde{\mathcal{F}} \subset \mathcal{G}$, which implies $\inf_{D_{q-1}} \varphi \ge \gamma$. Thus there is $l \in \{1, 2, \dots, n-1\}$ with $D_0D_1 \dots D_{q-1} \in \mathcal{P}_l$, since also $A \cap \bigcap_{i=0}^{q-1} T^{-i} D_i \ne \emptyset$ by (2.8). Hence $D_0D_1 \dots D_{q-1} \in \mathcal{P}_l(\mathcal{F})$. We have shown that $\mathcal{P}_n \subset \mathcal{P}_n(\mathcal{F}) \cup \bigcup_{l=1}^{n-1} \mathcal{R}_l$. Hence the lemma is proved if we have shown that $\operatorname{card} \mathcal{R}_l \le a e^{\alpha \varepsilon n/\gamma} \operatorname{card} \mathcal{P}_l(\mathcal{F}) e^{\alpha(n-l)}$ for $1 \le l \le n-1$ with $a = 4 \frac{e^{\alpha \Gamma/\gamma}}{e^\alpha - 1}$.

To this end consider some $D_0D_1 \dots D_{k-1} \in \mathcal{R}_l$. As $D_{k-1} \in \mathcal{G}$ and hence $\inf_{D_{k-1}} \varphi \ge \gamma$, there is j such that $D_qD_{q+1} \dots D_{k-1} \in \mathcal{Q}_j^{D_{q-1}}(\mathcal{D} \setminus \mathcal{E})$. Using Lemmas 1 and 2 we get

$$\begin{aligned} \gamma l + \gamma j &\le h(D_0D_1 \dots D_{q-1}) + h(D_qD_{q+1} \dots D_{k-1}) \le h(D_0D_1 \dots D_{k-1}) \\ &\le \tilde{h}(D_0D_1 \dots D_{k-2}) + \Gamma \le h(D_0D_1 \dots D_{k-2}) + \varepsilon n + \Gamma \\ &< \gamma n + \varepsilon n + \Gamma \end{aligned}$$

Hence $j \le n - l + \frac{\varepsilon n + \Gamma}{\gamma}$. Since $\operatorname{card} \mathcal{Q}_j^{D_{q-1}}(\mathcal{D} \setminus \mathcal{E}) \le 4e^{\alpha n}$ by Lemma 6 and since $D_0D_1 \dots D_{q-1} \in \mathcal{P}_l(\mathcal{F})$, we get $\operatorname{card} \mathcal{R}_l \le \operatorname{card} \mathcal{P}_l(\mathcal{F}) \sum_{j=0}^{n-l+u(\varepsilon)} 4e^{\alpha j}$, where $u(\varepsilon)$ is the largest integer less than or equal to $\frac{\varepsilon n + \Gamma}{\gamma}$. This easily implies the estimate for $\operatorname{card} \mathcal{R}_l$ stated above. □

Now we use again t-conformal measures.

Proposition 3. *Let m be a t-conformal measure with support A. For each finite subset $\mathcal{F}$ of $\mathcal{D}$ there is a constant b such that* $\operatorname{card} \mathcal{P}_n(\mathcal{F}) \le b e^{t\gamma n + t\varepsilon n}$ *for all $n \ge 1$.*

Proof. Set $q = \min\{m(D) : D \in \mathcal{F}, D \cap A \ne \emptyset\}$. Since $\mathcal{F}$ is finite and $\operatorname{supp} m = A$ we have $q > 0$. For $D_0D_1 \dots D_{k-1} \in \mathcal{P}_n(\mathcal{F}) \subset \mathcal{P}_n$ we have $D_{k-1} \cap A \ne \emptyset$ by (2.8), since A is invariant. Therefore we get

$$m\left(\bigcap_{i=0}^{k-1} T^{-i} D_i\right) \ge m(D_{k-1}) e^{-t(\gamma n + \varepsilon n)} \ge q e^{-t\gamma n} e^{-t\varepsilon n}$$

by Lemma 2. Since the intervals $\bigcap_{i=0}^{k-1} T^{-i} D_i$ are disjoint for different paths $D_0D_1 \dots D_{k-1}$ in $\mathcal{P}_n$ by Lemma 1 and (2.6), we get the desired result with $b = 1/q$. □

The three propositions together give now

Theorem. *Let A be an invariant subset of a weakly expanding piecewise monotonic transformation T with regular derivative. Suppose that there is a t-conformal measure with support A. Then* $\mathrm{BD}^+(A) \le t$.

Proof. Choosing $\alpha = t\gamma$ the three propositions imply that there is a constant c such that $N_{e^{-\gamma n}}(A) \le cn^2 e^{(2t+1)\varepsilon n} e^{t\gamma n}$ holds for all n. Hence

$$\limsup_{r\to 0} \frac{\log N_r(A)}{-\log r} \le \limsup_{n\to\infty} \frac{\log N_{e^{-\gamma n}}(A)}{-\log e^{-\gamma(n-1)}} \le t + \frac{(2t+1)\varepsilon}{\gamma}$$

Since ε can be chosen arbitrary small, the desired result follows. □

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