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THE PLANAR MOTION WITH BOUNDED DERIVATIVE OF THE CURVATURE AND ITS SUBOPTIMAL PATHS*

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ABSTRACT. We describe the construction of suboptimal trajectories of the problem of a planar motion with bounded derivative of the curvature and we prove their suboptimality. 'Suboptimal' means longer than the optimal by no more than a constant depending only on the bound B for the curvature's derivative. The initial and final coordinates, curvatures and tangent angles are given. The tangent angle and the curvature of the path are assumed to be continuous. The bound B and the distance d between the initial and final points satisfy an inequality of the kind $d \gg 1/\sqrt{B}$.

1. INTRODUCTION

We consider the problem of finding the shortest path connecting two given points of the Euclidian plane which has given initial and final tangent angles and initial and final curvatures, whose tangent angle and curvature vary continuously, the speed of changing the curvature being bounded by some constant B. We consider paths which contain no cusps.

The problem has a real background — this is the problem to find a (the) shortest path(s) for a car to go from one given point to another with the above mentioned initial and final conditions. One can turn the wheels of a car with a bounded speed. Hence, the speed of changing the curvature of the path of a real car is bounded.

This and similar problems have been the object of several efforts recently. Dubins (1957, see [**7**]) considers the problem of constructing the optimal trajectory between two given points with given tangent angles and with bounded curvature (cusps are not allowed). He proves that there exists a unique optimal trajectory which is a concatenation of at most three pieces: every piece is either a straight line segment or an arc of a circle of fixed radius. The same model is considered by

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Cockayne and Hall (1975, see [**6**]) but from another point of view: they provide the classes of trajectories by which a moving "oriented point" can reach a given point in a given direction and they obtain the set of all the points reachable at a fixed time.

Reeds and Shepp (1990, see [**14**]) solve a similar problem, when cusps are allowed. They obtain the list of all possible optimal trajectories. This list contains forty eight types of trajectories. Each of them is a finite concatenation of pieces each of which is either a straight line or an arc of a circle.

Laumond and Souères (1992, see [**11**]) obtain a complete synthesis for the Reeds-Shepp model in the case without obstacles.

A complete synthesis for the Dubins model in the case without obstacles is obtained by Boissonnat, Bui, Laumond and Souères (1994, see [**3**] and [**4**]).

All these authors use very particular methods in their proofs. It seems very difficult to generalize them. That is why the same problem is solved by Sussman and Tang (1991, see [**15**]) and by Boissonnat, Cérézo and Leblond (1991, see [**1**]) by means of simpler arguments based on the Maximum Principle of Pontryagin.

Using these arguments allows to treat more difficult models as the one considered in this paper. Here we consider a similar problem but now with a bounded derivative of the curvature (cusps are not allowed). The same problem is considered by Boissonnat, Cerezo and Leblond (1994, see [**2**]). After applying the Maximum Principle of Pontryagin they obtain the following result: any extremal path is the C^2 concatenation of line segments in one and the same direction and of arcs of clothoids with the same value of the parameter B (all of finite length). They study the possible variants of concatenation of arcs of clothoid and line segments and obtain that if an extremal path contains but is not reduced to a line segment, then it contains an infinite number of concatenated arcs of clothoids which accumulate towards each endpoint of the segment which is a switching point. Thus, in the generic case, an optimal path can have at most a finite number of switching points only if it is a finite concatenation of arcs of clothoids with the same value of the parameter B.

The readers familiar with chattering control theory can remark after examining Section 2 that the singular trajectories of our problem (i.e. the line segments in one and the same direction) have intrinsic order 2 (see the definition in [**12**]). The complete theory of such chattering controls known to the present day is exposed in the monograph of Zelikin and Borisov (1994, see [**16**]).

We solve the problem of the irregularity of optimal paths in the generic case (see [**10**]) and we obtain the following result: if the distance between the initial and the final points is greater than some constant C depending only on the parameter B of the clothoid, then, in the generic case, optimal paths have an infinite number of switching points. We prove this by showing that a path which is a finite concatenation of arcs of clothoids can be shortened while preserving the initial and final conditions, the continuity of the tangent angle and curvature and the boundedness of the curvature's derivative.

That is why in this paper we concentrate our attention on the explicit description of suboptimal trajectories (i.e. not more than a constant longer than the optimal one) and of their construction. Two students — A. Casta and Ph. Cohen — wrote a programme in MAPLE which draws such suboptimal paths.

We consider the same problem in the case when cusps are allowed in [**8**] (1993).

In §2 we consider the theoretical aspect of the problem, using the Maximum Principle of Pontryagin. We obtain that if an optimal trajectory is piecewise regular then it must be a concatenation of arcs of clothoids and of straight line segments. We construct suboptimal paths from such pieces in §4. We prove the suboptimality of the constructed path in §5 by means of some geometric properties of clothoids which are exposed in §3.

2. Statement of the Problem, Existence of an Optimal Solution and Application of the Maximum Principle of Pontryagin to This Problem

We study the shortest C^2 path on the plane joining two given points with given tangent angles and curvatures along which the derivative of the curvature remains bounded. The tangent angle $\alpha(t)$ between the axis Ox and the tangent-vector to the path is a continuous and piecewise C^1 function, the curvature $u(t)$ is continuous.

We have the following system (from now on we denote "d/dt" by "$\cdot$"):

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos\alpha(t) \\ \dot{y}(t) = \sin\alpha(t) \\ \dot{\alpha}(t) = u(t) \\ \dot{u}(t) = w(t) \qquad |w(t)| \leq B \end{cases} \tag{1}$$

with initial and final conditions:

$$X(0) = (x^0, y^0, \alpha^0, u^0), \qquad X(T) = (x^T, y^T, \alpha^T, u^T) \tag{2}$$

We control the derivative of the curvature by the control function w. The control function w is a measurable, real-valued function and $w \in W$, where $W = [-B, +B]$. We want to find such $X(t)$ that the associated control function $w(t)$ should minimize the length of the path

$$J(w) = T = \int_0^T dt \tag{3}$$

Here the variable t is the arc length but it will be called the time because the point moves with a constant speed 1, that is why this "minimum length problem" is also a "minimum time problem".

Of special interest are paths which are piecewise C^3 (whose tangent angle is piecewise C^2 and whose curvature is piecewise C^1). They are obtained for a piecewise continuous control w. At a point where the control function w is continuous the path is called **regular**. However as it was mentioned in the introduction optimal paths have, in general, infinitely many points of irregularity.

On the other hand-side, one can construct in practice a path with only finitely many points of irregularity.

We use the same ideas as the ones developed by Boissonnat et al. (1994, see [**2**]) to prove the existence of an optimal solution to system (1); as in [**2**] we apply the Maximum Principle of Pontryagin to obtain necessary conditions upon the control function in order the solution to be optimal.

Prove at first the controllability of system (1).

If $\mathrm{dist}\,((x^0,y^0),(x^T,y^T)) \gg 1/\sqrt{B}$ (see exact definition in Section 4), then one can construct a suboptimal path from (x^0,y^0,α^0,u^0) to (x^T,y^T,α^T,u^T) (see Section 4). If not, then one can construct a suboptimal path from (x^0,y^0,α^0,u^0) to a point (x^*,y^*,α^0,u^0) such that

$$\mathrm{dist}\,((x^0,y^0),(x^*,y^*)) \gg 1/\sqrt{B} \quad \text{and} \quad dist((x^T,y^T),(x^*,y^*)) \gg 1/\sqrt{B},$$

then a suboptimal path from (x^*,y^*,α^0,u^0) to (x^T,y^T,α^T,u^T). Both suboptimal paths belong to the class of paths under consideration. So, the controllability of system (1) is proved.

In order to prove the existence of an optimal solution we can use Filippov's existence theorem, see [**5**, Th. 5.1ii]. So, rewrite system (1) in the form

$$\dot{X} = F(X,w), \qquad X(t) \in \mathbf{R}^4, \qquad w \in W.$$

All requirements of the theorem of Filippov are satisfied: all functions $F(X,w)$ are continuous together with their partial derivatives; the function under the sign of the integral in (3) is continuous; the control function w is bounded and the range of control is convex; $X(t) \in \mathbf{R}^4$ ($\mathbf{R}^4$ is closed); the initial and final points $(X(0),X(T))$ are fixed; one can verify that there exists a constant $C>0$ such that for every $X(t) \in \mathbf{R}^4$ and $w \in W$ the following inequality is satisfied: $XF(X) \leq C(|X|^2+1)$. Thus we can assume the existence of an optimal solution and an optimal control for problem (1), (2), (3).

We are going to apply the Maximum Principle of Pontryagin to obtain necessary conditions for the control function $w(t)$ and for the trajectory $(x(t),y(t),\alpha(t),u(t))$ to be optimal. Rewrite system (1), (2) and integral (3) as the following system:

$$\begin{cases} \dot{x}(t) = \cos\alpha(t) & x(0)=x^0 & x(T)=x^T \\ \dot{y}(t) = \sin\alpha(t) & y(0)=y^0 & y(T)=y^T \\ \dot{\alpha}(t) = u(t) & \alpha(0)=\alpha^0 & \alpha(T)=\alpha^T \\ \dot{u}(t) = w(t) & u(0)=u^0 & u(T)=u^T & |w(t)| \leq B \\ \dot{x}_0(t) = 1 & x_0(0)=0 \end{cases}$$

If we denote by $\Psi(t) = (p, q, \beta, r, e)$ the vector of "dual" variables then the Hamiltonian H would be defined by

$$H(X, \Psi, w) = p(t)\cos\alpha(t) + q(t)\sin\alpha(t) + \beta(t)u(t) + r(t)w(t) + e, \quad \text{for every } t \in [0, T]. \tag{4}$$

So we have the adjoint system (for every $t \in [0, T]$):

$$\dot{\Psi}(t) = \begin{cases} \dot{p}(t) = 0 \\ \dot{q}(t) = 0 \\ \dot{\beta}(t) = p(t)\sin\alpha(t) - q(t)\cos\alpha(t) \\ \dot{r}(t) = -\beta(t) \\ \dot{e}(t) = 0 \end{cases} \tag{5}$$

Thus p, q, e, are constant on $[0, T]$. Setting $p = \lambda\cos\varphi$, $q = \lambda\sin\varphi$ (here $\lambda = \sqrt{p^2 + q^2}$, $\lambda \geq 0$, $\tan\varphi = q/p$) we can rewrite the adjoint system (5) and the Hamiltonian (4) as follows:

$$\begin{cases} p(t) \equiv \lambda\cos\varphi \\ q(t) \equiv \lambda\sin\varphi \\ \dot{\beta}(t) = \lambda\sin(\alpha(t) - \varphi) \\ \dot{r}(t) = -\beta(t) \\ e(t) \equiv e_0 \end{cases} \tag{6}$$

$$H(X, \Psi, w) = \lambda\cos(\alpha(t) - \varphi) + \beta(t)u(t) + r(t)w(t) + e_0. \tag{7}$$

Define

$$M(X, \Psi) = \min_{w \in [-B, +B]} H(X, \Psi, w)$$

where (p, q, β, r, e), (x, y, α, u), w are considered as independent variables.

We shall use the Maximum Principle of Pontryagin as it is formulated in [**5**, Th. 5.1i] and [**13**, Chapter 1, Th. 1]. It asserts that if w_* is an optimal control, then

(a) there exists a non-zero absolutely continuous vector-function $\Psi(t)$ which is a continuous solution to (5);

(b) for almost every fixed $t \in [0, T]$ the function $H(X, \Psi, w)$ (considered as a function of the variable $w \in [-B, +B]$ only) attains its minimum at the point $w = w_*$:

$$M(X(t), \Psi(t)) = H(X(t), \Psi(t), w_*(t)), \quad t \in [0, T];$$

(c) the function $M(t) = M(X(t), \Psi(t))$ is absolutely continuous in $[0, T]$ and

$$\frac{dM}{dt}(X(t), \Psi(t)) = \frac{\partial H}{\partial t}(X(t), \Psi(t), w(t));$$

(d) at any time $t \in [0, T]$ the relations $e_0 \geq 0$ and $M(X(t), \Psi(t)) = 0$ are satisfied.

From condition (b) with respect to $w(t)$ we obtain two cases:

1) $\partial H/\partial w \equiv 0$ for $t \in [t_1, t_2] \subset [0, T]$,

2) $\partial H/\partial w \not\equiv 0$ for $t \in (t_1, t_2) \subset [0, T]$.

Hence in case 1) $r(t) \equiv 0$ from (7), then from (6) we obtain that $\beta(t) \equiv 0$, hence $\dot\beta(t) \equiv 0$ and $\alpha(t) = \varphi(\mathrm{mod}\pi)$ for every $t \in [t_1, t_2]$ (we don't consider the case $\lambda = 0$ because it contradicts (a)). So $\dot\alpha(t) \equiv 0$, $u(t) \equiv 0$ and $w(t) \equiv 0$ for all $t \in [t_1, t_2]$.

The corresponding path is a line segment in the direction φ.

In case 2) $w = -B\,\mathrm{sgn}\,(r(t))$ (it follows from (7)). The corresponding path is a clothoid. A clothoid is a curve along which the curvature $u(t)$ depends linearly on the arc length t and varies continuously from $-\infty$ to $+\infty$. That is why $w(t) = \pm B$ determines a single clothoid (modulo the group of symmetries of the plane).

We can define the clothoid as the curve satisfying the following equation

$$u(t) = \pm Bt, \quad t \in (-\infty, +\infty).$$

We can also define the clothoid by its parametrized form (setting $x(0) = y(0) = 0$, $\alpha(0) = 0$, $u(0) = 0$)

$$\begin{cases} x(t) = \sqrt{2/B} \displaystyle\int_0^{t\sqrt{B/2}} \cos(\tau^2)\, d\tau \\ (t) = \pm\sqrt{2/B} \displaystyle\int_0^{t\sqrt{B/2}} \sin(\tau^2)\, d\tau \end{cases}$$

The two possible choices of the sign correspond to the two possible orientations of the clothoid.

Call B the **parameter of the clothoid**. The sign $\pm$ defines the orientation of the clothoid, the variable t is the natural parameter and the curvature equals $\pm Bt$. For $t = 0$ the clothoid has a (unique) inflexion point which is its centre of symmetry. Call **half-clothoid** its part corresponding to $t \in [0, +\infty)$ or to $t \in (-\infty, 0]$.

A measurable control function w and its associated trajectory of (1) satisfying all conditions of the Maximum Principle of Pontryagin will be called **extremal control** and **extremal trajectory**. A point $X(t_p)$ of an extremal trajectory will be called a **switching point** if at $t = t_p$ the control function $w(t)$ has a discontinuity.

From the preceding reasonings we can deduce

Lemma 2.1. *Any extremal path is the C^2 concatenation of the line segments in one and the same direction ($w = 0$) and of arcs of clothoids ($w = \pm B$), all of finite length.*

In [**10**] we prove the following theorem:

Theorem 2.2. *If the distance between the initial and the final point is greater than the constant C which depends only on the value of the parameter B, then, in the generic case, optimal paths have an infinite number of switching points.*

That is why in the present paper we construct (in §4) regular suboptimal paths in the case when the distance between the initial and the final point is much greater than $1/\sqrt{B}$ (the exact definition is given in §4).

The suboptimal path consists of a line segment and of four pieces of clothoids, its curvature and tangent angle are continuous, it has four switching points, see §4. One needs at least 4 switching points in order to be able to attain the 4 final conditions — coordinates, tangent angle and curvature. Using more switching points could lead to shortening of the path but it must certainly be connected with formulas more difficult to deal with. Therefore we have chosen the simplest possible way to construct suboptimal paths.

In order to prove the suboptimality of the path constructed in §4, i.e. that it is no more than a fixed constant (depending only on B) longer than the optimal one, we prove some geometric properties of clothoids in §3. Its suboptimality is proved in §5.

3. Geometric Properties of the Clothoid

Consider a half-clothoid

$$\begin{cases} \dot{x}(t) = \cos(Bt^2/2) & x(0) = 0 \quad t \geq 0 \\ \dot{y}(t) = \sin(Bt^2/2) & y(0) = 0 \quad B > 0\,. \end{cases} \tag{8}$$

Define as **the centre of the half-clothoid** the point O_c with coordinates (x_{O_c}, y_{O_c}) defined as follows:

$$\begin{cases} x_{O_c} = \displaystyle\int_0^\infty \cos\left(B\tau^2/2\right) d\tau = \sqrt{2/B} \int_0^\infty \cos \nu^2 \, d\nu = \sqrt{2/B}\sqrt{2\pi}/4 \\ \quad\;\; = \sqrt{\pi}/(2\sqrt{B}) \\ \\ y_{O_c} = \displaystyle\int_0^\infty \sin\left(B\tau^2/2\right) d\tau = \sqrt{2/B} \int_0^\infty \sin \nu^2 \, d\nu = \sqrt{2/B}\sqrt{2\pi}/4 \\ \quad\;\; = \sqrt{\pi}/(2\sqrt{B})\,. \end{cases} \tag{9}$$

Consider the circle with centre at the centre of the half-clothoid (8) and take the radius of this circle (denote it by r_B) to be equal to the distance between the centre of the half-clothoid (8) and its point with zero curvature. Then, from (9) we obtain (see Figure 1)

$$r_B = |\overrightarrow{OO_c}| = \sqrt{x_{O_c}^2 + y_{O_c}^2} = \sqrt{\pi/(2B)}\,. \tag{10}$$

Remark 3.1. A half-clothoid of the opposite orientation is defined by equations

$$\begin{cases} \dot{x}(t) = \cos(Bt^2/2) & x(0) = 0 \quad t \geq 0 \\ \dot{y}(t) = \sin(-Bt^2/2) & y(0) = 0 \quad B > 0\,. \end{cases}$$

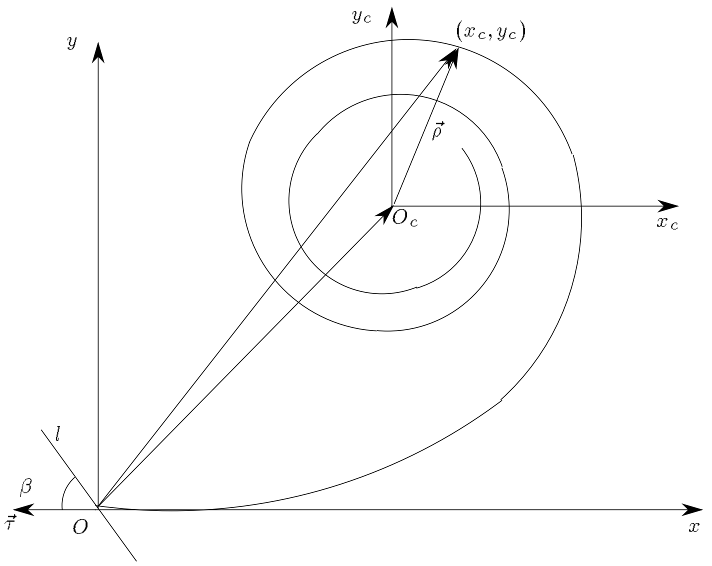

Figure 1. A half-clothoid and its corresponding polar coordinate system.

Further in the text we set $B = 2$ in the proofs of these statements whose formulations don't depend on the concrete value of the parameter B. If on the contrary, a statement or an estimation depends essentially on B, then we say this explicitely.

For $B = 2$ we consider the half-clothoid

$$\begin{cases} \dot{x} = \cos t^2 & x(0) = 0 \quad t \geq 0 \\ \dot{y} = \sin t^2 & y(0) = 0\,. \end{cases} \tag{11}$$

3.1 Fundamental Properties of an Individual Clothoid

Fix a direction α_* (modπ, not mod2π in $\mathbf{R}^2$ and let $P_1, P_2, \ldots$ denote the consecutive points on the half-clothoid with a tangent line at them of the chosen direction (with $t_1 < t_2 < \ldots$). Set $P_i = (x_i, y_i)$, $x_i = x(t_i)$, $y_i = y(t_i)$ (see Figure 2).

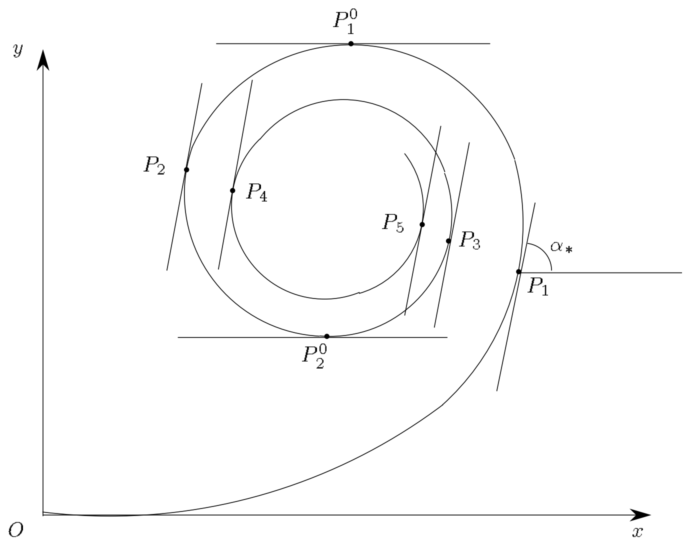

Figure 2. Consecutive tangent lines to a half-clothoid.

Proposition 3.2. *$\widehat{P_1P_2}$ is the longest among the arcs $\widehat{P_iP}_{i+1}$. Its length depends continuously and monotonously on the choice of the direction α_*.*

Proof.

$$|\widehat{P_iP}_{i+1}| = \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} \sqrt{\cos^2 t^2 + \sin^2 t^2}\, dt = \sqrt{\alpha_* + i\pi} - \sqrt{\alpha_* + (i-1)\pi}$$
$$= \frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}}\,.$$

Both statements follow directly from these equalities. The proposition is proved. □

For $B = 2$ the coordinates (x_{O_c}, y_{O_c}) is defined as follows:

$$\begin{cases} x_{O_c} = \displaystyle\int_0^\infty \cos\tau^2\, d\tau \\ y_{O_c} = \displaystyle\int_0^\infty \sin\tau^2\, d\tau \end{cases}$$

Consider the coordinate system with the centre at the centre O_c of the half-clothoid and with the axes O_cx_c, O_cy_c parallel to the corresponding axes of the coordinate system Oxy (see Figure 1). In the coordinate system $O_cx_cy_c$ the coordinates of the point (x_c, y_c) of the half-clothoid (11) are defined by the formulas:

$$\begin{cases} x_c(t) = x(t) - x_{O_c} = -\displaystyle\int_t^\infty \cos\tau^2\,d\tau \\ y_c(t) = y(t) - y_{O_c} = -\displaystyle\int_t^\infty \sin\tau^2\,d\tau \end{cases} \tag{12}$$

Denote by $\vec{\rho}$ the radius-vector of a point of the half-clothoid in the coordinate system $O_cx_cy_c$.

Then

$$\rho^2 = {x_c}^2 + {y_c}^2$$

and

$$\begin{aligned} \dot\rho(t) &= \frac{1}{\rho}(x_c\dot x_c + y_c\dot y_c) = \frac{1}{\rho}\left(-\cos t^2\int_t^\infty \cos\tau^2\,d\tau - \sin t^2\int_t^\infty \sin\tau^2\,d\tau\right) \\ &= -\frac{1}{\rho}\int_t^\infty \cos(\tau^2 - t^2)\,d\tau = -\frac{1}{\rho}\int_{t^2}^\infty \frac{\cos(\eta - t^2)\,d\eta}{2\sqrt{\eta}} = -\frac{1}{\rho}\int_0^\infty \frac{\cos\nu\,d\nu}{2\sqrt{\nu + t^2}}\,. \end{aligned}$$

Thus

$$\dot\rho(t) = -\frac{1}{2\rho}\int_0^\infty \frac{\cos\tau\,d\tau}{\sqrt{\tau + t^2}}\,. \tag{13}$$

Lemma 3.3. *The length of the radius-vector $\vec{\rho}(t)$ of a half-clothoid is a monotonously decreasing function of t:*

$$\dot\rho < 0\,.$$

Proof. Set

$$t^2 = a, \qquad \int_0^\infty \frac{\cos\tau\,d\tau}{\sqrt{\tau + a}} = I(a)\,.$$

The function $\cos\tau$ is periodic with period 2π. So using the property of the symmetry of the function $\cos\tau$ ($\cos(\pi - \tau) = \cos(\pi + \tau) = -\cos\tau$, $\cos(2\pi - \tau) = \cos\tau$) we can consider instead of the integral $I(a)$ the following integral:

$$\int_0^{\pi/2} \Sigma\cos\tau\,d\tau,$$

where

$$\Sigma = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} \right.$$
$$\left. - \frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} \right).$$

This series is convergent because

$$\frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} = O\left(\frac{1}{k\sqrt{k}}\right)$$

and

$$-\frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} = O\left(\frac{1}{k\sqrt{k}}\right).$$

Consider the first four terms of the series. The function $f(\xi) = 1/\sqrt{\xi}$ is convex and monotonously decreasing, see Figure 3.

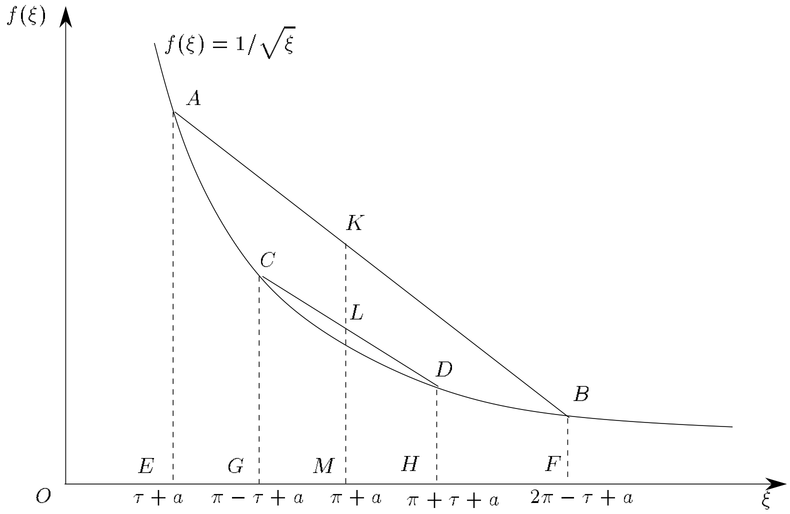

Figure 3. Application of the convexity of the function $1/\sqrt{\xi}$.

For the middle lines KM and LM of the trapezoids $EABF$ and $GCDH$ respectively we have $LM \subset KM$. We have the followings formulas:

$$\frac{1}{\sqrt{\tau+a}} + \frac{1}{\sqrt{2\pi-\tau+a}} = 2|KM|,$$
$$\frac{1}{\sqrt{\pi-\tau+a}} + \frac{1}{\sqrt{\pi+\tau+a}} = 2|LM|,$$
$$|LM| < |KM|.$$

Hence

$$\frac{1}{\sqrt{\tau+a}} - \frac{1}{\sqrt{\pi-\tau+a}} - \frac{1}{\sqrt{\pi+\tau+a}} + \frac{1}{\sqrt{2\pi-\tau+a}} > 0\,.$$

Every following sum of four terms in the series can be considered analogously. This proves that the sum of the series under consideration is positive. The function $\cos\tau$, $\tau \in [0, \pi/2]$ is non-negative. Hence, the integral $I(a)$ is positive and the derivative of the length of the radius-vector $\vec{\rho}(t)$ is negative.

The lemma is proved. $\square$

Lemma 3.4. *The derivative of the length of the radius-vector $\vec{\rho}(t)$ of a half-clothoid is a monotonously increasing function of t, i.e.*

$$\ddot{\rho} > 0\,. \tag{14}$$

The lemma is proved in Appendix A.

We give a geometric interpretation of the inequality $\ddot{\rho} > 0$. Denote by $\gamma(t)$ the angle between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ of the point of the clothoid (11). We have

$$\dot{\rho} = \cos\gamma \tag{15}$$

The angle γ is in the interval $(\pi/2, \pi)(\bmod 2\pi)$ (because $\dot{\rho} < 0$, see Lemma 3.3). Hence, the function $\sin\gamma$ is positive. We have

$$\ddot{\rho} = -\dot{\gamma}\sin\gamma \tag{16}$$

and obtain, from (14), that

$$\dot{\gamma} < 0\,. \tag{17}$$

So we obtain the geometric interpretation of Lemma 3.4:

Remark 3.5. The angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ is a monotonously decreasing function of t; $\gamma(t) \to 3\pi/4$ for $t \to 0$, $\gamma(t) \to \pi/2$ for $t \to +\infty$.

Corollary 3.6. *If we have an "unwinding" half-clothoid (i.e. half-clothoid with decreasing absolute value of the curvature) defined by the equations:*

$$\begin{cases} x(t) = \displaystyle\int_0^t \cos(\tau^2 + u_0\tau + \alpha_0)\,d\tau \quad x(0) = x_0 \quad u_0 < 0 \\ y(t) = \displaystyle\int_0^t \sin(\tau^2 + u_0\tau + \alpha_0)\,d\tau \quad y(0) = y_0 \quad\quad t \geq 0 \end{cases}$$

then for such a clothoid we have the following conditions:

$$\dot{\rho} > 0,$$
$$\ddot{\rho} > 0\,.$$

Corollary 3.7. *If two half-clothoids clA and clB have the same centre O_c, the same orientation and the same parameter B then either they coincide or they have no common point.*

Consider the circle $\mathcal{C}$ with centre at the centre O_c of clA and with radius equal to the distance between the centre of clA and its point of zero curvature. Denote by $\partial\mathcal{C}$ the circumference with centre at O_c and with the same radius. Then $\mathcal{C} \setminus O_c$ is the union of non-intersecting half-clothoids. The mapping which maps each half-clothoid on its point of zero curvature (lying on $\partial\mathcal{C}$) is a bijection from the set of half-clothoids onto $\partial\mathcal{C}$.

Proof. If clA and clB intersect, then at the intersection point they have equal radius-vectors, hence, equal curvatures (see Lemma 3.3), hence, equal values of $\dot{\rho}$ (see Lemma 3.4), hence, they must coincide, because they are obtained by integrating the equations $\dot{x} = \cos{(t-t_0)}^2$, $\dot{y} = \sin{(t-t_0)}^2$ with equal initial data (x_0, y_0, t_0).

The corollary is proved. □

Estimate now the maximal possible distance between two points of the half-clothoid (8). Consider some point E belonging to the half-clothoid (8) (see Figure 4). The length of the chord OE is defined as follows:

(18)
$$|OE| = \sqrt{x^2(t) + y^2(t)} = \sqrt{\left(\int_0^t \cos{(B\tau^2/2)}\, d\tau\right)^2 + \left(\int_0^t \sin{(B\tau^2/2)}\, d\tau\right)^2}\,.$$

Denote by K the point of a half-clothoid where the chord has maximal length.

Proposition 3.8. *The tangent angle α at the point K belongs to the interval $(\pi/2, 3\pi/4)$.*

Proof. At the point K the tangent vector $\vec{\tau}$ is perpendicular to the chord OK (because at K the derivative of the length of the chord is equal to zero). Denote by W the point of the half-clothoid where the tangent angle is equal to $\pi/2$ (see Figure 4). Evidently, $\alpha_K > \pi/2$, because $\frac{d|OW|}{dt} = \cos\gamma$, and $\gamma \in (0, \pi/2)$ at the point W, hence, $\frac{d|OW|}{dt} > 0$.

The angle OKL is equal to $\pi/2$. The angle O_cKL is smaller than $\pi/2$ (because $O_cKL = \pi - \gamma$ and $\gamma \in (\pi/2, \pi)$). Hence, the angle MOK is smaller than the angle MOO_c. But the angle MOO_c is equal to $\pi/4$, hence, the angle MOK is smaller than $\pi/4$ and the angle OMK is greater than $\pi/4$, i.e. the tangent angle α_K at the point K is smaller than $3\pi/4$.

The proposition is proved. □

In two following propositions (Proposition 3.9 and 3.10) we consider arbitrary B.

Proposition 3.9. *The maximal possible length of the chord $|OK|$ is smaller than $3r_B/2$.*

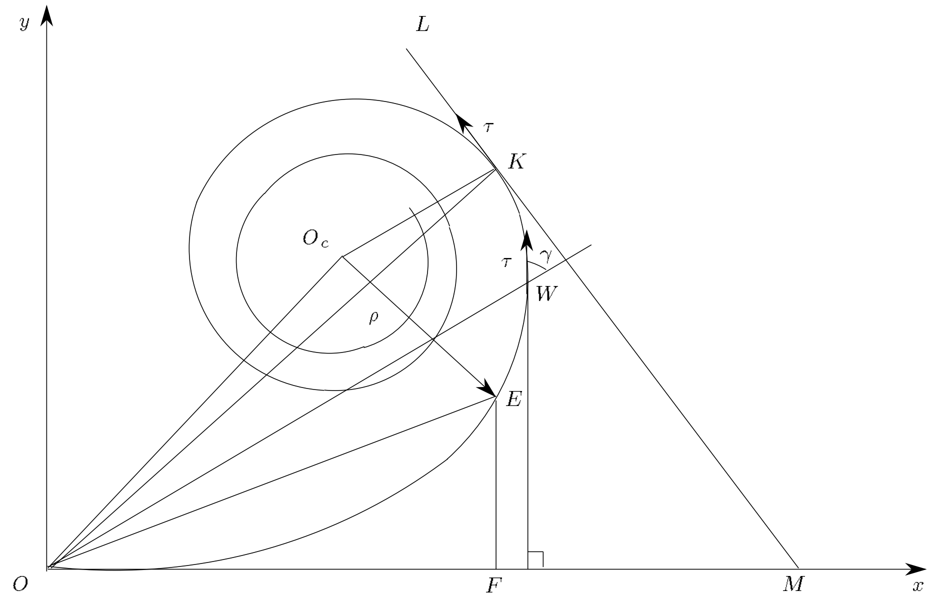

Figure 4. A half-clothoid and its point K where the chord has a maximal length.

Proof. From Lemma 3.3 we know that the length of the radius-vector $\vec{\rho}(t)$ of a half-clothoid is a monotonously decreasing function of t. Hence (see Figure 4),

$$|O_cK| < |O_cW|, \quad \text{i.e.} |OK| < |OO_c| + |O_cW| = r_B + |O_cW|.$$

But for $|O_cW|$ we have the following formulas (see (18)):

$$\begin{aligned}
|O_cW| &= \left(\left(\int_0^{\sqrt{\pi/B}} \cos\left(B\tau^2/2\right)\, d\tau - \sqrt{\pi} \Big/ \left(2\sqrt{B}\right) \right)^2 \right. \\
&\qquad \left. + \left(\int_0^{\sqrt{\pi/B}} \sin\left(B\tau^2/2\right)\, d\tau - \sqrt{\pi} \Big/ \left(2\sqrt{B}\right) \right)^2 \right)^{1/2} \\
&= \left(\left(\sqrt{2/B} \int_0^{\sqrt{\pi/2}} \cos\tau^2\, d\tau - \sqrt{\pi} \Big/ \left(2\sqrt{B}\right) \right)^2 \right. \\
&\qquad \left. + \left(\sqrt{2/B} \int_0^{\sqrt{\pi/2}} \sin\tau^2\, d\tau - \sqrt{\pi} \Big/ \left(2\sqrt{B}\right) \right)^2 \right)^{1/2} \\
&= 2r_B/\sqrt{\pi} \left(\left(\int_0^{\sqrt{\pi/2}} \cos\tau^2\, d\tau - \sqrt{\pi} \Big/ \left(2\sqrt{2}\right) \right)^2 \right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^{\sqrt{\pi/2}} \sin\tau^2\,d\tau - \sqrt{\pi}\Big/\left(2\sqrt{2}\right)\right)^2\Bigg)^{1/2} \\
&\approx 2r_B/\sqrt{\pi}\sqrt{(0.98-0.62)^2+(0.55-0.62)^2} \approx 2r_B\sqrt{0.14}/\sqrt{\pi} \\
&\approx 0.42r_B < r_B/2.
\end{aligned}
$$

Hence, we obtain the desired inequality:

$$|OK| < r_B + r_B/2 = 3r_B/2.$$

The proposition is proved. □

Proposition 3.10. *The maximal possible distance between two points of a half-clothoid is smaller than* $3r_B/2$.

Proof. Consider two points P and Q of a half-clothoid (8) (see Figure 5). We prove that for any points P and Q

$$|PQ| < |OK|\ ,$$

where K is defined before Proposition 3.8.

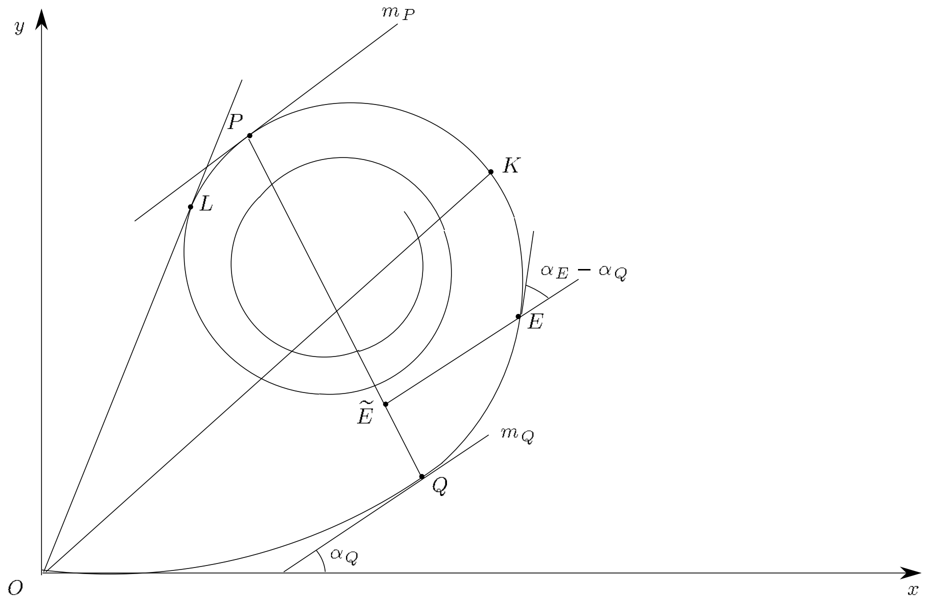

Figure 5. A half-clothoid and an arbitrary chord PQ.

Evidently, we must consider the case when if only one point (P or Q) belongs to the arc $\widehat{OL}$ (the tangent line at the point L passes through the point O). If the chord PQ has the maximal possible length then the tangent lines at the points P, Q (denote them by m_P, m_Q respectively) are perpendicular to the chord PQ.

Consider a point $E \in \widehat{QP}$. Denote by α_Q the tangent angle at the point Q, by α_E — the tangent angle at the point E. The line $E\widetilde{E}$ is parallel to the lines m_P and m_Q.

The length of the straight line segment $E\widetilde{E}$ can be defined by the following formula:

$$|E\widetilde{E}| = \int_{\sqrt{2\alpha_Q/B}}^{\sqrt{2\alpha_E/B}} \cos\left(B\tau^2/2 - \alpha_Q\right)\, d\tau$$

(because $\alpha_E = Bt_E^2/2$).

Assume that the point E coincides with the point P, i.e. $\alpha_E = \alpha_Q + \pi$. Hence, we have the following equality:

$$0 = \int_{\sqrt{2\alpha_Q/B}}^{\sqrt{(2\alpha_Q+2\pi)/B}} \cos\left(B\tau^2/2 - \alpha_Q\right)\, d\tau. \tag{19}$$

We can rewrite equality (19) as follows:

$$0 = \int_0^{\pi} \frac{\cos\tau\, d\tau}{\sqrt{2B(\alpha_Q+\tau)}}.$$

But

$$\int_0^{\pi} \frac{\cos\tau\, d\tau}{\sqrt{2B(\alpha_Q+\tau)}} > 0\ ,$$

because $\cos\tau = \cos(\pi-\tau)$ for any $\tau \in [0,\pi/2)$ and

$$\frac{1}{\sqrt{2B(\alpha_Q+\tau)}} > \frac{1}{\sqrt{2B(\alpha_Q+\pi-\tau)}}.$$

Hence, equality (19) isn't correct and, hence, the chord PQ can't have the maximal possible length, i.e.

$$|PQ| < |OK|.$$

From Proposition 3.9 we have

$$|OK| < 3r_B/2.$$

Hence, the maximal possible distance between two points of a half-clothoid is smaller than $3r_B/2$.

The proposition is proved. □

3.2 Properties of two arcs of clothoids at their concatenation point

Consider two clothoids $cl1$ and $cl2$ (see Figure 6) which for $t = 0$ have the same initial conditions $(x_0, y_0, \alpha_0, u_0)$, $u_0 < 0$, the absolute value of the curvature of $cl1$ is decreasing with t, the one of $cl2$ increasing with t; $cl1$ and $cl2$ are defined by equations:

$$cl1 : \begin{cases} x(t) = \displaystyle\int_0^t \cos\left(\tau^2 + u_0\tau + \alpha_0\right) d\tau + x_0 \\ y(t) = \displaystyle\int_0^t \sin\left(\tau^2 + u_0\tau + \alpha_0\right) d\tau + y_0 \end{cases} \tag{20}$$

$$cl2 : \begin{cases} x(t) = \displaystyle\int_0^t \cos\left(-\tau^2 + u_0\tau + \alpha_0\right) d\tau + x_0 \\ y(t) = \displaystyle\int_0^t \sin\left(-\tau^2 + u_0\tau + \alpha_0\right) d\tau + y_0 \end{cases} \tag{21}$$

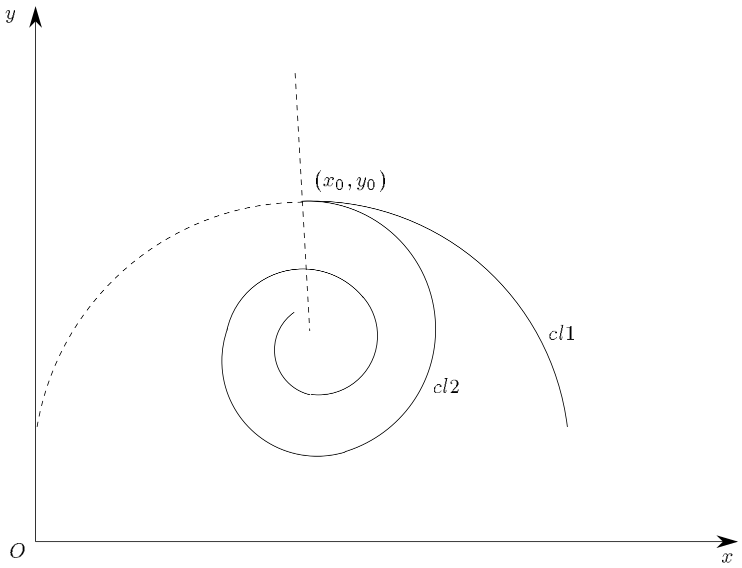

Figure 6. Two arcs of clothoids (with decreasing and with increasing curvatures) with equal curvatures and tangent angles at their common endpoint.

Consider clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ (see Figure 7).

On this figure the point O is the centre of $cl1$, the point A is the initial point, the points B and C belong to the clothoids $cl1$ and $cl2$ respectively and $|\widehat{AB}| =$

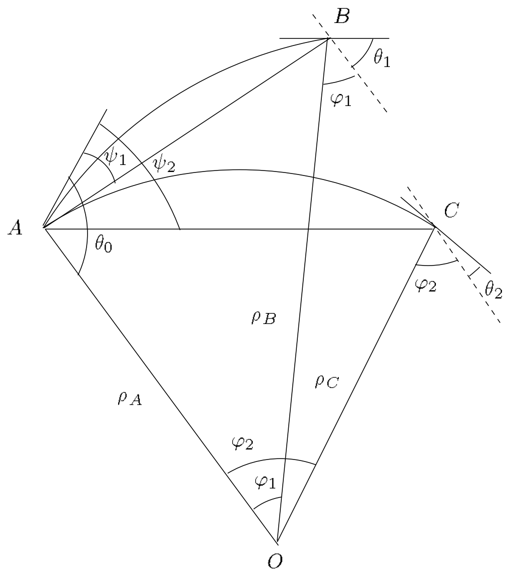

Figure 7. The pieces of half-clothoids $cl1$ and $cl2$ with denoted angles θ_0, θ_1, θ_2, φ_1, φ_2, ψ_1, ψ_2 and radius-vectors $\vec{\rho}_A$, $\vec{\rho}_B$, $\vec{\rho}_C$.

$|\widehat{AC}| = s$. The angle between the tangent vector to $cl1$ and $cl2$ at point A and the vector equal to $(-\vec{\rho}_A)$ ($\vec{\rho}_A$ is the radius-vector at point A) is denoted by θ_0. The angles between the tangent vectors to $cl1$ and $cl2$ at the points B and C and the vector equal to $(-\vec{\rho}_A)$ are denoted θ_1 and θ_2 respectively. The angles between the tangent vector at the point A and the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are denoted ψ_1 and ψ_2 respectively. And the angles between the radius-vector $\vec{\rho}_A$ and the radius-vectors $\vec{\rho}_B$ and $\vec{\rho}_C$ at the points B and C are denoted φ_1 and φ_2 respectively. Denote by δ_i $(i = 1, 2)$ the angles between the tangent lines at the points B and C and their radius-vectors $(\delta_i = \theta_i + \varphi_i,\ i = 1, 2)$.

Lemma 3.11. *For the clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ the following equalities hold:*

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A \sin\theta_0 s^3 + O(s^4), \tag{22}$$

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4). \tag{23}$$

See the proof of the lemma in Appendix B.

Corollary 3.12. *Denote by C_c the point of the clothoid $cl1$ with the same curvature as the point C belonging to clothoid $cl2$. Denote by C_ρ the point of the clothoid $cl1$ with the same length of the radius-vector $\vec{\rho}_C$ as the point C of the* *clothoid cl2; and denote by C_γ the point of the clothoid cl1 with the same angle γ between the radius-vector and the tangent vector as the point C of cl2. Denote by γ_A, γ_B, γ_C the angles γ at the points A, B, C. Then the points C_c, A, C_γ, C_ρ, B on a small interval $[0, s]$ are encountered in their order along cl1.*

This corollary is proved in Appendix B.

3.3 A property of a concatenation of several arcs of clothoids

Consider two paths with the same initial conditions $(x_0, y_0, \alpha_0, u_0)$ and whose graphs of the curvature as a function of the path length are shown on Figure 8.

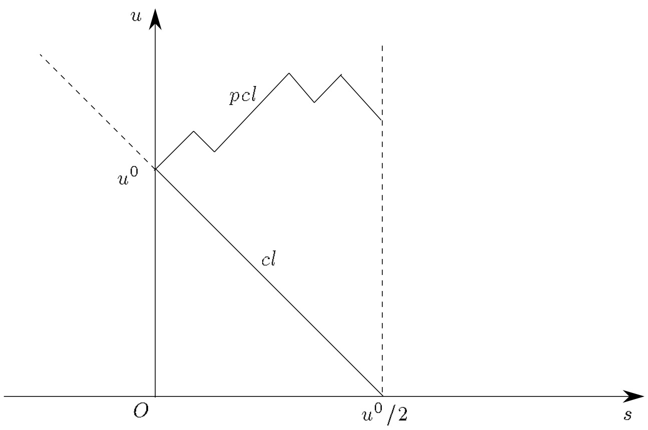

Figure 8. The graphs of the curvature of a piece of a half-clothoid (*cl*) and of a concatenation of several arcs of clothoids (*pcl*) with equal initial curvatures.

The path *cl* is a piece of a half-clothoid whose curvature is defined by the equation $u = -2s + u_0$ $(u_0 > 0)$. The path *pcl* consists of several pieces of clothoids whose curvatures are defined by equations of the kind $u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$ ($\tilde{u}^0 > 0$ and $\tilde{\tilde{u}}^0 > 0$), the sum of their lengths is equal to $u_0/2$. Denote by O_{cl} the centre of *cl*, by $\vec{\rho}_{cl}(t)$ the radius-vector of a point of *cl* in the coordinate system with centre at O_{cl}. Denote by $\vec{\rho}_{pcl}(t)$ the radius-vector of a point of the path *pcl* in this coordinate system. For $t = 0$ we have $\vec{\rho}_{cl}(0) = \vec{\rho}_{pcl}(0)$.

Lemma 3.13. *For any path pcl (defined as above) and for the path cl (both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_{pcl}(s), \textit{for every } s \in (0, u_0/2]. \tag{24}$$

See the proof of the lemma in Appendix C.

Denote by $\mathcal{D}$ the class of the paths with initial conditions $(x_0, y_0, \alpha_0, u_0)$, of length $u_0/2$ and whose graphs of the curvature u as a function of the path length s belong to the class Lip (2). Denote by $\vec{\rho}_L(t)$ the radius-vector of the point of some path L from the class $\mathcal{D}$ in the coordinate system with centre at O_{cl}. Then we have

Corollary 3.14. *For any path L from the class $\mathcal{D}$ and for the path cl from Lemma* 3.13 *(both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_L(s), \quad \textit{for every } s \in (0, u_0/2]$$

Really, the class of paths L belongs to the closure of the class of all paths *pcl* defined at the beginning of the subsection.

4. Construction of a Suboptimal Path

We construct a suboptimal path when dist $((x^0, y^0), (x^T, y^T)) \gg 1/\sqrt{B}$ (i.e. there exist constants $a > 1$, $c \geq 0$ such that the following inequality holds:

$$\text{dist}\,((x^0, y^0), (x^T, y^T)) \geq a/\sqrt{B} + c).$$

In the section we consider arbitrary B in Propositions 4.1 and 4.2 and we set $B = 2$ throughout the rest of the section.

We show that one can construct a path from the initial point X^0 with coordinates (x^0, y^0) to the final point X^T with coordinates (x^T, y^T) with four switching points which is a concatenation of four arcs of clothoids and a line segment (along the path the tangent angle and the curvature are continuous, their initial and final values are respectively α^0, α^T and u^0, u^T).

Construct the path from X^0 to X^T by means of the graph of the curvature as a function of the path length. Construct at first a part of the path which is a concatenation of two arcs of clothoids only, from the point X^0 to some point X'_D. For this purpose consider the graph of the curvature as a function of the path length, which is a piecewise linear and continuous function (the absolute values of the angular coefficients of these pieces are the same, i.e. every piece is of the kind $u = \pm 2t + u_{**}$).

This graph is shown on Figure 9. It is linear on $[0, \xi']$ and on $[\xi', \eta' + 2\xi']$, zero at the point $(\eta' + 2\xi')$. Here ξ' and η' are the path lengths, the number η' is defined by u^0 $(\eta' = u^0/2)$, ξ' can be considered as a parameter.

Construct the path corresponding to this graph from X^0 to some point X'_D (the point X'_D of the path corresponds to the point D of the graph of the curvature).

Increasing ξ' monotonously leads to increasing of the absolute value of the tangent angle α at the point X'_D (denote it by α'), because the curvature doesn't

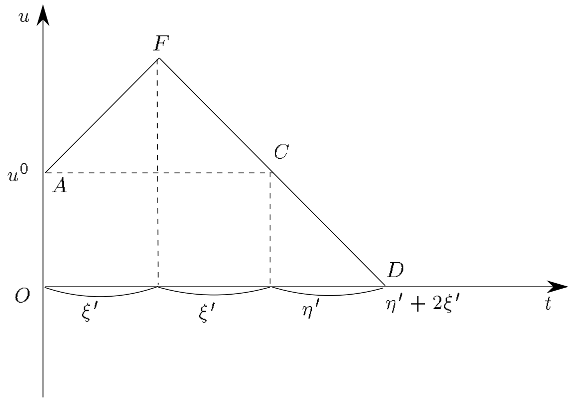

Figure 9. The graph of the curvature of the suboptimal path from 0 to $\eta' + 2\xi'$.

change sign on $[0, \eta' + 2\xi']$ and the angle $\alpha' - \alpha^0$ is the integral of the curvature on this interval:

$$\alpha' - \alpha^0 = \int_0^{\eta' + 2\xi'} u(t)\, dt.$$

Hence, there exist $d' > 0$ such that if ξ' varies in $[0, d']$, then the tangent angle α' at the point X'_D assumes continuously all the values from some interval of the kind $[\kappa_0, \kappa_0 + 2\pi]$ or $[\kappa_0, \kappa_0 - 2\pi]$, $\kappa_0 \in \mathbf{R}$, depending on the sign of u^0.

In conformity with Proposition 3.2 we can take for d' the maximal length of an arc of half-clothoid on which the tangent angle makes a full turn (i.e. 2π).

Estimate the area where the point X'_D can be if $\xi' \in [0, d']$.

Proposition 4.1. *If $\xi' \in [0, d']$ the point X'_D will be within some disc E'. For x^0, α^0 fixed the coordinates of its centre (which we assume to be the point X'_D for $\xi' = 0$) depend only on u^0; its radius doesn't depend on any of the constants x^0, α^0, u^0 and when B is not fixed then the radius and the coordinates of the centre depend only on the parameter B.*

Proof. Consider the circle with centre at the centre of the half-clothoid whose curvature is defined by the part AF of the graph of u as a function of t (see Figure 9). Denote this clothoid by cl_{in}. We take the radius of this circle to equal r_B (see (10)). Denote the point of cl_{in} corresponding to the point F of the graph shown on Figure 9 by X'_F. If we change $\xi' \in [0, d']$, then the point X'_F will remain within this circle. The point X'_D will be within the circle with centre at the point X'_F and with radius r_B. Thus, the point X'_D will be within the circle E' with centre at the centre of cl_{in} and with radius $2r_B$.

The proposition is proved. $\square$

We can use the same method for constructing the path from X^T to some point X''_D (from the right to the left). For this path we have a parameter ξ'', the interval $[0, d']$ and the disc E'' respectively.

Remind that we consider the case when $\mathrm{dist}\,((x^0, y^0), (x^T, y^T)) \gg 1/\sqrt{B}$. That is why $E' \cap E'' = \emptyset$.

In order to construct the path from X^0 to X^T vary ξ' and ξ'' so that the tangent lines at the points X'_D and X''_D should be parallel (i.e. ξ'' is a function of ξ'). For $\alpha' = \pi/2$, $\alpha'' = -\pi/2$ and for $\alpha' = -\pi/2$, $\alpha'' = \pi/2$ the angles between the tangent vector to the path at X'_D and the vector $\overrightarrow{X'_D X''_D}$ have different signs. Hence, thus varying ξ' and ξ'' in the interval $[0, d']$, we obtain that for some values ξ', ξ'' this angle equals 0. So, we obtain the desired path from X^0 to X^T. The thus constructed path satisfies all the initial requirements.

Proposition 4.2. *There are the following inequalities between the radius r_B and the parameters ξ', ξ'':*

$$\xi' \leq 2\sqrt{2} r_B,$$
$$\xi'' \leq 2\sqrt{2} r_B.$$

Proof. Remember that (from (10))

$$r_B = \sqrt{\pi/(2B)}\ ,$$

and $\xi' \in [0, d']$ where d' is the maximal length of an arc of a half-clothoid on which the tangent angle to the half-clothoid makes a full turn. To compute d' let the point P_1 coincide with the point O and let α_* be equal to zero (see Figure 2). Then

$$d' = \widehat{P_1 P_3} = \int_0^{\sqrt{4\pi/B}} \sqrt{\cos^2\left(\frac{Bt^2}{2}\right) + \sin^2\left(\frac{Bt^2}{2}\right)} dt = \sqrt{\frac{4\pi}{B}} = 2\sqrt{\frac{\pi}{B}}\ .$$

Hence

$$\xi' \leq 2\sqrt{\frac{\pi}{B}} = 2\sqrt{2} r_B.$$

Similarly, $\xi'' \leq 2\sqrt{2} r_B$.

The proposition is proved. □

Remark 4.3. The initial and final values of the curvature may be positive or negative. That is why the path constructed from X^0 to X^T may be of one of the forms shown on Figures 10a)–d). Figure 10 a) corresponds to $u^0 > 0$, $u^T < 0$; Figure 10b) — to $u^0 > 0$, $u^T > 0$; Figure 10c) — to $u^0 < 0$, $u^T > 0$ and Figure 10d) — to $u^0 < 0$, $u^T < 0$. The points X'_D, X''_D are the points of zero curvature.

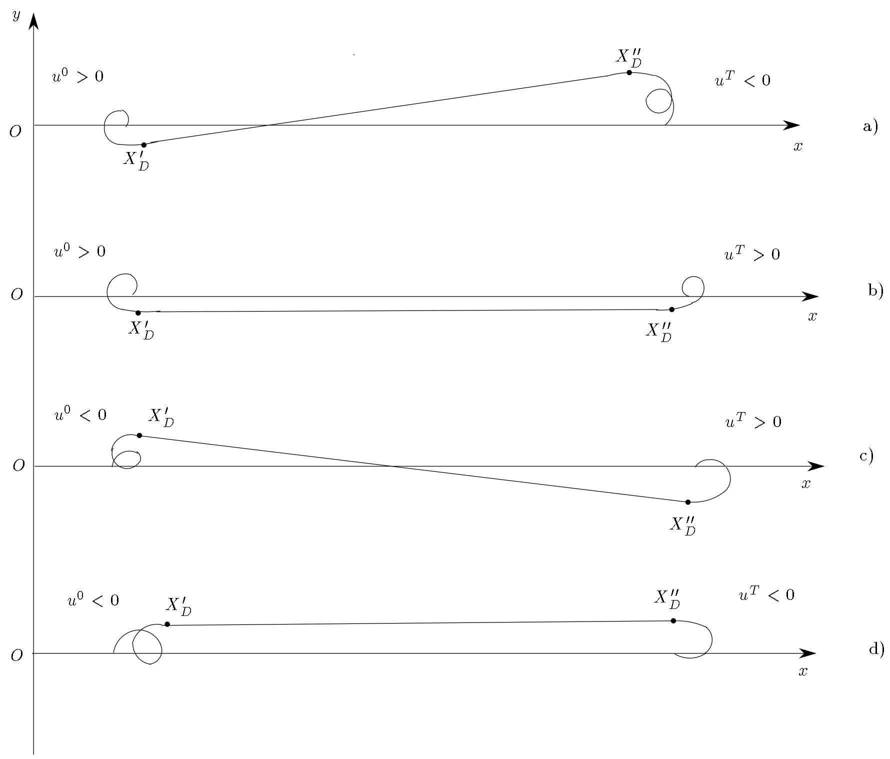

Figure 10. The four possible types of suboptimal paths.

It is practically impossible to feel the presence of a switching point between two clothoids on the path (Figures 10 a)–d)), because the first and the second derivatives are continuous there. On Figure 11 we show such a switching point — the path MKL contains an arc (MK) of the clothoid $\mathcal{C}_1$ and an arc (KL) of the clothoid $\mathcal{C}_2$.

Remark 4.4. Consider a path beginning at X^0 whose graph of the curvature as a function of the path length has the form shown on Figure 12 ($\zeta > 0$ is a parameter). Such a path will be longer than the path with $\zeta = 0$ (if the tangent angles at X'_D are equal for both paths, the initial angles and curvatures — too).

Really, the surfaces under both graphs of the curvature must be equal (because the tangent angle is the integral of the curvature). Hence, ξ is minimal when u_* is maximal, i.e. $\zeta = 0$. This observation makes us choose $\zeta = 0$ for the construction of the suboptimal path.

The condition $\operatorname{dist}((x^0, y^0), (x^T, y^T)) \gg 1/\sqrt{B}$ implies that the line segment between the points X'_D and X''_D is almost horizontal. Hence, if we change ζ the change of the length Δ_l of this segment is approximately equal to the change of

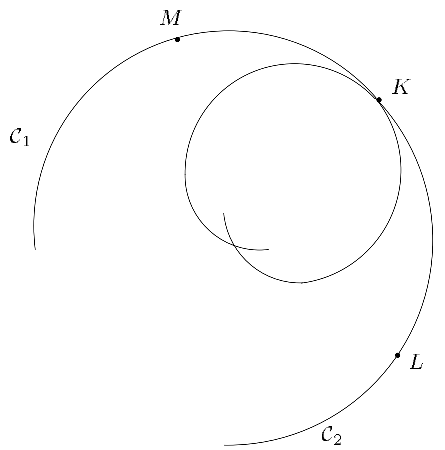

Figure 11. A switching point between two arcs of clothoids (represented together with their analytic continuations).

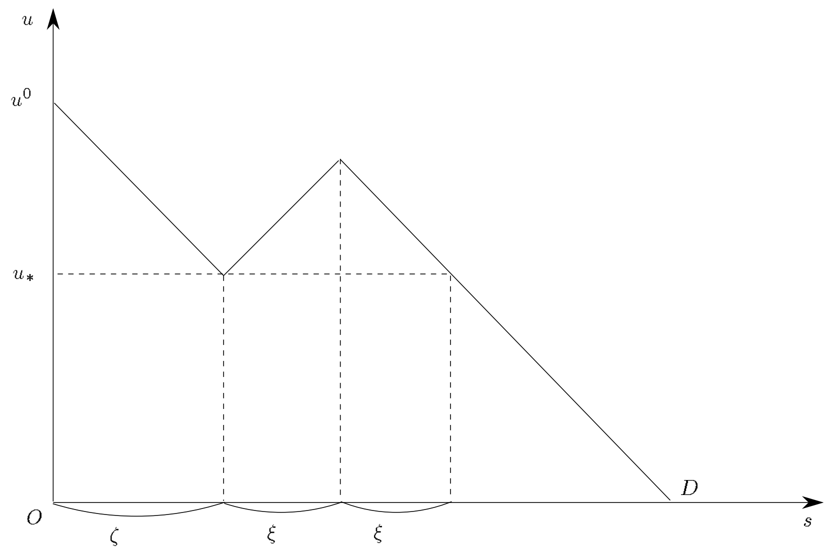

Figure 12. The graph of the curvature of the path from Remark 4.4.

the length of its projection Δ_{l_x} on Ox. Denote by Δ_s the change of the total length of the four arcs of clothoid. Denote by Δ_{s_x} the change of the total length of their projections on Ox. Then we have

$$\Delta_s \geq \Delta_{s_x} = -\Delta_{l_x} \approx \Delta_l .$$

Therefore one expects to have, in general, shorter paths for smaller values of ζ, because, in general, the left inequality should be strict.

Two students — A. Casta and Ph. Cohen — constructed suboptimal paths explicitly by means of MAPLE.

To construct a suboptimal path one has to express the coordinates and the tangent angles at the points X'_D and X''_D as functions, respectively, of ξ' and ξ''. One imposes the condition the tangent angles at X'_D and X''_D to be equal; this allows to express ξ'' by ξ'. After this one expresses the distance between the tangent lines at the points X'_D and X''_D as a function of ξ'. The necessary value of ξ' is a zero of this function. This zero can be found by means of MAPLE. A better result is obtained when the method of dichotomy is used, not Newton's one (it is not clear whether the latter is applicable or not).

5. Proof of the Suboptimality of the Path Constructed in §4

In the section we consider arbitrary B.

Theorem 5.1. *The optimal path for problem* (1)–(3) *is shorter than the suboptimal path constructed in* §4 *by no more than* $(10\sqrt{2}+10)r_B$ *(here r_B denotes the distance between the centre of the half-clothoid* (8) *and its point of zero curvature).*

Proof.

$\mathbf{1^0}$. Consider the suboptimal path as consisting of five pieces: the first piece is from the initial point X^0 to the point X'_C corresponding to the point C on the graph of the curvature u as a function of s (see Figure 9); the second piece is from the point X'_C to the point X'_D (remember that the point X'_D of the path corresponds to the point D on the graph of the curvature u); the third piece is a line segment between the points X'_D and X''_D; the forth and the fifth pieces are defined in the same way as the second and the first pieces respectively (the point X''_C corresponds to the point X'_C).

Consider the initial point X^0 with the initial values of the tangent angle and the curvature α^0 and u^0 as belonging to the unwinding half-clothoid. Then we can correctly define the centre of this half-clothoid, denoted by O_{X^0}. For the final point X^T with α^T, u^T we can define the unwinding half-clothoid with centre at the point O_{X^T} respectively.

Then we can consider the optimal path as consisting of three pieces: the first piece is the piece within the circle D_{X^0} with centre at the point O_{X^0} and with radius r_B (more precisely, the piece ends with the first point P which is out of the circle D_{X^0}; if the optimal path leaves D_{X^0} and then enters it again, its part after the point P belongs to the second piece). The third piece is the piece within the circle D_{X^T} with centre at the point O_{X^T} and with radius r_B (more precisely, from the last point belonging to D_{X^T} to the point X^T). The second piece is what is left between the first and the third one.

2^0. Remember that we use the folowing notations: we denoted by X'_F the point of the suboptimal path corresponding to the point F on the graph u as a function of t (see Figure 9), by X'_C — the point corresponding to the point C, by X'_D — the point corresponding to the point D and by X''_F, X''_C, X''_D we denoted the points belonging to the corresponding part of the path from the final point.

The point X'_C with $\tilde{\alpha}$, u_0 belongs to the unwinding half-clothoid whose centre is correctly defined. Denote it by $O_{X'_C}$. Denote by $D_{X'_C}$ the circle with centre at the point $O_{X'_C}$ and with radius r_B.

For the point X'_C we define similarly the point $O_{X''_C}$ and the circle $D_{X''_C}$.

3^0. Plan of the proof of the suboptimality of the path constructed in §4 (the suboptimal path).

We compare the length of the optimal path and the one constructed in §**4**. We can estimate the maximal possible difference of their lengths (denote it by σ). For this purpose we prove that the second (the forth) piece of the suboptimal path is no longer than the first (the third) piece of the optimal one (see **4^0**).

Then we estimate the maximal possible length of the pieces $\widehat{X^0 X'_C}$ and $\widehat{X^T X''_C}$ of the suboptimal path (see **5^0**). Their lengths are, respectively, $2\xi'$ and $2\xi''$.

In **6^0** we estimate the maximal possible difference between the distance between the circles defining the second and the forth pieces of the suboptimal one and the distance between the circles defining the first and the third pieces of the optimal one.

And then in **7^0** we estimate the difference between the shortest and the longest possible length of the line segment of the suboptimal path.

We summarise these results and obtain σ in **8^0**.

4^0. The first and the third pieces of the optimal path belong to the class $\mathcal{D}$ (see the definition in Subsection 3.3). Hence, we obtain from Corollary 3.14 that the second (the forth) piece of the suboptimal path is no longer than the first (the third) piece of the optimal one.

5^0. We obtain from Proposition 4.2 that $\xi' \leq 2\sqrt{2}r_B$ and $\xi'' \leq 2\sqrt{2}r_B$. Hence, adding the pieces $\widehat{X^0 X'_C}$ and $\widehat{X^T X''_C}$ we add no more than $2\xi' + 2\xi'' \leq 8\sqrt{2}r_B$ to the length of the suboptimal path.

6^0. The maximal possible distance between the points X^0 and X'_C is equal to $3r_B$, because $\widehat{X^0 X'_F}$ and $\widehat{X'_F X'_C}$ are two arcs of two half-clothoids, hence, the distance between the points X^0 and X'_F (the points X'_F and X'_C respectively) is smaller than $3/2r_B$ (see Proposition 3.10). Similarly for the point X^T and X''_C. Hence, the maximal possible distance between the points O_{X^0} and $O_{X'_C}$ is equal to $3r_B + r_B + r_B = 5r_B$ (see Figure 13). In the same way the maximal possible distance between the points O_{X^T} and $O_{X''_C}$ is equal to $5r_B$.

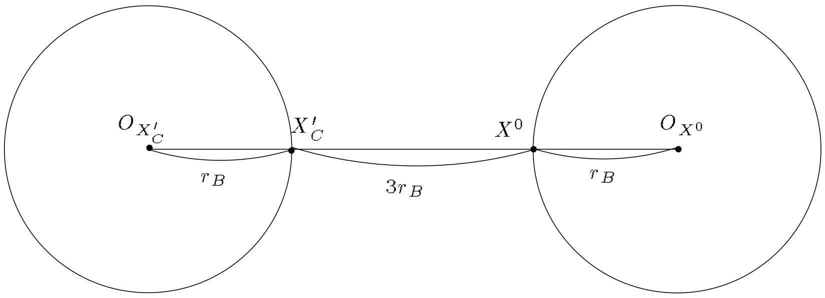

Figure 13. The circles of radius r_B with centres at the points O_{X^0} and $O_{X'_C}$.

Thus the distance between the circles defining the second and the forth pieces of the suboptimal path is no greater than the distance between the circles defining the first and the third pieces of the optimal one by no more than $5r_B+5r_B = 10r_B$.

7⁰. Estimate the difference between the shortest and the longest possible length of the line segment of the suboptimal path. Denote by RQ the line segment of the shortest possible length and by EW the one of the longest possible length (see Figure 14). Denote by G the point belonging to the border of the circle $D_{X'_C}$ and the segment $O_{X'_C}G$ is perpendicular to the line $O_{X'_C}O_{X''_C}$. For the circle $D_{X''_C}$ we have the point V respectively.

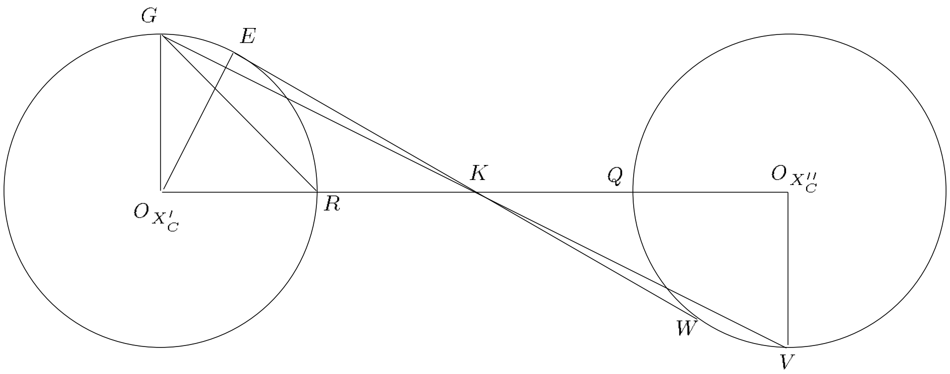

Figure 14. The circles of radius r_B with centres at the points $O_{X'_C}$ and $O_{X''_C}$.

Compute the angle $O_{X'_C}EK$. It is equal to the angle between the vectors $\overrightarrow{OO_c}$ and $\vec{\tau}$ (see Figure 1). The vector $\overrightarrow{OO_c}$ is the radius-vector of the centre O_c of the half-clothoid (11), the vector $\vec{\tau}$ is the tangent vector to this half-clothoid at the point O. The line l is perpendicular to the vector $\overrightarrow{OO_c}$ and the angle β is the angle between the line l and the tangent vector $\vec{\tau}$. From (9) we obtain that $x_{O_c} = y_{O_c}$, hence, the angle between the axis Ox and the vector $\overrightarrow{OO_c}$ is equal to $\pi/4$ and the angle β is equal to $\pi/4$, too. Thus the angle $O_{X'_C}EK$ is equal to $\frac{3}{4}\pi$.

Hence

$$|KE| < |KG|.$$

But $|GR| = \sqrt{2}r_B$ (because $|O_{X'_C}G| = |O_{X'_C}R| = r_B$ and $O_{X'_C}G \perp O_{X'_C}R$). Hence,

$$|KE| < |KG| < |GR| + |RK| = \sqrt{2}r_B + |RK|.$$

Analogously for the segment $|KW|$ we have the following inequality:

$$|KW| < \sqrt{2}r_B + |KQ|.$$

Thus

$$|EW| < |RQ| + 2\sqrt{2}r_B,$$

i.e. we obtain that the least possible length of the line segment of the suboptimal path is shorter than the greatest possible length by no more than $2\sqrt{2}r_B$.

$\mathbf{8^0}$. Summarising the results obtained in $\mathbf{4^0}--\mathbf{7^0}$, we can estimate the maximal possible difference of the lengths of the suboptimal and the optimal paths:

$$\sigma = 8\sqrt{2}r_B + 10r_B + 2\sqrt{2}r_B = (10\sqrt{2} + 10)r_B.$$

The theorem is proved. □

A Appendix: Proof of Lemma 3.4

An arbitrary point A of the clothoid (11) has a tangent vector $\vec{\tau}(t)$ with coordinates $(\cos t^2, \sin t^2)$ (see Figure 15).

Consider a point D of the clothoid (11) with tangent vector $\vec{\tau}_n = (0, 1)$. The point A is mapped onto the point D by means of the rotation on angle θ defined by the rotation matrix

$$\begin{pmatrix} \sin t^2 & -\cos t^2 \\ \cos t^2 & \sin t^2 \end{pmatrix}.$$

Hence, the radius-vector $\vec{\rho} = (-\int_t^\infty \cos\tau^2\, d\tau, -\int_t^\infty \sin\tau^2\, d\tau)$ (see (12)) is mapped into the radius-vector

$$\begin{aligned} \vec{\rho}_n = \Big(&-\sin t^2 \int_t^\infty \cos\tau^2\, d\tau + \cos t^2 \int_t^\infty \sin\tau^2\, d\tau, \\ &-\cos t^2 \int_t^\infty \cos\tau^2\, d\tau - \sin t^2 \int_t^\infty \sin\tau^2\, d\tau\Big) \\ = &\left(\int_t^\infty \sin(\tau^2 - t^2)\, d\tau, -\int_t^\infty \cos(\tau^2 - t^2)\, d\tau\right) \\ = &\left(\int_0^\infty \frac{\sin\nu\, d\nu}{2\sqrt{\nu + t^2}}, -\int_0^\infty \frac{\cos\nu\, d\nu}{2\sqrt{\nu + t^2}}\right). \end{aligned}$$

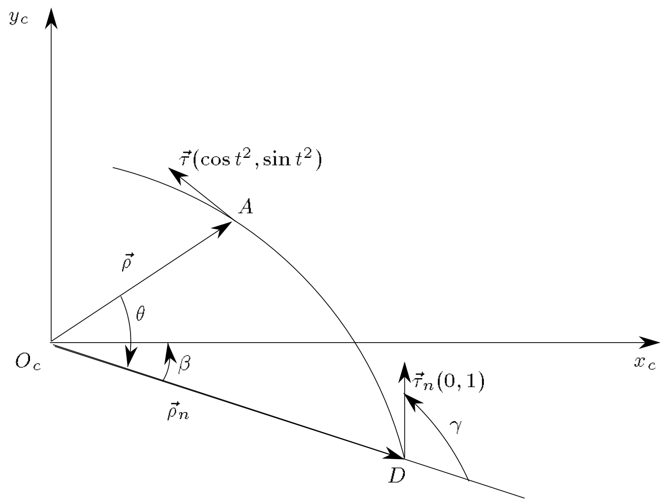

Figure 15. A piece of a clothoid with denoted raduis-vector $\vec{\rho}$ and angles $\gamma(t)$, $\beta(t)$.

We want to investigate the function $d\rho/dt$. Instead of it we can investigate the function $d\gamma/dt$ (see (15)). Denote by β the angle between the vector $\vec{\rho}_n$ and the axis $O_c x_c$. At the point D we have the following relations between the angles γ, β and the coordinates x_n, y_n of the vector $\vec{\rho}_n$:

$$\cot\gamma = -\tan\beta = -\frac{y_n}{x_n} = \int_0^\infty \frac{\cos\nu\, d\nu}{2\sqrt{\nu + t^2}} \Big/ \int_0^\infty \frac{\sin\nu\, d\nu}{2\sqrt{\nu + t^2}}\ .$$

Compute the derivative $d(\tan\beta)/dt$:

$$\begin{aligned}
\frac{d(\tan\beta)}{dt} &= -\frac{t}{4{x_n}^2}\left[\int_0^\infty \frac{\cos\tau d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau+t^2}}\right. \\
&\qquad \left. - \int_0^\infty \frac{\sin\tau d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau+t^2}}\right] \\
&= -\frac{t}{4{x_n}^2}\left[\left\{\frac{3}{2}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^5} - \frac{\sin\tau}{\left(\sqrt{\tau+t^2}\right)^3}\bigg|_0^\infty\right\}\int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau+t^2}}\right. \\
&\qquad \left. - \left\{\frac{1}{2}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3} - \frac{\sin\tau}{\sqrt{\tau+t^2}}\bigg|_0^\infty\right\}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\right] \\
&= -\frac{t}{8{x_n}^2}\left[3\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^5}\int_0^\infty \frac{\sin\tau\, d\tau}{\sqrt{\tau+t^2}} - \left(\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\right)^2\right]
\end{aligned}$$

(we use integration by parts).

Denote the expression in the brackets as $J(t^2)$. Consider $J(t^2)$ with ∞ changed to $2\pi p$ ($p \in \mathbf{N}$, $p > 1$). Consider the corresponding Riemann sums with step $\Delta = \pi/n$ instead of the integrals:

$$(25)\quad \int_0^{2\pi p} \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^i} \cong \sum_{k=1}^{2np} \frac{\sin\tau_k}{\left(\sqrt{\tau_k+t^2}\right)^i}\Delta + O(\Delta), \quad \tau_k = \pi k/n, \quad i = \{1,3,5\}$$

The function $\sin\tau$ is periodic with period 2π and $\sin(\pi+\tau) = -\sin\tau$.

Denote the three Riemann sums (corresponding to the three integrals) by

$$d_1 + \cdots + d_{np}, \quad g_1 + \cdots + g_{np}, \quad h_1 + \cdots + h_{np},$$

where if $j = s + \nu n$, $s = 1, \ldots, n$, $\nu = 0, \ldots, p-1$, then

$$\begin{aligned} d_j &= \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + t^2}} - \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + \pi + t^2}}\ , \\ g_j &= \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + t^2}\right)^3} - \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + \pi + t^2}\right)^3}\ , \\ h_j &= \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + t^2}\right)^5} - \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + \pi + t^2}\right)^5}\ . \end{aligned}$$

Show that

$$(26)\qquad I \equiv 3d_jh_j - {g_j}^2 \geq 0\,.$$

Set $\tau_s + 2\nu\pi = a$. Then rewrite I as follows:

$$3\left(\frac{\sin a}{\left(\sqrt{a+t^2}\right)^5} - \frac{\sin a}{\left(\sqrt{a+\pi+t^2}\right)^5}\right)\left(\frac{\sin a}{\sqrt{a+t^2}} - \frac{\sin a}{\sqrt{a+\pi+t^2}}\right) - \left(\frac{\sin a}{\left(\sqrt{a+t^2}\right)^3} - \frac{\sin a}{\left(\sqrt{a+\pi+t^2}\right)^3}\right)^2.$$

Denote $\sqrt{a+t^2}$ by α, $\sqrt{a+\pi+t^2}$ by β. Then

$$\begin{aligned} I &= 3\left(\frac{1}{\alpha^5} - \frac{1}{\beta^5}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) - \left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right)^2 = 3\frac{(\beta^5-\alpha^5)(\beta-\alpha)}{\alpha^6\beta^6} - \frac{\left(\beta^3-\alpha^3\right)^2}{\alpha^6\beta^6} \\ &= \frac{3(\beta-\alpha)^2(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4) - (\beta-\alpha)^2\left(\beta^2+\beta\alpha+\alpha^2\right)^2}{\alpha^6\beta^6} \\ &= \frac{\left(\beta^2-\alpha^2\right)^2\left[3(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4) - \left(\beta^2+\beta\alpha+\alpha^2\right)^2\right]}{\alpha^6\beta^6(\beta+\alpha)^2} \\ &= \frac{\pi^2(2\beta^4+2\alpha^4+\beta^3\alpha+\beta\alpha^3)}{\alpha^6\beta^6(\beta+\alpha)^2} > 0\,. \end{aligned}$$

Thus we prove (26). Show that

$$K \equiv 3(d_i h_j + d_j h_i) - 2g_i g_j \geq 0\,. \tag{27}$$

Set

$$\begin{aligned} \tau_s + 2\nu\pi &= a_i, & \tau_w + 2\nu\pi &= a_j,\\ \sqrt{a_i + t^2} &= \alpha, & \sqrt{a_i + \pi + t^2} &= \beta,\\ \sqrt{a_j + t^2} &= \gamma, & \sqrt{a_j + \pi + t^2} &= \delta. \end{aligned}$$

Rewrite K as follows:

$$\begin{aligned} K &= 3\left[\left(\frac{\sin a_i}{\alpha^5} - \frac{\sin a_i}{\beta^5}\right)\left(\frac{\sin a_j}{\gamma} - \frac{\sin a_j}{\delta}\right)\right.\\ &\left.+\left(\frac{\sin a_j}{\gamma^5} - \frac{\sin a_j}{\delta^5}\right)\left(\frac{\sin a_i}{\alpha} - \frac{\sin a_i}{\beta}\right)\right] - 2\left(\frac{\sin a_i}{\alpha^3} - \frac{\sin a_i}{\beta^3}\right)\left(\frac{\sin a_j}{\gamma^3} - \frac{\sin a_j}{\delta^3}\right)\\ &= \frac{\pi^2 \sin a_i \sin a_j}{(\beta+\alpha)(\gamma+\delta)\alpha\beta\gamma\delta}\\ &\qquad \times\left[\frac{3(\beta^4 + \beta^3\alpha + \beta^2\alpha^2 + \beta\alpha^3 + \alpha^4)}{\alpha^4\beta^4} + \frac{3(\delta^4 + \delta^3\gamma + \delta^2\gamma^2 + \delta\gamma^3 + \gamma^4)}{\gamma^4\delta^4}\right.\\ &\qquad \left. - \frac{2(\beta^2\delta^2 + \beta^2\delta\gamma + \beta^2\gamma^2 + \delta^2\beta\alpha + \beta\alpha\delta\gamma + \gamma^2\beta\alpha + \alpha^2\delta^2 + \alpha^2\delta\gamma + \alpha^2\gamma^2)}{\alpha^2\beta^2\gamma^2\delta^2}\right]. \end{aligned} \tag{28}$$

Estimate the expression in the brackets (denote it by L).

$$L = L_1 + L_2 + L_3 + L_4, \tag{29}$$

where

$$\begin{aligned} L_1 &= \left\{\frac{\alpha^4 + 2\alpha^2\beta^2 + \beta^4}{\alpha^4\beta^4} + \frac{\gamma^4 + 2\gamma^2\delta^2 + \delta^4}{\gamma^4\delta^4} - \frac{2(\beta^2\delta^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2)}{\alpha^2\beta^2\gamma^2\delta^2}\right\}\\ &= \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)^2 + \left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2 - 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right) > 0 \end{aligned}$$

(due to the inequality between the mean arithmetic and the mean geometric),

$$L_2 = \left\{\frac{1}{2}\left(\frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3}\right) - \frac{2}{\alpha\beta\gamma\delta}\right\} > 0$$

(for the same reason),

$$L_4 = \left\{\frac{1}{\alpha^2\beta^2} + \frac{1}{2}\left(\frac{1}{\alpha^3\beta} + \frac{1}{\alpha\beta^3} + \frac{1}{\gamma^3\delta} + \frac{1}{\gamma\delta^3}\right) + \frac{1}{\gamma^2\delta^2}\right\} > 0$$

(because $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$) and

$$L_3 = \left\{ 2\left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right) - M \right\},$$
$$M = \frac{2(\beta^2\delta\gamma + \delta^2\beta\alpha + \gamma^2\beta\alpha + \alpha^2\delta\gamma)}{\alpha^2\beta^2\gamma^2\delta^2}.$$

One has

$$M = \frac{2}{\gamma\delta}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) + \frac{2}{\alpha\beta}\left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right) < 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)$$
$$< \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)^2 + \left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2 < 2\left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right)$$

(due to the inequality between the mean arithmetic and the mean geometric). Hence, $L_3 > 0$.

So L (see (29)) is positive and then the expression K (see (28)) is positive because the points a_i, a_j belong to the interval $(0, \pi]$ and, hence, the functions $\sin a_i$, $\sin a_j$ are non-negative. So we prove (27).

From (26) and (27) when $n \to \infty$ it follows that

$$3\int_0^{2\pi p} \frac{\sin\tau\, d\tau}{(\sqrt{\tau + t^2})^5} \int_0^{2\pi p} \frac{\sin\tau\, d\tau}{\sqrt{\tau + t^2}} - \left(\int_0^{2\pi p} \frac{\sin\tau\, d\tau}{(\sqrt{\tau + t^2})^3}\right)^2 > 0.$$

If $2\pi p \to \infty$ and $n \to \infty$ we obtain that $J(t^2) > 0$ and, hence, $d(\tan\beta)/dt < 0$. Remember that $\tan\beta = -\cot\gamma$ and $\ddot\rho = -\dot\gamma\sin\gamma$ (see (16)). Hence,

$$\frac{d(\cot\gamma)}{dt} = -\frac{\dot\gamma}{\sin^2\gamma} > 0, \qquad \dot\gamma < 0$$

and

$$\ddot\rho > 0.$$

The lemma is proved. □

B Appendix: Proofs of Lemma 3.11 and of Corollary 3.12

B.1 Proof of Lemma 3.11

Consider a coordinate system $A\xi\eta$ (see Figure 16), the axis η coincides with the tangent vector to $cl1$ and $cl2$ at the point A, the axis ξ is a perpendicular to the axis η.

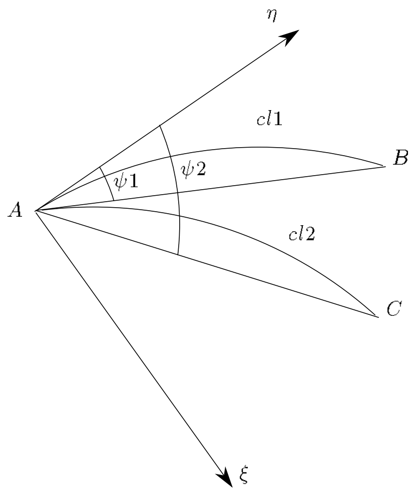

Figure 16. The coordinate system $A\xi\eta$.

In this coordinate system $cl1$ and $cl2$ are defined by the following equations:

$$cl1 : \begin{cases} \xi(t) = \displaystyle\int_0^t \cos\left(\tau^2 + u_0\tau + \pi/2\right) d\tau \\ \eta(t) = \displaystyle\int_0^t \sin\left(\tau^2 + u_0\tau + \pi/2\right) d\tau \end{cases}$$

$$cl2 : \begin{cases} \xi(t) = \displaystyle\int_0^t \cos\left(-\tau^2 + u_0\tau + \pi/2\right) d\tau \\ \eta(t) = \displaystyle\int_0^t \sin\left(-\tau^2 + u_0\tau + \pi/2\right) d\tau \end{cases}$$

So for the coordinates of the points B and C we have the following formulas:

$$\xi_B(s) = -\int_0^s \sin\left(\tau^2 + u_0\tau\right) d\tau, \qquad \eta_B(s) = \int_0^s \cos\left(\tau^2 + u_0\tau\right) d\tau,$$
$$\xi_C(s) = -\int_0^s \sin\left(-\tau^2 + u_0\tau\right) d\tau, \quad \eta_C(s) = \int_0^s \cos\left(-\tau^2 + u_0\tau\right) d\tau\,.$$

Then, using the Taylor series at 0 for the functions $\sin x$, $\cos x$ we obtain:

$$\xi_B(s) = -\frac{u_0}{2}s^2 - \frac{1}{3}s^3 + O(s^4), \qquad \eta_B(s) = s - \frac{u_0^2}{6}s^3 + O(s^4),$$
$$\xi_C(s) = -\frac{u_0}{2}s^2 + \frac{1}{3}s^3 + O(s^4), \qquad \eta_C(s) = s - \frac{u_0^2}{6}s^3 + O(s^4).$$

But $\tan\psi_1 = \xi_B(s)/\eta_B(s)$ and $\tan\psi_2 = \xi_C(s)/\eta_C(s)$. Now we use Taylor series again and obtain the following formulas for $\tan\psi_1$ and $\tan\psi_2$:

$$\tan\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3), \qquad \tan\psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3).$$

Since we consider the clothoids $cl1$ and $cl2$ on a small interval $[0,s]$, we can use for the angles ψ_1 and ψ_2 the following formulas:

$$\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3), \qquad \psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3). \tag{30}$$

Compute the values of ρ_B^2 and ρ_C^2. For this purpose we use the cosine theorem, the Taylor series and formulas (30):

$$\begin{aligned}
\rho_B^2 &= \rho_A^2 + s^2 - 2\rho_A s\cos(\theta_0 - \psi_1)\\
&= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\cos\left(-\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3)\right)\right.\\
&\qquad \left. + \sin\theta_0\sin\left(-\frac{u^0}{2}s - \frac{1}{3}s^2 + 0(s^3)\right)\right]\\
&= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\left(1 - \frac{1}{2}\left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) - \sin\theta_0\left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right]\\
&= \rho_A^2 - 2\rho_A\cos\theta_0 s + (1 + \rho_A u_0\sin\theta_0)s^2\\
&\qquad + \left(\rho_A\frac{u_0^2}{4}\cos\theta_0 + \frac{2}{3}\rho_A\sin\theta_0\right)s^3 + O(s^4),\\
\rho_C^2 &= \rho_A^2 + s^2 - 2\rho_A s\cos(\theta_0 - \psi_2)\\
&= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\cos\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3)\right)\right.\\
&\qquad \left. + \sin\theta_0\sin\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + 0(s^3)\right)\right]\\
&= \rho_A^2 + s^2 - 2\rho_A s\left[\cos\theta_0\left(1 - \frac{1}{2}\left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) + \sin\theta_0\left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right]\\
&= \rho_A^2 - 2\rho_A\cos\theta_0 s + (1 + \rho_A u_0\sin\theta_0)s^2\\
&\qquad + \left(\rho_A\frac{u_0^2}{4}\cos\theta_0 - \frac{2}{3}\rho_A\sin\theta_0\right)s^3 + O(s^4).
\end{aligned}$$

Thus we obtain (22):

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A\sin\theta_0 s^3 + O(s^4).$$

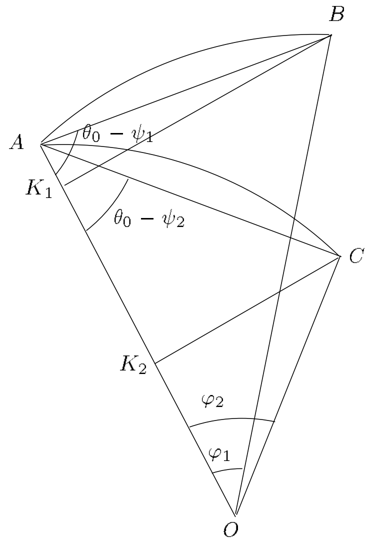

Figure 17. The clothoids $cl1$ and $cl2$ with the additional constructions (line segments BK_1 and CK_2) and with the angles $\theta_0 - \psi_1$, $\theta_0 - \psi_2$, φ_1, φ_2.

Compute the values of the angles δ_1 and δ_2. From (20) and (21) we obtain:

$$\begin{cases} \theta_1 = \theta_0 + u_0 s + s^2 \\ \theta_2 = \theta_0 + u_0 s - s^2 \end{cases} \tag{31}$$

To compute the angles φ_1 and φ_2 make the additional construction (see Figure 17): the segments BK_1 and CK_2 are perpendicular to the line OA.

We have

$$\begin{aligned} |K_1B| &= |AB| \sin(\theta_0 - \psi_1) = |OK_1| \tan \varphi_1 \,, \\ |K_2C| &= |AC| \sin(\theta_0 - \psi_2) = |OK_2| \tan \varphi_2 \,. \end{aligned}$$

Hence

$$\tan \varphi_1 = \frac{|AB|}{|OK_1|} \sin(\theta_0 - \psi_1), \qquad \tan \varphi_2 = \frac{|AC|}{|OK_2|} \sin(\theta_0 - \psi_2)$$

But

$$|AB| = s + O(s^2), \qquad |AC| = s + O(s^2),$$

$$\begin{aligned} |OK_1| &= |OA| - |AK_1| = \rho_A - |AB| \cos(\theta_0 - \psi_1) = \rho_A - s \cos(\theta_0 - \psi_1), \\ |OK_2| &= |OA| - |AK_2| = \rho_A - |AC| \cos(\theta_0 - \psi_2) = \rho_A - s \cos(\theta_0 - \psi_2). \end{aligned}$$

Thus we have that

$$\tan\varphi_1 = \frac{s\sin(\theta_0-\psi_1)}{\rho_A - s\cos(\theta_0-\psi_1)}, \qquad \tan\varphi_2 = \frac{s\sin(\theta_0-\psi_2)}{\rho_A - s\cos(\theta_0-\psi_2)}.$$

Now using (31) and Taylor series for the functions $\cos x$, $\sin x$ and $f(x) = 1/(1+x)$ at 0 we obtain the following expressions:

$$\begin{aligned}
\sin(\theta_0-\psi_1) &= \sin\theta_0\cos\psi_1 - \cos\theta_0\sin\psi_1 \\
&= \sin\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right) + + \cos\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) \\
&= \sin\theta_0 + \frac{u_0}{2}\cos\theta_0 s + \left(\frac{\cos\theta_0}{3} - \frac{u_0^2}{8}\sin\theta_0\right)s^2 + O(s^3), \\
\cos(\theta_0-\psi_1) &= \cos\theta_0\cos\psi_1 + \sin\theta_0\sin\psi_1 \\
&= \cos\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right) - \sin\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) \\
&= \cos\theta_0 - \frac{u_0}{2}\sin\theta_0 s - \left(\frac{\sin\theta_0}{3} + \frac{u_0^2}{8}\cos\theta_0\right)s^2 + O(s^3), \\
\tan\varphi_1 &= \frac{s\sin(\theta_0-\psi_1)}{\rho_A}\frac{1}{1-\frac{s}{\rho_A}\cos(\theta_0-\psi_1)} \\
&= \frac{s}{\rho_A}\sin(\theta_0-\psi_1)\left(1+\frac{s}{\rho_A}\cos(\theta_0-\psi_1)+\frac{s^2}{{\rho_A}^2}\cos^2(\theta_0-\psi_1)\right).
\end{aligned}$$

Hence after this series of transformations we obtain the formula for $\tan\varphi_1$:

$$\begin{aligned}
\tan\varphi_1 = {}& \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 \\
& + \left(\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4).
\end{aligned} \tag{32}$$

After analogous transformations we obtain the formula for $\tan\varphi_2$:

$$\begin{aligned}
\tan\varphi_2 = {}& \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 \\
& + \left(-\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4).
\end{aligned} \tag{33}$$

In a small neighbourhood of the initial point A $\tan\varphi_i = \varphi_i + O(\varphi_i^3)$ $(i = 1, 2)$. Hence, from the definitions of the angles δ_1 and δ_2 and from (32)–(33) we obtain equality (23):

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4).$$

The lemma is proved. □

B.2 Proof of Corollary 3.12

It follows from (21) that the absolute value of the curvature at the point C is greater than the one at the point A. That is why the point C_c is located before the point A.

Note that the angles γ_i and δ_i are connected by the following equations: $\gamma_i = \pi - \delta_i$ $(i = 1, 2)$. Hence from (31) and (32) we obtain that

$$\gamma_A - \gamma_B = \delta_1 - \theta_0 = \left(u_0 + \frac{\sin\theta_0}{\rho_A}\right) s + \left(1 + \frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right) s^2 + O(s^3).$$

From Remark 3.5 we obtain that the angle γ is a monotonously decreasing function, hence

$$\gamma_B < \gamma_A\,.$$

From (23) we have

$$\gamma_C - \gamma_B = \delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A} s^3 + O(s^4)\,.$$

So, $\gamma_B < \gamma_C$ and $\gamma_B < \gamma_A$. But the difference between γ_B and γ_A is of order s, and the difference between γ_B and γ_C is of order s^2. Hence, we obtain the following inequalities

$$\gamma_A > \gamma_C > \gamma_B,$$

and the point C_γ is located between the points A and B.

The difference between ρ_B^2 and ρ_C^2 is of order s^3 (see (22)). The difference between γ_C and γ_B is of order s^2. Hence, the point C_ρ is located between the points C_γ and B.

The corollary is proved. □

C Appendix: Proof of Lemma 3.13

a) Consider the path cl. We parametrise it by the natural parameter s, setting $s = 0$ for the point $(x_0, y_0, \alpha_0, u_0)$. Hence, the graph of the curvature u as a function of the path length s looks like the one shown on Figure 8 (for $s < 0$ it is given by the dotted line). In the proof we consider the path cl only on $[-u_0/2, u_0/2]$.

Denote by O the point of the path cl with zero curvature (i.e. $s = u_0/2$), by A — the point with curvature $2u_0$ (i.e. $s = -u_0/2$), by S — the point with curvature

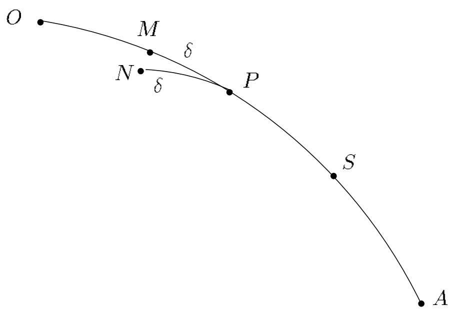

Figure 18. The piece of the path cl from the point with zero curvature to the point with curvature $2u_0$.

u_0 (i.e. $s = 0$) and by P — an arbitrary point corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$ $(u_P(s) \in (0, 2u_0))$, see Figure 18.

Consider a small δ-half-neighbourhood $(s, s+\delta)$ of the point P and consider a path beginning at the point P which is piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length δ and with the same values of x, y, α,u at the point P as the ones of the point P of cl. Denote the final point of this path by N, the final point of the corresponding piece of the clothoid cl by M (the lengths of the arcs $\widehat{PM}$ and $\widehat{PN}$ are equal to δ, the curvature of cl is decreasing from P to M). Denote by N_c the point of the clothoid cl with the same curvature as the point N, by N_ρ — the point of the clothoid cl with the same length of the radius-vector $\vec{\rho}(t)$ as the point N, by N_γ — the point of the clothoid cl with the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ as the point N. Then for every point P there exists a small δ-half-neighbourhood where the points N_c, P, N_γ, N_ρ, M are encountered in this order along cl (see Corollary 3.12). Denote this disposition of the points N_c, P, N_γ, N_ρ, M by **disposition**$(*)$. The number δ can be chosen the same for all values of $s \in [-u_0/2, u_0/2]$; assume that δ is fixed.

b) Consider some path $\mathcal{P}$ of the class $\mathcal{A}$ of all paths beginning at the point P, piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length $\leq \nu(s) = u_0/2 - |s|$ and consisting of n pieces ($n > 1/\delta$, each piece being of length $1/n$ except the first one which is of length $\leq 1/n$).

We prove the lemma for paths $\mathcal{P} \in \mathcal{A}$ first, by induction on n. For paths pcl defined at the beginning of the subsection the lemma will be proved in **c)**.

For the first piece of the path $\mathcal{P}$ we have **disposition**$(*)$ (because the length of this piece is $\leq \delta$ and for the δ-half-neighbourhood of the point P we have this disposition). Suppose that **disposition**$(*)$ doesn't hold at some moment s'. If s' is the very first moment when it happens, then 3 cases can occur:

1) **If at the moment s' the point N_γ coincides with the point N_ρ.** Then at the next moment we shall have **disposition**$(*)$. Really, using the Taylor series, as in Lemma 3.11, we shall obtain the result of Corollary 3.12, because at the moment s' both paths cl and $\mathcal{P}$ have the same value of the radius-vector $\vec{\rho}(t)$ and the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$, and the curvature at the point N_γ of the path $\mathcal{P}$ is greater than the curvature at the point N_γ of the path cl.

2) **If at some moment s' the points N_γ, N_ρ and N_c coincide.** Then this means that we move along a half-clothoid cl but with a delay; hence, we either continue like that and come with a delay, or at some moment we have again **disposition**$(*)$.

3) **If at some moment s' the point N_γ coincides with the point N_c and the point N_ρ is situated after them (see Figure 19).** Then it doesn't happen in the first piece of the path $\mathcal{P}$ (see the definition of δ). Hence, if it happens in the k-th piece of the path $\mathcal{P}$ then for the $(k-1)$-st piece of the path $\mathcal{P}$ **disposition**$(*)$ holds. Prove that in this case

$$\rho_{cl}(s) - \rho_{cl}(s') \geq \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s') \quad \text{for} s \geq s'.$$

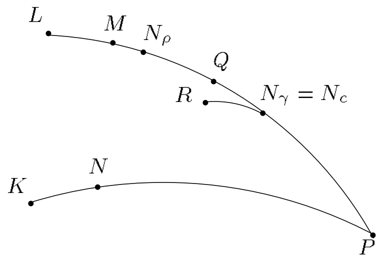

Figure 19. The piece of the path cl from the point P to the point L, the piece of the path $\mathcal{P}$ from the point P to the point K and additionally constructed arc $\widehat{N_\gamma R}$.

We denote by M the point belonging to the path cl and corresponding to the moment s', by N — the point belonging to the path $\mathcal{P}$ and corresponding to the moment s' (see Figure 19). Note that the notation is the same as the one of Figure 18. Denote by $\widehat{ML}$ an arc of the path cl corresponding to the interval $[s', s'+s^*]$ for some $s^* > 0$ and by $\widehat{NK}$ — an arc of the path $\mathcal{P}$ corresponding to the same interval $[s', s'+s^*]$. Denote by s_{pc} $(s_{pc} < s')$ the moment to which the point $N_\gamma = N_c$ corresponds and denote by $\widehat{N_\gamma Q}$ an arc of the path cl corresponding to the interval $[s_{pc}, s_{pc}+s^*]$. Translate the arc $\widehat{NK}$ so that the point N should coincide with the point N_γ, then rotate the image so that the tangent vector to the image at the point N_γ should coincide with the tangent vector to the arc $\widehat{N_\gamma Q}$ at the point N_γ. Denote the obtained arc by $\widehat{N_\gamma R}$.

For the lengths of the radius-vectors $\vec{\rho}(s)$ at the points N_γ, N_ρ and M we have the following inequalities:

$$\rho_{N_\gamma} < \rho_{N_\rho} < \rho_M.$$

This follows from Corollary 3.6 ($\dot{\rho}(s) > 0$).

Rotate the arcs $\widehat{N_\gamma Q}$, $\widehat{N_\gamma R}$ and $\widehat{NK}$ around O_{cl} on different angles so that the points M, N_γ and N should be on the line $O_{cl}M$, see Figure 20a).

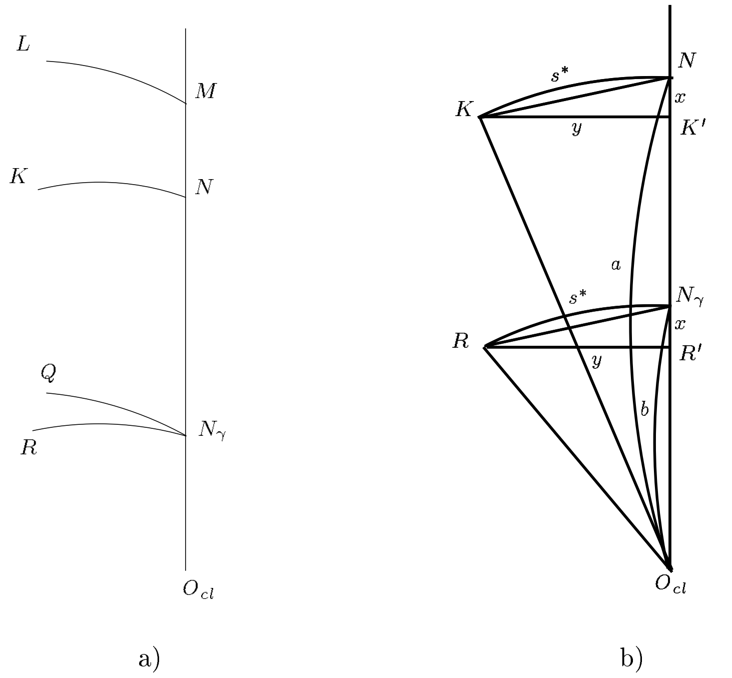

Figure 20. The arcs $\widehat{ML}$, $\widehat{NK}$, $\widehat{N_\gamma Q}$ and $\widehat{N_\gamma R}$ after a rotation around O_{cl} such that the points M, N_γ and N should be on the line $O_{cl}M$.

Denote

$$\Delta\rho_{\mathcal{P}} = |\overrightarrow{O_{cl}K}| - |\overrightarrow{O_{cl}N}|, \qquad \Delta\rho_{\mathcal{P}_{tr}} = |\overrightarrow{O_{cl}R}| - |\overrightarrow{O_{cl}N_\gamma}|,$$
$$\Delta\rho_{clpr} = |\overrightarrow{O_{cl}Q}| - |\overrightarrow{O_{cl}N_\gamma}|, \qquad \Delta\rho_{cl} = |\overrightarrow{O_{cl}L}| - |\overrightarrow{O_{cl}M}|.$$

We know that for the $(k-1)$-st piece of the path $\mathcal{P}$ **disposition**$(*)$ holds. Hence,

$$\Delta\rho_{\mathcal{P}_{tr}} < \Delta\rho_{clpr} \tag{34}$$

(by the inductive assumption, as $k < n$).

Using Corollary 3.6 ($\ddot{\rho} > 0$) we obtain

$$\Delta\rho_{clpr} < \Delta\rho_{cl}\ . \tag{35}$$

Prove that

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{\mathcal{P}_{tr}}\ . \tag{36}$$

The tangent angles at the points N and N_γ are the same, the curvatures — too. Hence (see Figure 20b)),

$$|KK'| = |RR'| = y, \qquad |NK'| = |N_\gamma R'| = x.$$

Denote $|O_{cl}N| = a$, $|O_{cl}N_\gamma| = b$. We have the following equalities:

$$\Delta\rho_{\mathcal{P}} = \sqrt{(a \pm x)^2 + y^2} - a, \qquad \Delta\rho_{\mathcal{P}_{tr}} = \sqrt{(b \pm x)^2 + y^2} - b.$$

Inequality (36) is equivalent to

$$\sqrt{(a \pm x)^2 + y^2} - a < \sqrt{(b \pm x)^2 + y^2} - b, \quad \text{or to}$$
$$(a \pm x)^2 - (b \pm x)^2 < (a - b)\left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right),$$
$$(a + b \pm 2x) < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right).$$

Thus we have

$$a+b\pm 2x = (a\pm x)+(b\pm x) \le |a\pm x|+|b\pm x| < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right).$$

This chain of inequalities is correct, hence, inequality (36) is also correct. Thus, from inequalities (34)—(36) we obtain the desired inequality:

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{cl}\ ,$$

i.e. $\rho_{cl}(s) - \rho_{cl}(s') > \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s')$ for $s > s'$.

Thus we proved that if at some moment s' **disposition**$(*)$ doesn't hold then for the moments $s > s'$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ for the point belonging to *cl* is greater than the length of the radius-vector $\vec{\rho}_{\mathcal{P}}(s)$ for the point belonging to $\mathcal{P}$. This holds for any path of the class $\mathcal{A}$ for any point P corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$.

c) Assume that the point P coincides with the point S (see Figure 18). The curvature of the path *pcl* and the curvature of any path of the class $\mathcal{A}$ are continuous functions. Hence, if $n \to \infty$, then we can uniformly approximate the path *pcl* by a sequence of paths of the class $\mathcal{A}$. Hence, for $s \in [0, u_0/2]$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ is greater than the length of the radius-vector $\vec{\rho}_{pcl}(s)$, i.e. inequality (24) is proved.

The lemma is proved. ☐

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