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NUMERICAL SOLUTION OF NONLINEAR DIFFUSION WITH FINITE EXTINCTION PHENOMENON

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Abstract. The implementation of the numerical method of W. Jäger and J. Kačur for solving the porous-medium type problems with strong absorption is disscussed. The computed numerical results concerning the extinction of the solution in finite time and the interface motions are presented.

1. Introduction

The aim of this paper is to present the numerical computations of the porous-medium type problems with strong absorption by the approximation scheme suggested by W. Jäger and J. Kačur in [**JK1**].

Let u be a function which satisfies the nonlinear partial differential equation

$$u_t = \Delta u^m - cu^p \qquad \text{in} \qquad Q_T := I \times \Omega, \tag{1.1}$$

together with the following boundary and initial conditions

$$u|_{I\times\partial\Omega} = 0, \tag{1.2}$$

$$u(0,x) = u_0(x), \tag{1.3}$$

where $\Omega \subset R^N$ is a bounded domain, $I = (0,T)$, $T < \infty$ is a time interval, and $m \geq 1$, $p > 0$, and $c \geq 0$ are real constants.

In case $m > 1$, the diffusion coefficient mu^{m-1} vanishes at the points where $u \equiv 0$ and the governing parabolic equation degenerates there. The set of such points is called the interface. The equation (1.1) exhibits the finite speed of propagation of initially compact support. If the absorption term cu^p is absent and if $m > 1$ the equation (1.1) is well-known as porous medium equation. In the presence of this reaction term the equation (1.1) desribes the nonlinear fluid-transfer process with an absorption. The influence of both absorption and degenerate diffusion terms makes the dynamics of interfaces complicated and difficult to study from

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both theoretical and computational point of view ([**HV**], [**CMM**], [**Ke**], [**RK**], [**MNT**], [**N**], [**T**], [**TM**]).

Let us consider the problem (1.1)–(1.3) with $m > 1$, $c > 0$ and a nonnegative initial function $u_0(x)$ with the compact support $S(0)$. For one dimensional case, the qualitative behaviour of $S(t)$ is classified into the following three cases depending on the relation bettween m and p (see [**HV**], [**N**]):

(i) for $p \geq m$, $S(t)$ expands as t inscreases and $S(t) \to R^1$ as $t \to \infty$,

(ii) for $1 \leq p < m$, $S(t)$ also expands and there exists a bounded set $B \subset R^1$ satisfying $S(t) \subset B$, for all $t \geq 0$,

(iii) for $0 < p < 1$, $S(t)$ is a compact set in R^1 and there exists a positive number $T^* < \infty$ such that $S(t) \neq \emptyset$ on $[0, T^*)$ and $S(t) = \emptyset$ on (T^*, ∞).

T^* is called an **extinction time** of a solution u and the behaviour, such as (iii), is known as **finite extinction phenomenon**. There are several numerical schemes treating this type of behaviour. Interface tracking scheme given in [**T**], aproximation from [**RK**] and Nakaki's scheme from [**N**] are successfull in the case (iii), but its effectiveness is limited by $N = 1$. Our implementation of the approximation scheme of W. Jäger and J. Kačur yealds the results of comparable degree of accuracy in one dimensional case and we compute effectively the finite extinction phenomenon also for higher dimensional problems.

2. Used Approximation Scheme

For numerical computations presented in part III we have used the approximation scheme suggested in [**JK1**] for solving general slow diffusion systems. This scheme originates in the work [**MNV**] in which the nonlinear Chernoff formula was used to treating the nonlinearity (see also [**BBR**]). The approximation has been later addapted to solve Stefan-like problems ([**KHK**]), fast diffusion and degenerate doubly-nonlinear problems ([**JK2**], [**JK3**]), and the convergence results, concerning our problem, have been improved ([**KHK**]), too.

Let us denote

$$\beta(s) \equiv s^m, \qquad f(s) \equiv s^{\frac{p}{m}},$$

$$\gamma_K(s) = \begin{cases} s, & \text{if } 0 \leq s \leq K, \\ K, & \text{if } s > K, \end{cases} \qquad K \equiv \text{const.}$$

The approximation scheme from [**JK1**] is the following:

Let $n \in \mathbb{N}$, $\tau = \frac{T}{n}$, $t_i = i\tau$ for $i = 0, \ldots, n$, $u_0 = u_0(x)$, $\theta_0 = \beta(u_0)$ and $0 < \alpha < 1$. For $i = 1, \ldots, n$ we look for functions $\mu_i \in L_\infty(\Omega)$ $(0 < \mu_i(x) \leq K)$, θ_i such that

$$\mu_i(\theta_i - \beta(u_{i-1})) - \tau\Delta\theta_i = \tau f(\beta(u_{i-1})) \tag{2.1}$$

satisfying the "convergence condition"

$$|\beta(u_{i-1}+\mu_i(\theta_i-\beta(u_{i-1})))-\beta(u_{i-1})|<\alpha|\theta_i-\beta(u_{i-1})|. \tag{2.2}$$

Then the function u_i is obtained by the algebraic correction

$$u_i := u_{i-1}+\mu_i(\theta_i-\beta(u_{i-1})). \tag{2.3}$$

The scheme in this form is implicit according to a couple θ_i, μ_i satisfying simultaneously (2.1) and (2.2). We seek them iteratively in a finite number of steps. For this purpose, we use the following iterations suggested in [**JK1**]

$$\mu_{i,k-1}(\theta_{i,k}-\beta(u_{i-1}))-\tau\Delta\theta_{i,k}=\tau f(\beta(u_{i-1})), \tag{2.1$'$}$$

$$\mu_{i,k}=\gamma_K\left[\frac{\beta^{-1}(\alpha\theta_{i,k}+(1-\alpha)\beta(u_{i-1}))-u_{i-1}}{\theta_{i,k}-\beta(u_{i-1})}\right], \tag{2.2$'$}$$

starting with

$$\mu_{i,0}=\gamma_K\left[\frac{1}{\beta'(u_{i-1})}\right]. \tag{2.2$''$}$$

These iterations converge to the required couple — see [**JK1**] (in practice the convergence is very fast); then we construct the Rothe functions

$$u^{(n)}(t)=u_{i-1}+(t-t_{i-1})\frac{u_i-u_{i-1}}{\tau}, \quad \text{for } t_{i-1}\le t\le t_i, i=1,\dots,$$
$$\theta^{(n)}(t)=\theta_{i-1}+(t-t_{i-1})\frac{\theta_i-\theta_{i-1}}{\tau}, \quad \text{for } t_{i-1}\le t\le t_i, i=1,\dots,$$

for which it has been proved (see [**JK1**], [**KHK**])

$$u^{(n)}\to u \text{ in } L_2(I,L_2), \quad \theta^{(n)}\to\beta(u) \text{ in } L_2(I,V)$$

where u is a weak solution of (1.1)–(1.3), and V is the corresponding Sobolev space.

Starting with a nonnegative initial function, nonnegativeness of a solution of (1.1) is conserved for all later time ([**HV**]). For expanding core region, the property is guaranteed also in practical numerical computations (we don't use any supplementary meanings for this purpose). The situation is more difficult if $c>0$ and the support of the solution is shrinking and the solution finally extincts in a finite time. For shrinking interfaces the original scheme (2.1)–(2.3) yealds numerical solutions which are slightly negative beyond the theoretical interfaces. In that case we cut the solution and in each time step we use the "modified algebraic correction"

$$u_i := [u_{i-1}+\mu_i(\theta_i-\beta(u_{i-1})]_+. \tag{2.4}$$

After time semidiscretization, the scheme requires to solve a linear reaction-diffusion equation (2.1$'$). At this point we can use the finite difference, finite element or finite volume method for space discretization to obtain the corresponding system of linear equations. We solve the arising system using direct linear solvers or iteration techniques.

3. DISSCUSSION ON NUMERICAL RESULTS

First we have applied the approximation scheme (2.1)–(2.4) to solve the problem (1.1)–(1.3) with initial condition given by

$$u_0(x) = \begin{cases} ((m-1)a)^{-1/(m-1)}(1-x^2)^{1/(m-1)} & , \quad |x| < 1, \\ 0 & , \quad |x| \geq 0, \end{cases}$$

where the constant a is choosen such that $u_0(0) = 1$. In case $m + p = 2$, R. Kersner found the exact solution of the previous problem in the explicit form ([**Ke**]). Support of his solution is bounded during evolution, so we choose the domain $\Omega = [-l, l]$ in that way to cover its maximal expansion.

Let us consider $m = 1.5$, $p = 0.5$, $c = 5.0$, $l = 1.5$ and use the method parameters $K = 10^{20}$, $\alpha = 0.999999$, numerical time step $\tau = 0.001$, spatial grid step $h = 0.01$ (the classical three points approximation of second order derivative is used) and 3 iterations (2.1′)–(2.2′). With these computational parameters we obtain numerical extinction time $T_N^* = 0.316$, while the exact one is $T^* = 0.31829$. The relative $L_1(I, L_1)$ error of the solution was 0.2431%.

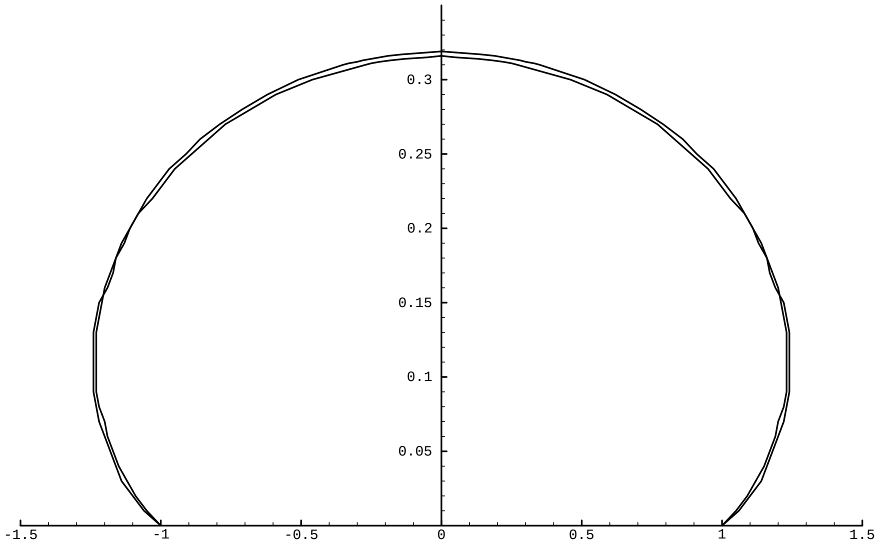

Figure 1. Comparison in x-t plane of the exact and numerical supports, starting from special Kersner solution.

Figure 1 shows the time evolution of the numerical and exact Kersner's interfaces. Initially, the support of both solutions is interval $[-1, 1]$, then supports expand and after certain time moment both supports shrink to the extinction point (numerical extinction is slightly faster).

In the repeated experiments with smaller time step $\tau = 0.0001$ the numerical extinction time T_N^* was 0.3179 and the relative $L_1(I, L_1)$ error was 0.0261 % and with $\tau = 0.00001$ we obtain $T_N^* = 0.31826$ (very near to the exact one), the relative $L_1(I, L_1)$ error 0.0048 %.

In the next experiment for $m = 1.9$, $p = 0.1$, $c = 0.5$, $l = 2$, $\tau = 0.001$ and with the same other method parameters we obtain the numerical extinction time equal to 1.194 while the exact one is 1.1939.

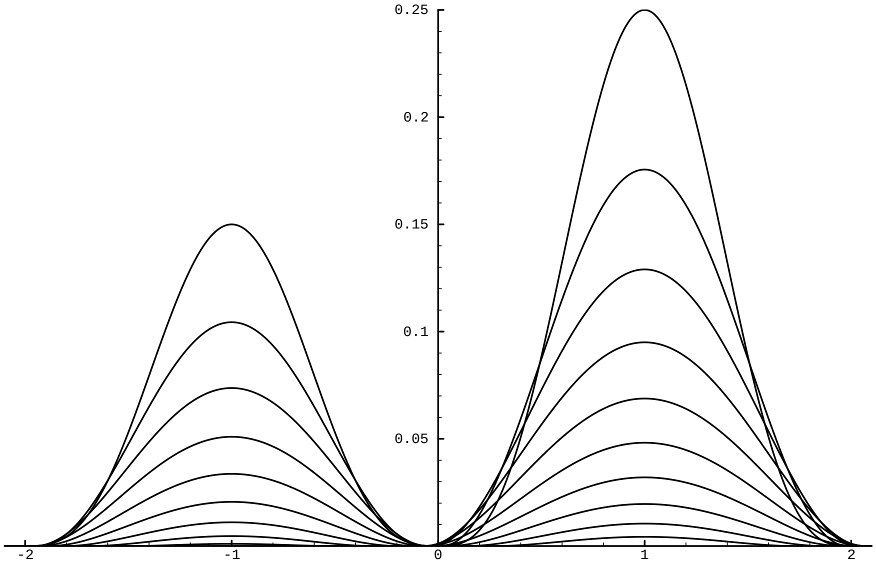

Figure 2a. Time evolution of the special initial profile.

Figures 2a–b show another interesting behaviour — nonmonotonous splitting of the support. This problem was studied theoretically e.g. in [**CMM**]. Let us choose the initial condition as the highest profile in Figure 2a. So we have $S(0) = [-2, 2]$ and $u_0(x)$ has small positive values in the neighbourhood of 0. Then the number of connected subregions of the support may change several times during evolution (Figure 2b). First splitting appears in time 0.014, the support is again connected at 0.094, the second splitting occurs in time moment 0.225. The solution finally extincts at two points in different times. The copmputational parameters for this

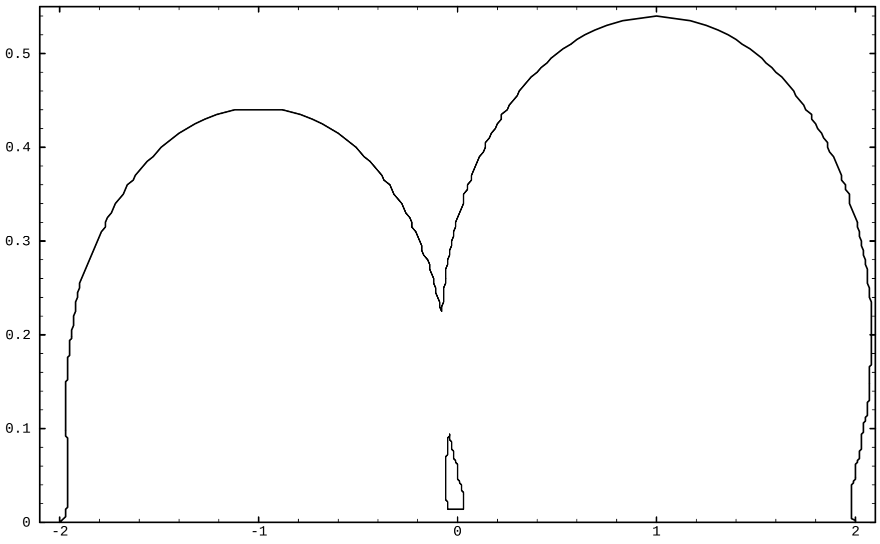

Figure 2b. Double splitting of the initially connected support from Figure 2a, plotted in x-t plane.

experiment: $m = 1.5$, $\nu = 0.5$, $c = 1.4$, $l = 2.1$; $\alpha = 0.999999$, $K = 10^{30}$, $h = 0.01$, $\tau = 0.001$, 5 iterations (2.1′)–(2.2′).

Let us consider the problem (1.1)–(1.3) in $\Omega \subset R^2$ (Ω be a square $[-l,l]\times[-l,l]$).

Let $m = 2$, $p = 0.5$, $c = 5$, $l = 1.2$. Starting computation from initial function given by

$$u_0(x) = \begin{cases} 1 & , \quad x = 0, y = 0, \\ [1 - \frac{(x^2+y^2)^2}{(x^6+y^6)^{1/2}}]_+ & , \quad x \neq 0, y \neq 0, \end{cases} \tag{3.5}$$

(plotted in Figure 3a), we obtain evolution of the initial support graphically documented in Figure 3b. During small expansion the support is smoothed and then successively decreases. Obtaining the circle shape, it eventually shrinks to the central point. The computational parameters:$\alpha = 0.999999$, $K = 10^{25}$, 3 iterations (2.1′)–(2.2′), $\tau = 0.001$, $h = 0.03$ (we use five point scheme to approximate Laplacian).

Choosing initial condition as in Figure 4a (four hills with nonequal heights), the initially connected support successively separates to the several parts. Each of them finally extincts in a finite time (Figure 4b). For this example $m = 2$, $\nu = 0.5$, $c = 10$, $l = 2.8$, $\tau = 0.002$, $h = 0.035$ and other parameters are the same as in previous one.

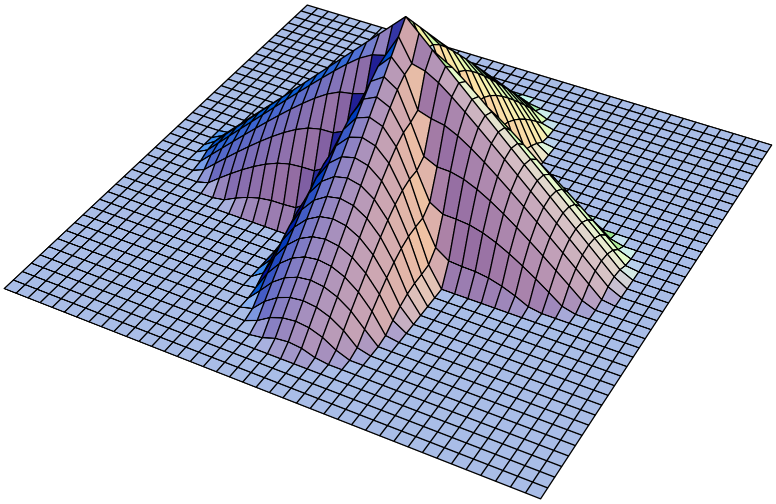

Figure 3a. Initial profile.

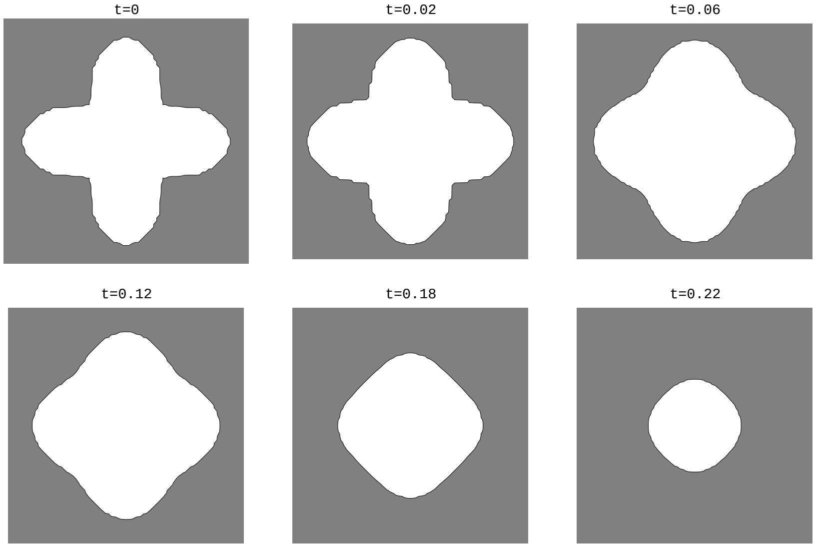

Figure 3b. Support evalution of the numerical solution.

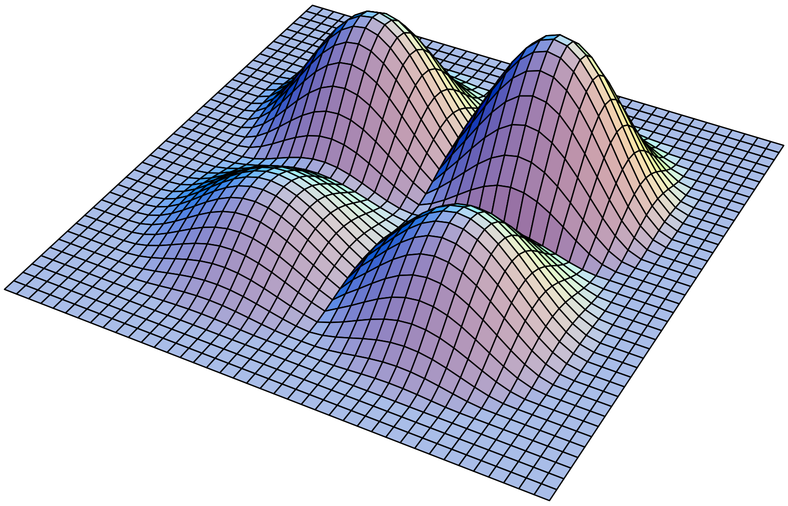

Figure 4a. Initial profile.

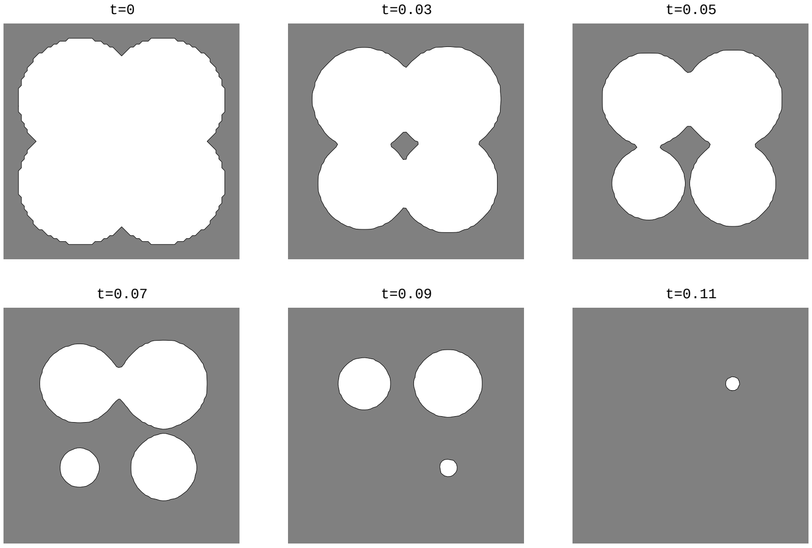

Figure 4b. Support evalution of the numerical solution.

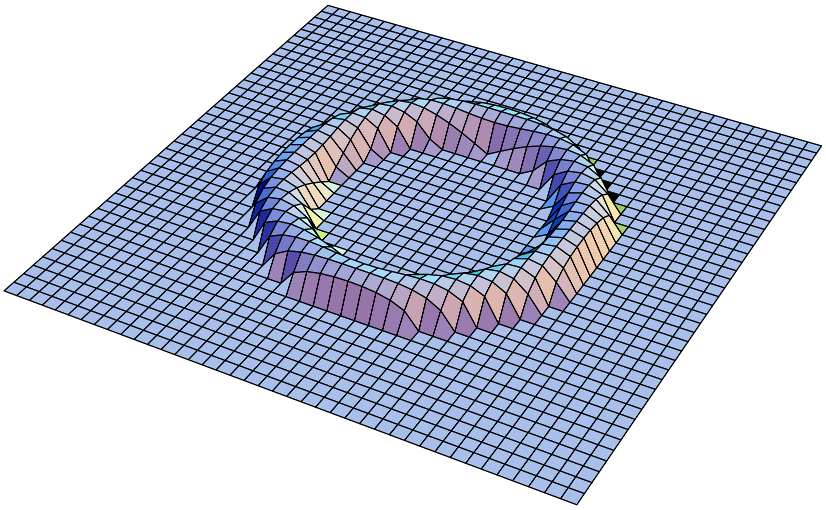

Figure 5a. Initial profile.

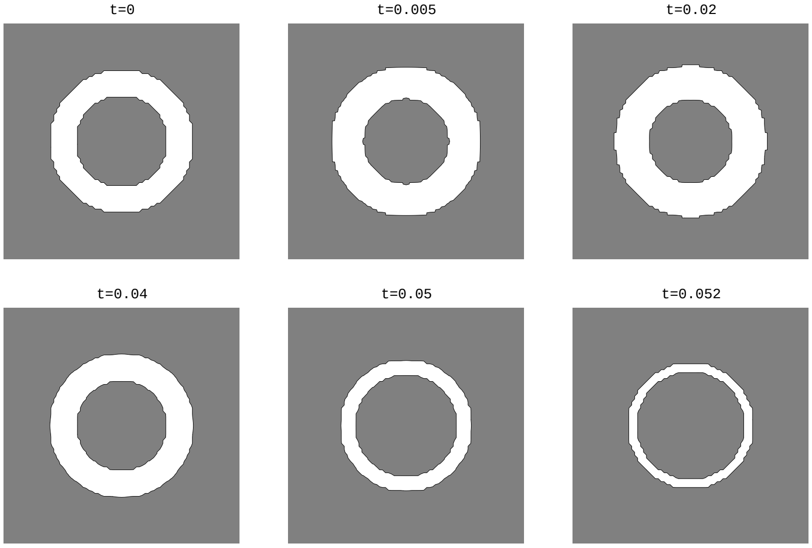

Figure 5b. Support evalution of the numerical solution.

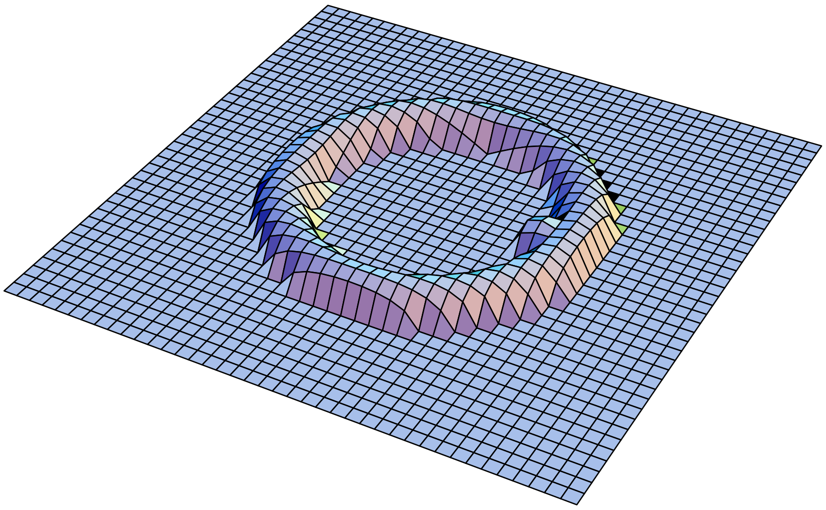

Figure 6a. Initial profile.

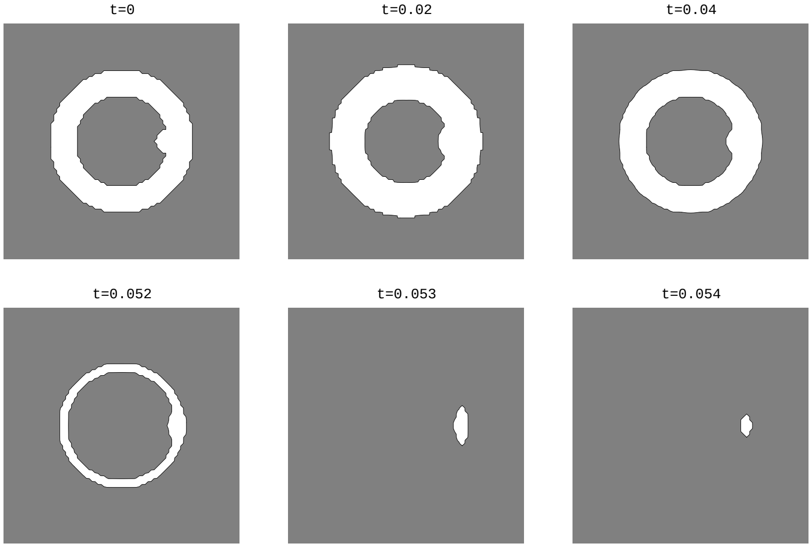

Figure 6b. Support evalution of the numerical solution.

Figures 5a–6b show the interesting dynamics property computed by approximation scheme (2.1)–(2.4). If the solution starts from the special initial condition such as in Figure 5a, the extinction set of its support is the circle. This process can be seen in Figure 5b. On the other hand, if we make a small perturbation of the initial condition such as in Figure 6a, the dynamics is different. Figure 6b indicates that in this case the support shrinks to a point. The analytical conjecture of H. Matano, that generically the point is an extinction set for arbitrary initial condition, has been tested by these experiments. This effect is also a demonstration of the property of a finite speed of propagation of disturbances in the model (1.1)–(1.3) (for $m > 1$). Because of this fact the perturbation has no influence to certain part of the domain during evolution and there exists a time moment when the numerical solution extincts locally in a part of the circle and then the rest extincts in a point. In both experiments we have $m = 2$, $\nu = 0.5$, $c = 10$, $l = 2$, $\tau = 0.001$, $h = 0.025$, $K = 10^{25}$, $\alpha = 0.999999$, 3 iterations (2.1′)–(2.2′). In the first case the numerical extinction time was equal to 0.053, in the second case it was 0.055.

References

[BBR] Berger A. E., Brezis H. and Rogers J. C. W., *A numerical method for solving the problem* $u_t - \Delta f(u) = 0$, R.A.I.R.O. Anal. Numer. **13**, 297–312.

[HV] Herrero M. A. and Vazquez J. L., *The one-dimensional nonlinear heat equation with absorption: Regularity of solutions and interfaces*, SIAM J. Math. Anal. **18** (1987), 149–167.

[CMM] Chen X., Matano H. and Mimura M., *Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption*, Preprint CDSNS90, Atlanta (1989).

[JK1] Jäger W. and Kačur J., *Solution of porous medium type systems by linear approximation schemes*, Num. Math. **60** (1991), 407–427.

[JK2] ______, *Approximation of porous medium type systems by nondegenerate elliptic systems*, Preprint, University of Heidelberg (1991).

[JK3] ______, *Approximation of degenerate elliptic-parabolic problems by nondegenerate elliptic and parabolic problems*, Preprint, University of Heidelberg (1991).

[K1] Kačur J., *Method of Rothe in evolution equations, Teubner-Texte zur Mathematik 80*, Teubner, Leipzig, 1985.

[KHK] Kačur J., Handlovičová A. and Kačurová M., *Solution of nonlinear diffusion problems by linear approximation schemes*, SIAM J. Num. Anal. **30** (1993), 1703–1722.

[Ke] Kersner R., *The behavior of temperature fronts in media with nonlinear thermal conductivity under absorption*, Vest. Mosk. Univ. **33** (1978), 44–51.

[MNV] Magenes E., Nochetto R. H. and Verdi C., *Energy error estimates for a linear scheme to approximate nonlinear parabolic problems*, Math. Mod. Num. Anal. **21** (1987), 655–678.

[MNT] Mimura M., Nakaki T. and Tomoeda K., *A numerical approach to interface curves for some nonlinear diffusion aquations*, Japan J. Appl. Math. **1** (1984), 93–139.

[N] Nakaki T., *Numerical interfaces in nonlinear diffusion with finite extinction phenomena*, Hiroshima Math. J. **18** (1988), 373–397.

[P] Patankar S., *Numerical heat transfer and fluid flow*, Hemisphere Publ. Corp., New York, 1980.

[**RK**] Rosenau P. and Kamin S., *Thermal waves in an absorbing and convecting medium*, Physica **8D** (1983), 273–283.

[**T**] Tomoeda K., *Convergence of numerical interface curves for nonlinear diffusion equation, Advances in computational methods for boundary and interior layers*, Lecture notes of an international short course held in association with the BAIL III conference 18-19 June 1984, Trinity college, Dublin, Ireland.

[**TM**] Tomoeda K. and Mimura M., *Numerical approximations to interface curves for a porous media equation*, Hiroshima Math. J. **13** (1983), 273–294.

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