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LINEAR INDEPENDENCES IN BOTTLENECK ALGEBRA AND THEIR COHERENCES WITH MATROIDS

J. PLÁVKA

Abstract. Let $(B, \leq)$ be a dense, linearly ordered set with maximum and minimum element and $(\oplus, \otimes) = (\max, \min)$. We say that an (m, n) matrix $A = (a_1, a_2, \dots, a_n)$ has: (i) weakly linearly independent (WLI) columns if for each vector b the system $A \otimes x = b$ has at most one solution; (ii) regularly linearly independent columns (RLI) if for each vector b the system $A \otimes x = b$ is uniquely solvable; (iii) strongly linearly independent columns (SLI) if there exist vectors $d_1, d_2, \dots, d_r$, $r \geq 0$ such that for each vector b the system $(a_1, \dots, a_n, d_1, \dots, d_r) \otimes x = b$ is uniquely solvable. For these linear independences we derive necessary and sufficient conditions which can be checked by polynomial algorithms as well as their coherences with definition of matroids.

1. Introduction

The aim of this paper is to review the results concerning some types of linear independences in Bottleneck algebras (some of them and the others were studied in [**1**]–[**10**]) and suggest their coherences with matroidal properties where matroid was formally introduced by Welsh in the following definition.

Definition. Let $\mathcal{S}$ be a finite set and $\mathcal{I}$ a family of its subsets, called independent sets. Then $(\mathcal{S}, \mathcal{I})$ is a matroid if
(i) $\mathcal{I} \neq \phi$ has hereditary property (if $\mathcal{A} \in \mathcal{I}$ and $\mathcal{B} \subseteq \mathcal{A}$ then $\mathcal{B} \in \mathcal{I}$)
(ii) $\mathcal{A}, \mathcal{B} \in \mathcal{I}$ such that $|\mathcal{A}| = |\mathcal{B}| + 1$ then there exists $a \in \mathcal{A} \setminus \mathcal{B}$ such that $\mathcal{B} \cup \{a\} \in \mathcal{I}$.
If there only (i) is fulfilled we say that $(\mathcal{S}, \mathcal{I})$ is hereditary system.

A notion of linear independence fulfilling (i), (ii) properties, ensures that all maximal independent sets will have the same cardinality and, hence serves a good starting point for the notion of rank and dimension.

2. Definitions and Notations

The quadruple $\mathcal{B} = (B, \oplus, \otimes, \leq)$, or B itself, is called bottleneck algebra (BA) if $(B, \leq)$ is a nonempty, linearly ordered set with a maximum element (denoted by ϵ

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and called zero) and a minimum element (denoted by σ and called unit), whereby $\epsilon \neq \sigma$ and $\oplus, \otimes$ are binary operations on B defined by formulas

$$a \oplus b = \max\{a, b\}$$
$$a \otimes b = \min\{a, b\}.$$

In the following we will deal with (m, n) matrices, and we assume everywhere that m and n are given positive integers. For short we denote $\{1, 2, \ldots, n\}$ by N and $\{1, 2, \ldots, m\}$ by M. The system of all (m, n) matrices over B will be denoted by $B(m, n)$. The elements of $B(m, 1)$ will be called vectors. The elements of B will be represented by letters of Greek alphabet, a matrix with vectors $a_1, \ldots, a_n$ as its columns will be denoted by $A = (a_1, \ldots, a_n)$ or $A = (a_{ij})$. If $A = (a_{ij}) \in B(m, n)$, $m \geq n$ and $a_{ij} > \sigma$ for $i = j$ and $a_{ij} = \sigma$ otherwise then we say that matrix A is trapezoidal one and we will denote it as $A = \operatorname{trap}\{a_{11}, \ldots, a_{nn}\}$. If $m = n$ we say that the trapezoidal matrix A is diagonal and denote $A = \operatorname{diag}\{a_{11}, \ldots, a_{nn}\}$.

Two matrices A, B are said to be equivalent (abbr. $A \sim B$) if one can be obtained from the other by permutations of its rows and columns. If matrix A is equivalent to a diagonal matrix then we say A is a permutation matrix.

Extend $\oplus, \otimes$ and $\leq$ to matrices over B as in conventional algebra. The main results are proved under the assumption of density of the ordering $\leq$, that is to say,

$$(\forall x, y \in B)\ x < y \implies (\exists\, z \in B)\ x < z < y.$$

We say that a matrix $A = (a_1, \ldots, a_n) \in B(m, n)$, $n \leq m$ has
(i) weakly linearly independent (WLI) columns if for each vector b the system $A \otimes x = b$ has at most one solution;
(ii) regularly linearly independent (RLI) columns if for each vector b the system $A \otimes x = b$ is uniquely solvable;
(iii) strongly linearly independent (SLI) columns if there exist vectors $d_1, \ldots, d_r \in B(m, 1)$, $r \geq 0$ such that for each vector b the system $(a_1, \ldots, a_n, d_1, \ldots, d_r) \otimes x = b$ is uniquely solvable.

The definition WLI was introduced under the name $2B$-**independence** in [**4**], for which was formulated open problem to find necessary and sufficient conditions and cheking algorihm for testing of it. RLI has a motivation in the conventional linear algebra. The SLI is introduced originally in this paper to give a definition of independence with matroidal properties.

3. Necessary and Sufficient Conditions for Linear Independences

Before statement of the main results of this paper we establish some notations. If in i-th row of A only one maximum element exists i.e. there exists $j \in N$ such that $a_{ij} > a_{is}$ for all $s \neq j$ we will denote it by π_i and second maximum element of i-th row we will denote by τ_i i.e. $\tau_i = \oplus_{\substack{j \\ a_{ij} \neq \pi_i}} a_{ij}$. By M_A^* we denote the set of all row indices for which only one maximum exists. Denote the sets $\{j \in M_A^*;\ \pi_j = a_{ji} > \tau_j > \sigma\}$ and $\{j \in M_A^*;\ \pi_j = a_{ji} > \tau_j = \sigma\}$ by R_i and C_i, respectively.

Theorem 1. *Let $A \in B(m,n)$. Then A has WLI columns if and only if*
(i) A contains a permutation submatrix of order n
(ii) A contains a square submatrix of order n which has in each row and each column exactly one unit entry
(iii) for all $i \in N' = \{s \in N;\ R_s \neq \phi\}$

$$\bigotimes_{j \in R_i} \tau_j \leq \bigoplus_{j \in C_i} \pi_j$$

holds.

Proof. Suppose that $A = (a_{ij}) \in B(m,n)$.

(i) Denote $M_j = \{i \in M;\ a_{ij} > \sigma\}$. Suppose that a matrix A is different from the zero-matrix and all zero rows are removed since they do not have any influence on WLI and it implies $\cup_{k \in N} M_k = M$. Then it is clear that A contains a permutation submatrix of order n if and only if $\cup_{k \neq j} M_k \neq M$ holds for all $j \epsilon N$. Now suppose that A doesn't contain a permutation submatrix of order n i.e. according to foregoing discussion there exists $j \in N$ (say $j = n$) such that $\cup_{k \neq j} M_k = M$. Then the system $A \otimes x = b$ for $b = (b_1, \dots, b_m)^T \in B(m,1)$, and $b_i = \otimes_{a_{rs} > 0} a_{rs}$ has solutions $x = (x_1, \dots, x_n)^T$ where $x_i = b_i$ for $i = 1, 2, \dots, n-1$ and x_n is arbitrary element from closed interval $[\sigma, \otimes_{a_{ij} > \sigma} a_{ij}]$.

(ii) Suppose that A contains a column with all entries less than ϵ. W.l.o.g. let $a_{i1} < \epsilon$ for all $i \in M$. Set the right-hand side vector b equal to the first column of A. It is easy to see that the vector $x = (\oplus a_{i1}, \sigma, \dots, \sigma)^T$ is a solution of $A \otimes x = b$, moreover, $x' = (\epsilon, \sigma, \dots, \sigma)$ is another solution. Therefore each column of A must contain at least one unit entry. If a submatrix of order n with exactly one unit in each row and column does not exist then A contains a row with at least two unit entries (say) in r-th and s-th position for $r < s$, $k \leq s \leq n$. Then system $A \otimes x = b$ for $b_i = \oplus_j a_{ij}$ has solutions $x = (x_1, \dots, x_n)^T$, $x_i = \epsilon$ for all $i \neq s$ and $x_s \in [\oplus_{a_{rs} < \epsilon} a_{rs}, \epsilon]$.

(iii) The case $N' = \phi$ is clear since since according to (i), (ii) the matrix A contains a submatrix of order n equivalent to a diag$\{\epsilon, \dots, \epsilon\}$ and consequently it follows that the system $A \otimes x = b$ has at most one solution for each vector b. Suppose that there exists $i \in N'$ (say $i = 1$) such that

$$\bigotimes_{j \in R_1} \tau_j > \bigoplus_{j \in C_1} \pi_j .$$

Then the system $A \otimes x = b$ for $b = (b_1, \dots, b_m)^T$ where $b_i = \tau_i$ for $i \in R_1$, $b_i = \pi_i$ for $i \in C_1$, otherwise $b_i = \oplus_j a_{ij}$ has solutions $x = (x_1, \dots, x_n)^T$ whereby

$x_2 = x_3 = \ldots = x_n = \epsilon$ and $x_1 \in [\oplus_{j \in C_1} \pi_j, \otimes_{j \in R_1} \tau_j]$. From the density a contradiction follows.

Conversely, we suppose that (i), (ii), (iii) hold. By analysis of cases we will show that for arbitrary vector b the system $A \otimes x = b$ either doesn't have solution or $\oplus_{j \in C_i} \pi_j \leq x_i \leq \otimes_{j \in R_i} \tau_j$ or $x_i = b_j$ but this fact together with (iii) imply the assertion. Suppose that $j \in C_i$. If $b_j < \pi_j$ then $x_i = b_j$ and if $b_j = \pi_j$ then $x_i \geq \pi_j$ and otherwise the system is not solvable. From foregoing inequality follows that $\oplus_{j \in C_i} \pi_j \leq x_i$. The second part we will prove similarly. Let $j \in R_i$. If $\pi_j > b_j > \tau_j$ then $x_i = b_j$. If $b_j \leq \tau_j$ then $x_i \leq b_j \leq \tau_j$ and again otherwise the system is not solvable. Thus, $x_i \leq \otimes_{j \in R_i} \tau_j$ and the assertion results. $\square$

The previous theorem gives a clear hint to the testing of WLI-it suffices to look for first and second maxima (in $O(mn)$ steps) for each $i \in N'$ to check whether $\otimes_{j \in R_i} \pi_j \leq \oplus_{j \in C_i} \tau_j$. For this purpose suppose that rows of A which have only one maximum element precede the others and denote $p = (\pi_1, \ldots, \pi_j)$, $s = (\tau_1, \ldots, \tau_j)$, $j \leq m$ then we are led on finding the sets R_i and C_i and then minima and maxima of elements of p and s over R_i and C_i, respectively (in $O(2m)$ steps) $-O(mn) + nO(2m) = O(mn)$.

Lemma 1. *Let $A \in B(m,n)$. If A has RLI columns then $m = n$.*

Proof. Suppose that A has RLI (implies WLI) columns. The part (ii) of Proof of Theorem 1 suggests that a matrix A having WLI columns contains a square submatrix of order n which has in each row and each column exactly one unit entry. For $n < m$ we will construct a vector b which implies the system $A \otimes x = b$ is not solvable. Denote $b_i = \oplus_j a_{ij}$. If there exists $i \in M$ such that $b_i < \epsilon$ then for $b' = (b_1, \ldots, b_{i-1}, \epsilon, b_{i+1}, \ldots, b_m)^T$ the system $A \otimes x = b'$ doesn't have a solution. If $b_i = \epsilon$ for all $i \in M$ then the matrix A contains a column with at least two unit entries (say) in r-th and s-th positions, $s > n$. The system $A \otimes x = b$ is not solvable for $b = (b_1, \ldots, b_m)$ where $b_i = \sigma$ for all $i \in M \setminus \{s\}$ and $b_s = \epsilon$. $\square$

Theorem 2. *Let $A \in B(n,n)$. Then A has RLI columns if and only if $A \sim$ diag $\{\epsilon, \ldots, \epsilon\}$.*

Proof. The part "if" is trivial. For a converse, suppose that A has RLI columns then A has WLI columns and according to (i) and (ii) of Theorem 1 we have the assertion. $\square$

Theorem 3. *Let $A \in B(m,n)$. Then A has SLI columns if and only if $A \sim$ trap $\{\epsilon, \ldots, \epsilon\}$.*

Proof. Suppose that $A \sim trap\{\epsilon, \ldots, \epsilon\}$. Denote for $i = 1, 2, \ldots, r$; $r = m - n$ a vector d_i which has on $(n+i)$-th position unit entry and otherwise entries are equal to σ. Then the matrix $(a_1, \ldots, a_n, d_1, \ldots ., d_r) \sim$ diag $\{\epsilon, \ldots, \epsilon\}$ and according to Theorem 2 the system $(a_1, \ldots, a_n, d_1, \ldots, d_r) \otimes x = b$ has only one solution for arbitrary vector b.

Conversely, suppose that there exist vectors $d_1, \dots, d_r$ such that the system

$$(a_1, \dots, a_n, d_1, \dots, d_r) \otimes x = b$$

is unique solvable. But again using the Theorem 2

$$(a_1, \dots, a_n, d_1, \dots, d_r) \sim \operatorname{diag}\{\epsilon, \dots, \epsilon\}$$

implies $A \sim \operatorname{trap}\{\epsilon, \dots, \epsilon\}$. □

The last assertions enable to immediately compile an $O(mn)$ algorithm for testing RLI and SLI columns of a matrix A.

4. Coherence of the Linear Independences with Matroids

Let $A = (a_1, \dots, a_n) \in B(m, n)$. If $A' = (a_{i_1}, \dots, a_{i_k})$, $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ has WLI (RLI, SLI) columns then the system of vectors $\{a_{i_1}, \dots, a_{i_k}\}$ is said to be WLI (RLI, SLI) subset of $\mathcal{A}$.

Theorem 4. *Let $\mathcal{S} = \{a_1, \dots, a_n\}$. Then $(\mathcal{S}, \mathcal{I})$ is hereditary system where $\mathcal{I}$ is a family of WLI subsets of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{A} = \{a_1, \dots, a_r\} \in \mathcal{I}$ and $\mathcal{B} = \{a_{i_1}, \dots, a_{i_s}\} \subseteq \mathcal{A}$ and $A = (a_1, \dots, a_r)$, $B = (a_{i_1}, \dots, a_{i_s})$.

Denote

$$\pi_j = \bigoplus_{i \in M_A^*} a_{ji}, \quad \pi_j' = \bigoplus_{i \in M_B^*} a_{ji}$$

$$\tau_j = \bigoplus_{\substack{i \in M_A^* \\ a_{ji} \neq \pi_j}} a_{ji}, \quad \tau_j' = \bigoplus_{\substack{i \in M_B^* \\ a_{ji} \neq \pi_j}} a_{ji}$$

$$R_t' = \{j \in M_B^*;\ \pi_j' = a_{jt} > \tau_j' > \sigma\}$$

and

$$C_t' = \{j \in M_B^*;\ \pi_j' = a_{jt} > \tau_j' = \sigma\}.$$

Since $\{i_1, \dots, i_s\} \subseteq \{1, \dots, r\}$ we have $C_i \subseteq C_i'$ and

$$\bigoplus_{j \in C_t} \pi_j \leq \bigoplus_{j \in C_t'} \pi_j'$$

holds. Therefore for all $t \epsilon \{i_1, \dots, i_s\} \setminus \{p; R_p' = \phi\}$ is fulfilled either

$$\bigoplus_{j \in C_t'} \pi_j' = \epsilon$$

or

$$\bigotimes_{j \in R'_t} \tau'_j \leq \bigotimes_{j \in R_t} \tau_j .$$

In each of both cases the assertion follows. □

Since the structure of a matrix A which has SLI columns is very simple

$$A \sim \begin{pmatrix} \epsilon & \sigma & \dots & \sigma \\ \sigma & \epsilon & \dots & \sigma \\ \dots & \dots & \dots & \dots \\ \sigma & \sigma & \dots & \epsilon \\ \sigma & \sigma & \dots & \sigma \\ \dots & \dots & \dots & \dots \\ \sigma & \sigma & \dots & \sigma \end{pmatrix}$$

straightforwartly from definitions the following assertion results.

Theorem 5. *Let $\mathcal{S} = \{a_1, \dots, a_n\}$. Then $(\mathcal{S}, \mathcal{I})$ is matroid where $\mathcal{I}$ is a family of SLI subset of $\mathcal{S}$.*

To summarise the results of this article we give the following table.

Independence	The order	Complexity	Matroid
WLI	$m \geq n$	$O(mn)$	hereditary system
RLI	$m = n$	$O(n^2)$	
SLI	$m \geq n$	$O(mn)$	matroid

Table 1.

In conclusion two examples.

Example 1. Let $B = [0, 1] \subset R$ and

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 0.3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and $\mathcal{I}$ be a family of WLI subsets of $\mathcal{S}$. Then this example shows that $(\mathcal{S}, \mathcal{I})$ is not matroid since $\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0.4 \end{pmatrix}, \begin{pmatrix} 0.3 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ are maximal independent sets and their cardinality are not equal.

Example 2. Let $B = [0, 1] \subset R$

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

and $\mathcal{I}$ be a family of RLI subsets of $\mathcal{S}$. Then for this example $(\mathcal{S}, \mathcal{I})$ is not hereditary system since the condition (i) of the definition of matroids is not fulfilled using of Theorem 2.

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J. Plávka, Department of Mathematics, Technical University, Hlavná 6, 040 01 Košice, Slovakia