Acta Math. Univ. Comenianae
Vol. LXIV, 2(1995), pp. 247–254

MEASURABILITY OF SOME SETS OF BOREL MEASURABLE FUNCTIONS ON $[0,1]$

M. ŠMÍDEK

Abstract. In the paper we show that the space of injective Borel measurable functions and the space of functions, which norm attains supremum at exactly one point, with supremum metric are coanalyticly hard by using the space of trees.

In this paper we show that the set of injective functions is not Suslin in the space of Borel measurable functions $f\colon [0,1] \longrightarrow [0,1]$ with the supremum metric. This answers a question of A. H. Stone posed after the problem of [**DS**], whether the set of injective functions is Borel measurable in the space of Lebesgue measurable functions $f\colon [0,1] \longrightarrow [0,1]$ with the supremum metric, was solved by Miroslav Chlebík.

We say that M is **a Polish** space if M is a complete separable metric space. Let M be a topological space and P be a metric space. Then $\mathcal{B}_b(M,\ P)$ denotes the space of all bounded Borel measurable functions $f\colon M \longrightarrow P$ with the supremum metric. The space of continuous bounded functions is denoted by $\mathcal{C}_b(M,P)$ for $P=\mathbb{R}$ it is a normed linear space endowed with the supremum norm. Further, we put $\boldsymbol{M}_{\mathcal{B}}(M,\mathbb{R}) = \{f \in \mathcal{B}_b(M,\mathbb{R});\ \exists!\, x \in M\colon |f(x)| = \|f\|\}$, $\boldsymbol{M}_{\mathcal{C}}(M,\mathbb{R}) = \boldsymbol{M}_{\mathcal{B}}(M,\mathbb{R}) \cap \mathcal{C}_b(M,\mathbb{R})$, $\boldsymbol{I}_{\mathcal{B}}(M,P) = \{f \in \mathcal{B}_b(M,P);\ f \text{ is injective}\}$, and $\boldsymbol{I}_{\mathcal{C}}(M,P) = \boldsymbol{I}_{\mathcal{B}}(M,P) \cap \mathcal{C}_b(M,P)$.

Let $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ denote the Baire space of sequences of natural numbers and $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathbb{N}^n \cup \{\emptyset\}$ denote the set of all finite sequences of element of $\mathbb{N}$.

For $s = (s^1,\dots,s^i) \in \mathcal{S}$ let $|s| = i$ denote the length of sequence s and for $\mu = (\mu^1,\mu^2,\dots) \in \mathcal{N}$ and $k \in \mathbb{N}$ let $\mu|k = (\mu^1,\dots,\mu^k) \in \mathcal{S}$ denote the first k members of the sequence μ. We say that $t = (t^1,\dots,t^i) \in \mathcal{S}$ is a extension of $s = (s^1,\dots,s^j) \in \mathcal{S}$ if $j \le i$ and $(t^1,\dots,t^j) = s$. By the metric on the Baire space we understand $\varrho(\mu,\nu) = \bigl(\min\{k \in \mathbb{N};\ \mu|k \ne \nu|k\}\bigr)^{-1}$ for $\mu \ne \nu$ and $\varrho(\mu,\mu) = 0$. For $s \in \mathcal{S}$ denote $\mathcal{N}(s) = \{\nu \in \mathcal{N};\ \nu\big||s| = s\}$. Let $G \subset \mathcal{N}$ be an open nonempty set in $\mathcal{N}$, then spaces $\mathcal{N}$, $\mathcal{N}\times\mathcal{N}$ and G are homeomorphic, denote $\mathcal{N} \sim \mathcal{N}\times\mathcal{N} \sim G$.

Let M be a metric space. We say that $S \subset M$ is **a Suslin set** if it can be written in the form $S = \bigcup_{\nu\in\mathcal{N}} \bigcap_{n\in\mathbb{N}} F(\nu|n)$, where $F(s) \subset M$ is closed for $s \in \mathcal{S}$.

Received July 27, 1994; revised August 2, 1995.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 04A15, 54H05.

The set $C \subset M$ is **co-Suslin** if $M \setminus C$ is Suslin. The preimages of Suslin sets under a Borel measurable mapping are Suslin.

Let A be a subset of a metric space M. We say that a point $x \in M$ is **a condensation point of the set** A if, for every neighbourhood U of the point x, $A \cap U$ is uncountable. The set A is **condensed** if it is nonempty and each of its points is a condensation point. For separable A, let B be a set of all condensation points of A. Then the set B is condensed and the set $A \setminus B$ is countable, [**K**, Chapter 2.B, §23, III, p. 260].

Proposition 1. *For every separable absolute Borel metric space A (i.e. Borel in its completion), which is condensed, there is a continuous one-to-one mapping f of $\mathcal{N}$ onto A.*

The proof of this proposition in the special case $A \subset \mathbb{R}$ is in [**S**]. To prove Proposition 1 in the general case we follow closely the procedure of [**S**] using the following two lemmas. The proof of Lemma B can follow the case $M = \mathbb{R}$ from [**S**] (Lemma 3) almost word by word and we omit it.

Lemma A. *Let M be a metric space, a subset A of the space M be an injective continuous image of the space $\mathcal{N}$ and $x \in M$ be a condensation point of the set A. Then $A \cup \{x\}$ is an injective continuous image of $\mathcal{N}$.*

Lemma B. *Let M be a Polish space and A be a condensed Borel measurable subset of M. Then there exists a family of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset M$ which are condensed, Borel measurable, dense in A and $A = \bigcup_{n\geq 1} A_n$.*

Proof of Proposition 1. Let us denote $M = \widetilde{A}$. Let $(A_n)_{n\in\mathbb{N}} \subset M$ be a family from Lemma B. There exists sets $D_n \subset M$ and $B_n \subset M$ such that $A_n = B_n \cup D_n$, D_n are at most countable and B_n are continuous injective images of $\mathcal{N}$, [**K**, Chapter 3, §37, II, consequence 1c, p. 462]. Denote $D = \bigcup_{n\geq 1} D_n$ and $\{x_1, x_2, \dots\} = D$ finite or infinite sequence and $C_n = B_n \cup \{x_n\}$ for $n \in \mathbb{N}$ if card $D = \infty$ and, if card $D = n_0$, $C_n = B_n$ for $n > n_0$. The sets D_n are at most countable, A_n are condensed and dense in A, hence B_n are condensed and dense in A. Then each point of D is a condensation point of B_n. Then, by Lemma A, the set C_n is injective continuous image of $\mathcal{N}$. And now we easily obtain that $A = \bigcup_{n\in\mathbb{N}} C_n$ is injective continuous image of the space $\mathcal{N}$ as $\mathcal{N} \sim \mathcal{N}(k)$ for every $k \in \mathcal{N}$. □

Proof of Lemma A. Let $x \notin A$, otherwise the proof is easy, and $f \colon \mathcal{N} \longrightarrow M$ be an injective continuous mapping such that $f(\mathcal{N}) = A$. Since x is a condensation point of A, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ such that $x_n \to x$. Denote $\nu_n = f^{-1}(x_n)$, $r_n = \frac{1}{3}\varrho\left(x_n, \{x\} \cup \bigcup_{i\neq n}\{x_i\}\right)$, where ϱ is metric of the space M.

Then $A_n = \mathcal{U}(x_n, r_n)$ (open ball of centre x_n and radius r_n) are pairwise disjoint. As $r_n \to 0$, for arbitrary sequence $y_n \in A_n$, we get $y_n \to x$. Since f is continuous, there is a sequence $(l_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ so that $f\big(\mathcal{N}(\nu_n|l_n)\big) \subset A_n$. Denote $H_n = \mathcal{N}(\nu_n|l_n)$ for $n \geq 1$ and $H_0 = \mathcal{N} \setminus \bigcup_{n\geq 1} H_n$. For $n \geq 1$, the set H_n is open and closed, hence H_0 is closed.

The set H_0 is open. If it was not open, then there exist $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$, $(\mu_i)_{i\in\mathbb{N}} \subset \mathcal{N}$ and $\mu \in \mathcal{N}$ such that $\mu \in H_0$, $\mu_i \in H_{n_i}$ and $\mu_i \to \mu$. If there is an $m \in \mathbb{N}$ so that $n_i < m$ for each $i \in \mathbb{N}$, then $(\mu_i)_{i\in\mathbb{N}} \subset \bigcup_{j=1}^m H_j$. Hence $\mu \in \bigcup_{j=1}^m H_j$, and $\mu \notin H_0$ as the set $\bigcup_{j=1}^m H_j$ is closed. Thus there exists a subsequence $n_{i_j} \to \infty$. Since $f(\mu_{i_j}) \in A_{n_{i_j}}$, we get $f(\mu_{i_j}) \to x$, and it implies $f(\mu) = x$ and $x \in A$ what is contradiction. The sets H_n are open, hence $H_n \sim \mathcal{N}$ for each $n \geq 0$.

Let us choose an arbitrary $\mu \in \mathcal{N}$ and denote $K_n = \mathcal{N}(\mu|n) \setminus \mathcal{N}(\mu|n+1)$ for $n \geq 0$. The sets K_n are pairwise disjoint open sets and $\bigcup_{n\geq 0} K_n = \mathcal{N} \setminus \{\mu\}$. Hence $K_n \sim \mathcal{N}$ and $K_n \sim H_n$, denote by $\varphi_n \colon K_n \longrightarrow H_n$ some homeomorphism. Now let us define a mapping $g \colon \mathcal{N} \setminus \{\mu\} \longrightarrow M$ by $g(\varrho) = f(\varphi_n(\varrho))$ for $\varrho \in K_n$. The mapping g is injective, continuous on $\mathcal{N} \setminus \{\mu\}$ and $g(\mathcal{N} \setminus \{\mu\}) = f(\mathcal{N}) = A$. It is easy to see that we can extend the function g to the point μ by $g(\mu) = x$ and g is continuous. □

A **metric** space C is called **coanalyticly hard** if for every Polish space P and every its co-Suslin subset $E \subset P$ there exists a Borel measurable mapping $f \colon P \longrightarrow \widetilde{C}$ into the completion $\widetilde{C}$ of the space C such that $E = f^{-1}(C)$.

Recall that usually **a subset** C of Polish space M is said to be **coanalyticly hard** (in M), if for every Polish space P and every its co-Suslin subset $E \subset P$ there exists a Borel measurable mapping $f \colon P \longrightarrow M$ so that $E = f^{-1}(C)$, [**KL**]. A subset C of a Polish space P is coanalyticly hard if and only if C is a coanalyticly hard space.

Moreover, C is a coanalyticly hard space if and only if it contains a separable subset E such that $C \cap \overline{E}^{\widetilde{C}}$ is a coanalyticly hard subset of Polish space $\overline{E}^{\widetilde{C}}$. If C is coanalyticly hard space, then by Lemma 4 bellow the set $\mathcal{L}$ of well-founded trees is coanalyticly hard and co-Suslin in the Polish space $\mathcal{T}$. Hence there is Borel measurable mapping $f \colon \mathcal{T} \longrightarrow \widetilde{C}$ such that $f^{-1}(C) = \mathcal{L}$. By [**F**, Theorem 1] a set $E = f(\mathcal{T})$ is separable and by Lemma 3 bellow the set $C \cap \overline{E}^{\widetilde{C}}$ is coanalyticly hard.

Lemma 2. *Let M be a complete metric space and $A \subset M$ be a coanalyticly hard space. Then A is not the Suslin subset of M.*

Proof. There exists a co-Suslin set $C \subset \mathcal{N}$, which is not Suslin in $\mathcal{N}$, [**K**, Chapter 3, §38, VI, p. 472]. Since A is a coanalyticly hard space and M is a complete metric space, there exists a Borel measurable mapping $f \colon \mathcal{N} \longrightarrow M$ such that $C = f^{-1}(A)$. If A was Suslin in M, $f^{-1}(A) = C$ would be Suslin in $\mathcal{N}$.□

Lemma 3. *Let $f \colon P \longrightarrow M$ be a Borel measurable mapping of a complete metric space P to a metric space M and $B \subset M$ be a set such that $f^{-1}(B) = A$ is a coanalyticly hard space. Then B is a coanalyticly hard space.*

Proof. Let E be a co-Suslin subset of a Polish space L. Then there exists Borel measurable mapping $g\colon L \longrightarrow P$ so that $E = g^{-1}(A)$. Let us define a mapping h from L into the completion $\widetilde{M}$ of the space M by $h = f \circ g$. The mapping h is Borel measurable and $h^{-1}(B) = g^{-1}(A) = E$. We need to find a Borel measurable mapping $\widetilde{h}\colon L \longrightarrow \overline{B}^{\widetilde{M}}$ such that $\widetilde{h}^{-1}(B) = E$.

There is a point $x \in \overline{B}^{\widetilde{M}} \setminus B$, otherwise the set B is closed in $\widetilde{M}$, hence $f^{-1}(B) = A$ is Borel measurable which is a contradiction with Lemma 1. Let us define the mapping $\widetilde{h}\colon L \longrightarrow \overline{B}^{\widetilde{M}}$ by $\widetilde{h}(z) = h(z)$ if $h(z) \in \overline{B}^{\widetilde{M}}$ and $\widetilde{h}(z) = x$ otherwise. $\square$

We say that $T \subset \mathcal{S}$ is a tree if for every $t \in T$ and for every $s \in \mathcal{S}$ such that t is an extension of s, $s \in T$. Let us denote the space of trees by $\mathcal{T}$. Recall that the space $\mathcal{T}$ is a compact metric space endowed with such a metric that $T_n \to T$ in its metric means that $s \in T$ if and only if there exists $n_0 \in \mathbb{N}$ so that $s \in T_n$ for $n \geq n_0$. The space $\mathcal{T}$ corresponds to the stopping times defined in [**D**, p. 235].

For $s \in \mathcal{S}$, $\nu \in \mathcal{N}$ and $T \in \mathcal{T}$ let us denote:

$$\mathcal{T}(s) = \{T \in \mathcal{T};\ s \in T\} \quad \text{and} \quad \mathcal{T}(\nu) = \bigcap_n \mathcal{T}(\nu|n),$$

$$T(\nu) = \infty \quad \text{if} \quad \nu|i \in T \text{ for every } \ i \in \mathbb{N} \quad \text{and}$$

$$T(\nu) = \min\{i;\ \nu|i \notin T\} \quad \text{otherwise.}$$

We put $\mathcal{P} = \{T \in \mathcal{T};\ \exists\, \nu \in \mathcal{N}\colon T(\nu) = \infty\}$ the set of ill-founded trees, $\mathcal{L} = \mathcal{T} \setminus \mathcal{P}$ the set of well-founded trees, $\mathcal{M} = \{T \in \mathcal{T};\ \exists!\, \nu \in \mathcal{N}\colon T(\nu) = \infty\ \}$, and, finely, $B_k(T) = \{s;\ s \in T\ \ \&\ \ |s| \leq k\}$. The family $\{\mathcal{T}(s);\ s \in \mathcal{S}\} \cup \{\mathcal{T} \setminus \mathcal{T}(s);\ s \in \mathcal{S}\}$ is a countable subbasis of topology of $\mathcal{T}$.

Lemma 4. *The spaces $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{L} \cup \mathcal{M}$ are coanalyticly hard and they are co-Suslin subsets of $\mathcal{T}$.*

Proof. Let us denote $F = \{(\nu, T) \in \mathcal{N} \times \mathcal{T};\ T(\nu) = \infty\}$ and $\pi\colon \mathcal{T} \times \mathcal{N} \longrightarrow \mathcal{T}$ be the projection. The set F is obviously closed in the space $\mathcal{T} \times \mathcal{N}$. Since $\pi(F) = \mathcal{P} = \mathcal{T} \setminus \mathcal{L}$ and the spaces $\mathcal{T}$ and $\mathcal{T} \times \mathcal{N}$ are Polish, the set $\mathcal{L}$ is co-Suslin, [**K**, Chapter 3, §39, II, p. 493].

Denote $f = \pi \restriction F$. Then $\mathcal{M} = \big\{T \in \mathcal{T};\ \operatorname{card}\big(f^{-1}(T)\big) = 1\big\}$ is co-Suslin in space $\mathcal{T}$, [**K**, Chapter 3, §39, VII, p. 504]. The set $\mathcal{L} \cup \mathcal{M}$ is the union of two co-Suslin sets, hence it is co-Suslin.

For every co-Suslin subset E of a Polish space P, exists a upper semicontinuous mapping $f\colon P \longrightarrow \mathcal{T}$ such that $f^{-1}(\mathcal{P}) = P \setminus E$, [**D**, p. 239]. Since mapping f is Borel measurable and $f^{-1}(\mathcal{L}) = E$, the space $\mathcal{L}$ is coanalyticly hard.

Let us define a continuous mapping $H\colon \mathcal{T} \longrightarrow \mathcal{T}$ by

$$H(T) = \big\{(2, s);\ s \in T\big\} \cup \bigcup_{i \in \mathbb{N}} \mu|i,$$

where $\mu = (1, 1, \dots)$. It holds that $\mathcal{L} = H^{-1}(\mathcal{M})$ and, moreover, $H(\mathcal{T}) \subset \mathcal{P}$, hence $H^{-1}(\mathcal{M} \cup \mathcal{L}) = H^{-1}(\mathcal{M}) = \mathcal{L}$. Since $\mathcal{L}$ is coanalyticly hard, both sets $\mathcal{M}$ and $\mathcal{L} \cup \mathcal{M}$ are coanalyticly hard. □

Let us define a mapping $\Phi \colon \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \longrightarrow \mathcal{T}$ by

$$\Phi(f) = \Big\{ s \in \mathcal{S};\ \big\| f \restriction \mathcal{N}(s) \big\| = \|f\| \Big\}.$$

Obviously $\Phi(f) \in \mathcal{T}$ for $f \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$.

Lemma 5. *The mapping Φ is Borel measurable of the first class.*

Proof. Since the family $\{\mathcal{T}(s);\ s \in \mathcal{S}\} \cup \{\mathcal{T} \setminus \mathcal{T}(s);\ s \in \mathcal{S}\}$ forms a countable subbasis of topology $\mathcal{T}$, it is sufficient to prove that, for every $s \in \mathcal{S}$, the set $\Phi^{-1}(\mathcal{T}(s))$ is closed. Let $f_n \in \Phi^{-1}(\mathcal{T}(s))$ and $f_n \rightrightarrows f$. For arbitrary $\varepsilon > 0$, there exists an $i \in \mathbb{N}$ such that $\|f_i - f\| < \frac{\varepsilon}{3}$. Since, for every $n \in \mathbb{N}$, $\big\| f_n \restriction \mathcal{N}(s) \big\| = \|f_n\|$, there exists a $\mu \in \mathcal{N}(s)$ such that $\|f_i\| - \big|f_i(\mu)\big| < \frac{\varepsilon}{3}$. Hence

$$\Big| |f(\mu)| - \|f\| \Big| \le \Big| |f(\mu)| - |f_i(\mu)| \Big| + \Big| |f_i(\mu)| - \|f_i\| \Big| + \Big| \|f_i\| - \|f\| \Big| < 2\|f_i - f\| + \frac{\varepsilon}{3} < \varepsilon.$$

For every $\varepsilon > 0$, we found a $\mu \in \mathcal{N}(s)$ such that $\|f\| - \big|f(\mu)\big| < \varepsilon$. It means that $\big\| f \restriction \mathcal{N}(s) \big\| = \|f\|$, and $f \in \Phi^{-1}(\mathcal{T}(s))$. □

Proposition 6. *The sets $\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$ and $\boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$ are co-Suslin in the space $\mathcal{C}_b(\mathcal{N}, \mathbb{R})$.*

Proof. It is easy to see that $\Phi^{-1}(\mathcal{M}) = \boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$ because $\big|f(\mu)\big| = \|f\|$ if and only if $\Phi(f)(\mu) = \infty$. The mapping is Borel measurable and $\mathcal{T} \setminus \mathcal{M}$ is Suslin. So

$$\Phi^{-1}(\mathcal{T} \setminus \mathcal{M}) = \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \setminus \Phi^{-1}(\mathcal{M}) = \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \setminus \boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$$

is Suslin in $\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$.

The spaces $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}$ are homeomorphic, let $\varphi \colon \mathcal{N} \longrightarrow \mathcal{N} \times \mathcal{N}$ be a homeomorphism. Let us denote $D = \{(\nu, \nu) \in \mathcal{N} \times \mathcal{N}; \nu \in \mathcal{N}\}$. As $\mathcal{N} \setminus \varphi^{-1}(D)$ is an open set in $\mathcal{N}$, the spaces $\mathcal{N}$ and $\mathcal{N} \setminus \varphi^{-1}(D)$ are homeomorphic, let $\psi \colon \mathcal{N} \longrightarrow \mathcal{N} \setminus \varphi^{-1}(D)$ be a homeomorphism.

Now, let us define a continuous mapping $F_1 \colon \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \longrightarrow \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ by

$$F_1(f)(\nu) = \Big| f \circ \pi_1 \circ \varphi \circ \psi(\nu) - f \circ \pi_2 \circ \varphi \circ \psi(\nu) \Big|,$$

where π_1 is the projection on the first coordinate and π_2 on the second coordinate of the space $\mathcal{N} \times \mathcal{N}$.

It is easy to see that a function $f \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ is injective if and only if the function $F_1(f)$ does not attain zero. Moreover, for every $f \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$, $\inf_{\nu \in \mathcal{N}} F_1(f)(\nu)$

$= 0$ because there exist $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ and $\mu \in \mathcal{N}$ such that $\mu_n \neq \mu$ and $\mu_n \to \mu$. For $\nu_n = \psi^{-1}(\varphi^{-1}(\mu, \mu_n))$, it is $F_1(f)(\nu_n) = \Big|f(\mu) - f(\mu_n)\Big| \to 0$.

Let us define a continuous mapping $F_1'\colon \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \longrightarrow \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ by $F_1'(f)(\nu) = \|f\| - f(\nu)$ and denote $F_2 = F_1' \circ F_1$. A function $f \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ is injective if and only if the function $F_2(f)$ does not attain its norm. Hence $F_2^{-1}(\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})) = \boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$ and because F_2 is a continuous mapping, the set

$$F_2^{-1}\Big(\mathcal{C}_b(\mathcal{N}, \mathbb{R}) \setminus \boldsymbol{M}_c(\mathcal{N}, \mathbb{R})\Big) = \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \setminus \boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$$

is Suslin. □

Let us define a mapping $\Theta\colon \mathcal{T} \longrightarrow \mathbb{R}^{\mathcal{N}}$. Given $T \in \mathcal{T}$ and $\nu \in \mathcal{N}$ put

$$\begin{aligned} &\Theta(T)(\nu) = 2^{-T(\nu)} && \text{if } T(\nu) < \infty \text{ and} \\ &\Theta(T)(\nu) = 0 && \text{otherwise.} \end{aligned}$$

The mapping Θ is obviously injective. For $S, T \in \mathcal{T}$, $S \neq T$, there exists a $\nu \in \mathcal{N}$ so that $S(\nu) \neq T(\nu)$, thus $\Theta(S)(\nu) \neq \Theta(T)(\nu)$.

Lemma 7. *For every $T \in \mathcal{T}$, $\Theta(T) \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$. The mapping Θ is Borel measurable of the first class.*

Proof. Let $T \in \mathcal{T}$ and $\nu, \nu_n \in \mathcal{N}$, $\nu_n \to \nu$, denote $f = \Theta(T)$. For any $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $\nu_n|k = \nu|k$ for $n \geq n_0$. If $T(\nu) \leq k$, then $f(\nu_n) = f(\nu)$. If $T(\nu) > k$ or $T(\nu) = \infty$, then $f(\nu_n) < 2^{-k}$ and $f(\nu) < 2^{-k}$. Therefore in both cases we have $\big|f(\nu_n) - f(\nu)\big| < 2 \cdot 2^{-k}$ for $n \geq n_0$. Hence $f(\nu_n) \to f(\nu)$ and $f \in \mathcal{C}_b(\mathcal{N}, \mathbb{R})$.

Now we show that $\Theta(\mathcal{T})$ is separable. Let $T \in \mathcal{T}$, $k \in \mathbb{N}$ be arbitrary. For every $\nu \in \mathcal{N}$ it holds that $0 \leq \Theta\big(B_k(T)\big)(\nu) - f(\nu) \leq 2^{-k-1}$. Thus $\Theta(\mathcal{R})$ is dense in $\Theta(\mathcal{T})$, where $\mathcal{R} = \{T \in \mathcal{T};\ \exists\, n \in \mathbb{N}\colon \forall\, s \in T\colon |s| \leq n\}$ is a countable set.

Since $\Big\{\mathcal{U}\left(\Theta(T), 2^{-k}\right);\ T \in \mathcal{R}, k \in \mathbb{N}\Big\}$ is a countable basis of $\Theta(\mathcal{T})$, it is obviously sufficient to prove that, for every $T \in \mathcal{T}$ and $k \in \mathbb{N}$, $\Theta^{-1}\Big(\mathcal{U}\left(\Theta(T), 2^{-k}\right)\Big)$ is closed. Denote $f = \Theta(T)$, $U = \mathcal{U}\left(f, 2^{-1-k}\right)$ and

$$\begin{aligned} A &= \Big\{S \in \mathcal{T};\ B_k(S) = B_k(T)\Big\} \\ &= \Big\{S \in \mathcal{T};\ \forall\, \nu \in \mathcal{N}\ \ \forall\, j \leq k\colon \nu|j \in S \iff \nu|j \in T\Big\}. \end{aligned}$$

A tree $S \in \mathcal{T}$ belongs to A if and only if, for every $\nu \in \mathcal{N}$, $\min\big(k, S(\nu) - 1\big) = \min\big(k, T(\nu) - 1\big)$. This is equivalent to $\Big|2^{k-S(\nu)} - 2^{k-T(\nu)}\Big| < \frac{1}{2}$ and hence also to

$$\Big|\Theta(S)(\nu) - \Theta(T)(\nu)\Big| = \Big|2^{-S(\nu)} - 2^{-T(\nu)}\Big| < 2^{-1-k}.$$

Both happens if and only if $S \in \Theta^{-1}(U)$. Thus $A = \Theta^{-1}(U)$.

It remains to prove that the set A is closed. Let $S_n \to S$ and $B_k(S_n) = B_k(T)$. We will prove that $B_k(S) = B_k(T)$. Let $s \in \mathcal{S}$ be so that $|s| \le k$. If $s \in S$, then, for some $n \in \mathbb{N}$, it is $s \in S_n$, hence $s \in T$. Conversely, if $s \in T$, then, for every $n \in \mathbb{N}$, it is $s \in S_n$, hence $s \in S$. That means $B_k(S) = B_k(T)$. ☐

Proposition 8. *The spaces $\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$ and $\boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$ are coanalyticly hard.*

Proof. Let us define an injective Borel measurable mapping $\Psi_1 : \mathcal{T} \longrightarrow \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ by $\Psi_1(T)(\nu) = 1 - 2^{-\nu^1}\Theta(T)(\nu)$ for $\nu = (\nu^1, \nu^2, \dots) \in \mathcal{N}$. It is $\|\Psi_1(T)\| = 1$ for every $T \in \mathcal{T}$. Because $\Psi_1(T) \in \boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$ if and only if $T \in \mathcal{M}$, it holds that

$$\boldsymbol{M}_c(\mathcal{N}, \mathbb{R}) \cap \Psi_1(\mathcal{T}) = \Psi_1(\mathcal{M}) \quad \text{and} \quad \Psi_1^{-1}(\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})) = \mathcal{M}.$$

The space $\mathcal{M}$ is coanalyticly hard, hence $\boldsymbol{M}_c(\mathcal{N}, \mathbb{R})$ is coanalyticly hard.

Let $\varphi\colon \mathcal{N} \longrightarrow (1, 2)$ be an injective continuous function. Let us define an injective Borel measurable mapping $\Psi_2\colon \mathcal{T} \longrightarrow \mathcal{C}_b(\mathcal{N}, \mathbb{R})$ by $\Psi_2(T)(\nu) = \varphi(\nu)\Theta(T)(\nu)$. Let $T \in \mathcal{T}$ be an arbitrary tree and denote $f = \Psi_2(T)$.

We show that if a function f attains zero in at most one point, then f is injective. It means that a function $\Psi_2(T)$ is injective if and only if $T \in \mathcal{L} \cup \mathcal{M}$. For suppose not. Then there exist sequences $\mu, \nu \in \mathcal{N}$ such that $\mu \ne \nu$ and $f(\mu) = f(\nu) \ne 0$. Since $f(\mu) \ne 0 \ne f(\nu)$ it must be $\Theta(T)(\mu) \ne 0 \ne \Theta(T)(\nu)$. Moreover φ is injective, it means that $\Theta(T)(\mu) \ne \Theta(T)(\nu)$. Thus there exist $i, j \in \mathbb{N}$ so that $i \ne j$, $\Theta(T)(\mu) = 2^{-i}$ and $\Theta(T)(\nu) = 2^{-j}$, hence $f(\mu) \in (2^{-i}, 2^{1-i})$ and $f(\nu) \in (2^{-j}, 2^{1-j})$, which are two disjoint intervals, but $f(\mu) = f(\nu)$.

This means that $\Psi_2^{-1}\big(\boldsymbol{I}_c(\mathcal{N}, \mathbb{R})\big) = \mathcal{L} \cup \mathcal{M}$. Thus the space $\boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$ is coanalyticly hard. ☐

Proposition 9. *Let M and L be absolute Borel, separable, uncountable spaces. Then the spaces $\boldsymbol{M}_{\mathcal{B}}(M, \mathbb{R})$ and $\boldsymbol{I}_{\mathcal{B}}(M, L)$ are coanalyticly hard. Thus the sets $\boldsymbol{M}_{\mathcal{B}}(M, \mathbb{R})$ and $\boldsymbol{I}_{\mathcal{B}}(M, L)$ are not Suslin subsets of $\mathcal{B}_b(M, \mathbb{R})$ and $\mathcal{B}_b(M, L)$.*

Proof. Let $M_1 \subset M$ and $L_1 \subset L$ be some countable sets, $M_2 \subset M \setminus M_1$ and $L_2 \subset L \setminus L_1$ be sets of points from the sets $M \setminus M_1$ and $L \setminus L_1$ which are condensation points of M and L. There exist injective continuous mappings $\varphi_1 : \mathcal{N} \longrightarrow M_2$, $\psi_1 : \mathcal{N} \longrightarrow L_2$ and $g\colon \mathcal{N} \longrightarrow \mathbb{R}$ so that $\varphi_1(\mathcal{N}) = M_2$, $\psi_1(\mathcal{N}) = L_2$ and $g(\mathcal{N}) = \mathbb{R}$, Let $D \subset \mathcal{N}$ be a countable closed set. The spaces $\mathcal{N}$ and $\mathcal{N} \setminus D$ are homeomorphic; let $\eta\colon \mathcal{N} \setminus D \longrightarrow \mathcal{N}$ be a homeomorphism. Denote $\varphi = \varphi_1 \circ \eta$ and $\psi = \psi_1 \circ \eta$. As the sets $M \setminus M_2$ and $L \setminus L_2$ are countable, we can define the mapping φ and ψ on the set D such that $\varphi(\mathcal{N}) = M$, $\psi(\mathcal{N}) = L$. Then the mappings φ and ψ are Borel measurable. As φ^{-1} and g^{-1} are Borel measurable ([**K**, Chapter 3, §39, V, Theorem 3, p. 500]), we can define an injective Borel measurable mapping $F\colon \mathcal{C}_b(\mathcal{N}, \mathbb{R}) \longrightarrow \mathcal{B}_b(M, L)$ by $F(f) = \psi \circ g^{-1} \circ f \circ \varphi^{-1}$. As the mappings φ^{-1}, g^{-1} and ψ are injective, $F^{-1}(\boldsymbol{I}_{\mathcal{B}}(M, L)) = \boldsymbol{I}_c(\mathcal{N}, \mathbb{R})$, and it means that the space $\boldsymbol{I}_{\mathcal{B}}(M, L)$ is coanalyticly hard.

For $\boldsymbol{M}_{\mathcal{B}}(M,\mathbb{R})$ we define the mapping F by $F(f) = f\circ\varphi^{-1}$. Again, it holds that $F^{-1}(\boldsymbol{M}_{\mathcal{B}}(M,L)) = \boldsymbol{M}_{\mathcal{C}}(\mathcal{N},\mathbb{R})$, and the space $\boldsymbol{M}_{\mathcal{B}}(M,\mathbb{R})$ is coanalyticly hard. $\square$

This paper was written on base of my thesis. I would like to thank my supervisor Petr Holický for his helpful comments.

References

[**D**] Dellacherie C., *Un cours sur les enumbles analytiques*, Analytic Sets, Academic press, 1980, pp. 182–316.

[**DS**] van Douwen E. K. and Stone A. H., *The topology of close approximation*, Topology and its Applications **35** (1990), 261–275.

[**F**] Frolík Z., *A Measurable Map with Analytic Domain and Metrizable Range is Quotient*, Bulletin of the American Mathematical Society **76** no. 5, 1112–1117.

[**JR**] Rogers C. A. and Jayne J. E., *K-analytic Sets*, Analytic Sets, Academic press, 1980, pp. 1–181.

[**K**] Kuratovskij K., *Topologija*, Izdavatelstvo Mir, Moskva, 1966. (Russian)

[**KL**] Kechris A. S. and Louveau A., *Descriptive Set Theory and Sructure of Sets of Uniqueness*, London Math. Soc. Lecture Note Ser., vol. 128, Cambridge Univ. press, 1987.

[**S**] Sierpinski W., *Sur les images continues et biunivoques de l'ensemble de tous les nombres irrationnels*, Mathematica (Cluj) **1** (1929), 18–21.

M. Šmídek, Department of Mathematical Analysis, Charles University, Sokolovsk´ 83, 186 00 Praha, Czech Republic