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ON DECOMPOSABILITY OF NAMBU–POISSON TENSOR

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Abstract. In this paper we find some interesting algebraic structure of Nambu Poisson manifold and also we prove Takhtajan's conjecture that Nambu-Poisson tensor which defines Nambu bracket in Nambu mechanics is decomposable.

1. Introduction

Nambu mechanics is a natural generalization of Hamiltonian mechanics [**1, 2, 3, 4**]. It is defined by Nambu bracket, $\mathbb{R}$-multilinear completely antisymmetric operation $\{f_1,\dots,f_m\}$ in the space $C^\infty(M)$ of functions on a manifold M, which generalizes the bilinear Poisson bracket $\{f_1,f_2\}$. Any $m-1$ functions $H_1,\dots H_{m-1}\in C^\infty(M)$ (Nambu-Hamiltonians) determine a Nambu-Hamiltonian flow

$$\frac{df}{dt}=\{f,H_1,\dots,H_{m-1}\}$$

on the manifold M. The Jacobi identity for Poisson bracket is replaced by fundamental (or generalized Jacobi) identity which states that a Nambu-Hamiltonian flow preserves the Nambu bracket.

An example of Nambu bracket is the canonical Nambu bracket on $M=\mathbb{R}^m$ with the standard coordinates $x_1,\dots,x_m$ given by

$$\{f_1,\dots,f_m\}=\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)},$$

where the right hand side stands for the Jacobian of the mapping

$$\tilde f=(f_1,\dots,f_m):\mathbf{R}^m\longmapsto\mathbf{R}^m.$$

It is clear from the definition of Nambu bracket that it contains an infinite family of "subordinated" Nambu structure of lower degree, including Poisson structure. Fundamental identity imposes strong condition on the possible form of Nambu bracket, hence the structure of Nambu bracket is more rigid than Poisson bracket. In addition to quadratic differential equations, it also satisfy an overdetermined system of quadratic algebraic equations for Nambu bracket tensor.

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We prove than in fact any Nambu bracket is locally isomorphic to the canonical Nambu bracket of the above example, as it was conjectured by L. Takhtajan [**5**].

Let us begin with the definition of Nambu-Poisson manifold.

Definition 1. Let M be a smooth finite n-dimensional manifold with algebra of functions $C^\infty(M)$ and Lie algebra of vector fields $\chi(M)$. M is called Nambu-Poisson manifold if there exists a multi-linear map

$$X : [C^\infty(M)]^{\otimes(m-1)} \longrightarrow \chi(M)$$

$\forall f_1, f_2, \dots, f_{2m-1} \in C^\infty(M)$,

$$(f_1, \dots, f_{m-1}) \longmapsto X_{f_1,\dots,f_{m-1}}.$$

such that the bracket defined by

$$\{f, f_1, \dots, f_{m-1}\} := X_{f_1\dots f_{m-1}} f$$

is skew symmetric in all arguments and is invariant under any Hamiltonian vector fields $X = X_{f_1\dots f_m}$, i.e.

$$X\{g_1, \dots, g_m\} = \{Xg_1, \dots, g_m\} + \dots + \{g_1, \cdots, Xg_m\}. \tag{1}$$

Similar to a Poisson structure, Nambu-Poisson structure is defined by a m-multivector

$$P = P^{i_1,\dots,i_m} \in \Gamma(\wedge^m TM)$$

by

$$\begin{aligned} X_{f_1,\dots,f_{m-1}} f &= \{f, f_1, \dots, f_{m-1}\} = P(df, df_1, \dots df_{m-1}) \\ &= P^{i_0 i_1,\dots i_{m-1}} \partial_{i_0} f \partial_{i_1} f_1 \dots \partial_{i_{m-1}} f_{m-1}, \end{aligned}$$

where $(x_1, \dots, x_m)$ are local coordinates and $\partial_{i_n} := \frac{\partial}{\partial x^n}$.

The equation (1) means that the bracket $\{f_1, \dots, f_{m-1}, f_m\}$ satisfies the following fundamental identity:

$$\begin{aligned} &\{\{f_1, \dots, f_{m-1}, f_m\}, f_{m+1}, \dots, f_{2m-1}\} \\ &\quad + \{f_m, \{f_1, \dots f_{m-1}, f_{m+1}\}, f_{m+2}, \dots, f_{2m-1}\} \\ &\quad + \dots + \{f_m, \dots, f_{2m-2}, \{f_1, \dots, f_{m-1}, f_{2m-1}\}\} \\ &= \{f_1, \dots, f_{m-1}, \{f_m, \dots, f_{2m-1}\}\}. \end{aligned} \tag{2}$$

Incidentally Takhtajan has written fundamental identity in this form.

Takhtajan [**5**] proved that the fundamental identity (2) is equivalent to the following differential and algebraic constraint equations of Nambu-Poisson tensor $P^{i_1,\cdots,i_m}(x)$:

$$\sum_{k=1}^{m}\Big(P^{ki_2\dots i_m}\frac{\partial P^{j_1\cdots j_m}}{\partial x_k}+P^{j_m k i_3\dots i_m}\frac{\partial P^{j_1 j_2 j_3\cdots j_{m-1} i_2}}{\partial x_k} + \dots + P^{j_m i_2\dots i_{m-1}k}\frac{\partial P^{j_1\cdots j_{m-1} i_m}}{\partial x_k}\Big) = \sum_{k=1}^{M} P^{j_1 j_2\cdots j_{m-1}k}\frac{\partial P^{j_m i_2\dots i_m}}{\partial x_k}, \tag{3}$$

for all $i_2,\dots,i_m,\, j_1,\dots,j_m = 1,\cdots,n$, and

$$S_{ij}+\mathcal{P}(S_{ij})=0, \tag{4}$$

where

$$S_{ij}=P^{i_1\cdots i_m}P^{j_1\cdots j_m}+P^{j_m i_1 i_3\cdots i_{m-1}}P^{j_1\cdots j_{m-1} i_2} + \dots + P^{j_m i_2\dots i_{m-1}}P^{j_1\cdots j_{m-1} i_1} - P^{j_m i_2\cdots}P^{j_1\cdots j_{m-1} i_1} \tag{5}$$

and $\mathcal{P}$ is the permutation operator which interchanges the indices i_1 and j_1 of $2m$ dimensional tensor S. He proved that any decomposable multivector

$$P = X_1\wedge\ldots\wedge X_m, \qquad X_i\in\chi(M),$$

whose support is an integrable distribution, satisfies these constraints and hence defines a Poisson-Nambu tensor and conjectured that any multivector P which satisfies the algebraic equation (4) is decomposable. To prove this conjecture we reformulate (4) in coordinate free way.

Earlier Larry Lambe using symbolic computations technique varified in some cases the decomposability of Nambu tensor. Anyway, before leaving this section let us de-emphasised the main slogan of Takhtajan:

Conjucture 2. *Any Nambu-Poisson tensor $P\in\Gamma(\wedge^m TM)$ for $m>2$ is decomposable.*

Notation. In this paper we shall denote wedge product by $\wedge$ and symmetric product by $\vee$.

2. Reformulation of Fundamental Identity

Now we write the algebraic Takhtajan identity (4) for multivector P in a point $o\in M$ in coordinate free way. Let us denote by $V=T_oM$ the tangent space at the point o and by

$$P_\eta = \langle P,\eta\rangle \in \wedge^{m-k}V$$

result of the natural pairing between a multivector $P\in\wedge^m V$ and k-form $\eta\in\wedge^k V^*$, $k\le m$.

Lemma 3. *The algebraic Takhtajan identity* (4) *for m-multivector $P \in \wedge^m V$ is equivalent to the identity*

$$\sum_{i=1}^{m} (P_{\alpha\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\beta\wedge\phi} + P_{\beta\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\alpha\wedge\phi}) = 0,$$

for any $\alpha, \beta, \eta_1, \ldots \eta_m \in V^*$, $\phi \in \wedge^{m-2}V^*$, $\eta = \eta_1 \wedge \ldots \wedge \eta_m$, where

$$\partial_{\eta_i}\eta = (-1)^{i-1}\eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_m.$$

Proof. Given any 1-forms $\alpha, \beta, \xi^2, \ldots, \xi^{m-1}$, $\eta^1, \ldots, \eta^m \in V^* = T_0^* M$, we choose functions $f_1, \ldots, f_{m-1}$, $g_1, \ldots, g_m$ such that

$$\begin{aligned} \eta^i = dg_i|_0, \qquad d^2 g_i|_0 = 0, \qquad \xi^j = df_j|_p, \quad j > 1, \\ d^2 f_i|_0 = 0 \quad \forall\, i > 0, \qquad df_1|_0 = 0, \\ d^2 f_1|_0 = \alpha \otimes \beta + \beta \otimes \alpha = \alpha \vee \beta. \end{aligned}$$

The fundamental identity can be written as

$$\begin{aligned} (6) \qquad & X \cdot P(dg_1, \ldots, dg_m) \\ &= P(d(X \cdot g_1), dg_2, \ldots, dg_m) + \cdots + P(dg_1, \ldots, d(X \cdot g_m)) \\ &= P(d(X \cdot g_1), dg_2, \cdots, dg_m) - P(d(X \cdot g_2), dg_1, \hat{d}g_2, \ldots dg_m) \\ &\quad + P(d(X \cdot g_3), dg_1, dg_2, \hat{d}g_3, \ldots, dg_m) \\ &\quad + \cdots + (-1)^{m-1} P(d(X \cdot g_m), dg_1, \cdots, \hat{d}g_m) \end{aligned}$$

where

$$X \cdot g_i = P(dg_i, df_1, \ldots, df_{m-1}).$$

Hence we obtain

$$\begin{aligned} (7) \qquad d(X \cdot g_1)|_0 &= dP(dg_1, df_1, \ldots, df_{m-1})|_0 = P(\eta^1, \alpha \vee \beta, \xi^2, \cdots, \xi^{m-1}) \\ &= P(\eta^1, \alpha, \xi^2, \cdots, \xi^{m-1})\beta + P(\eta^1, \beta, \xi^2, \ldots, \xi^{m-1})\alpha \\ &= P_{\eta^1\wedge\alpha\wedge\phi}\beta + P_{\eta^1\wedge\beta\wedge\phi}\alpha, \end{aligned}$$

where $\phi = \xi^2 \wedge \ldots \wedge \xi^{m-1}$. Similarly we get,

$$\begin{aligned} (8) \qquad d(X \cdot g_i)|_0 &= P(dg_i, d^2 f_1, df_2, \ldots, df_{m-1})|_0 \\ &= P(\eta^i, \alpha \vee \beta, \xi^2, \ldots, \xi^{m-1}) = P_{\eta^i\wedge\alpha\wedge\phi}\beta + P_{\eta^i\wedge\beta\wedge\phi}\alpha. \end{aligned}$$

Taking into account that $X|_0 = 0$ we obtain

$$
\begin{aligned}
0 &= P(\alpha, dg_2, \dots, dg_m)|_0 P_{\eta^1\wedge\beta\wedge\phi} + P(\beta, dg_2, \dots, dg_m)|_0 P_{\eta^1\wedge\alpha\wedge\phi} \\
&\quad + \dots + P(dg_1, \dots, dg_{i-1}, \alpha, dg_{i+1}, \dots, dg_m)|_0 P_{\eta^i\wedge\beta\wedge\phi} \\
&\quad + P(dg_1, \dots, dg_{i-1}, \beta, dg_{i+1}, \dots, dg_m)|_0 P_{\eta^i\wedge\alpha\wedge\phi} + \cdots \\
&= P_{\alpha\wedge\eta^2\wedge\ldots\wedge\eta^m} P_{\eta^1\wedge\beta\wedge\phi} + P_{\beta\wedge\eta^2\wedge\ldots\wedge\eta^m} P_{\eta^1\alpha\wedge\phi} \\
&\quad + \dots + (-1)^{i-1} P_{\alpha\wedge\eta^1\wedge\ldots\wedge\hat{\eta}^i\wedge\ldots\wedge\eta^m} P_{\eta^1\wedge\beta\wedge\phi} \\
&\quad + (-1)^{i-1} P_{\beta\wedge\eta^1\wedge\ldots\hat{\eta}^i\wedge\ldots\wedge\eta^m} P_{\eta^1\wedge\alpha\wedge\phi} + \cdots \\
&= \sum_{i=1}^{m} (P_{\alpha\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\beta\wedge\phi} + P_{\beta\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\alpha\wedge\phi}).
\end{aligned}
\tag{9}
$$

This proves the lemma. □

To rewrite the identity in more simple way we introduce the following Koszul type operator:

$$d : \wedge^m V \vee \wedge^m V \longrightarrow S^2 V \otimes \wedge^{m-2} V \otimes \wedge^m V,$$

by the formula

$$d(P \otimes P)(\alpha \vee \beta \otimes \phi) = P_\alpha \wedge P_{\beta\wedge\phi} + P_\beta \wedge P_{\alpha\wedge\phi},$$

for $\alpha, \beta \in V^*$ and $\phi \in \wedge^{m-2} V^*$. Here P_α denotes contraction of P by α etc. Hence P_α, P_β are $m-1$ multivectors and $P_{\beta\wedge\phi}$, $P_{\alpha\wedge\phi}$ are vectors.

Note that $d = 0$ for $m = 2$.

Let us make a remark that for a decomposable m polyvector $\eta = \eta_1 \wedge \dots \wedge \eta_m$ we have

$$
\begin{aligned}
d(P \otimes P)(\alpha \vee \beta \otimes \phi \otimes \psi) &= \langle (P_\alpha \wedge P_{\beta\wedge\phi} + P_\beta \wedge P_{\alpha\wedge\phi}), \psi \rangle \\
&= \sum_{i=1}^{m} (P_{\alpha\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\beta\wedge\phi} + P_{\beta\wedge\partial_{\eta_i}\eta} P_{\eta_i\wedge\alpha\wedge\phi}).
\end{aligned}
$$

Hence we have

Corollary 4. *A multivector $P \in \wedge^m V$ satisfies the algebraic Takhtajan identity iff*

$$d(P \otimes P) = 0.$$

3. Properties of Nambu-Poisson Operator

In fact d is composed of two operators d_1 and d_2,

$$d_1 : \wedge^m V \otimes \wedge^m V \longrightarrow S^2 V \otimes \wedge^{m-1} V \otimes \wedge^{m-1} V$$

and

$$d_2 : S^2V \otimes \wedge^{m-1}V \otimes \wedge^{m-1}V \longrightarrow S^2V \otimes \wedge^{m-2}V \otimes \wedge^m V,$$

defined by

$$d_1(P \otimes Q) = \sum e_k \vee e_l \otimes P_{e^k} \otimes Q_{e^l}$$

and

$$d_2(S \otimes P \otimes Q) = S \otimes \sum P_{e^k} \otimes e_k \wedge Q.$$

Here $S \in S^2V$ and $\{e_i\}$ is a basis of V and $\{e^i\}$ is the dual basis of V^*. Hence d is written as

$$d = d_2 \circ d_1.$$

Given any contravariant m-tensor $T \in V^{\otimes m}$ we will denote by supp T its support, that is subspace of V generated by contructions of T with all covariant $(m-1)$ tensors. Since the operator d is the sum of the permutations of tensor factors, we have

$$\operatorname{supp} d(P \vee Q) = \operatorname{supp}(P \vee Q)$$

for any $P, Q \in \wedge^m V$.

Let now $T \in V^{\otimes m}$ is a contravariant tensor and e is a non-zero vector. We say that T contains factor e with multiplicity k if any non-zero coordinate $T^{i_1,\cdots,i_m}$ of T with respect to a base $e_0 = e, e_1, \ldots, e_{n-1}$ of V has at least k zero indices and there is a coordinate which has exactly k zero indices. In other terms, T is decomposed as a linear combination of decomposable tensors $e_{i_1} \otimes \cdots \otimes e_{i_m}$ each of them has at least k-factors $e_0 = e$.

It is clear that this definition is correct (i.e. does not depend on the choice of the base $e_0 = e, \ldots, e_{n-1}$ and the multiplicity k of a factor e in T does not change after any transformation of T which is a linear combination of permutations.

Using this argument, we get:

Lemma 5. *Let $Q \in \wedge^{m-1}V$, $R \in \wedge^m V$ be multivectors and e is a vector such that $e \notin \operatorname{supp} Q + \operatorname{supp} R$. Then the multivector $P = e \wedge Q + R$ satisfies the equation $d(P \otimes P) = 0$ iff*

$$d(Q \otimes Q) = 0, \quad d((e \wedge Q) \vee R) = 0, \quad d(R \otimes R) = 0.$$

Proof. We know

$$d(P \otimes P) = d(e \wedge Q \otimes e \wedge Q) + d(e \wedge Q \vee R) + d(R \otimes R) = 0.$$

Since the summands contain the vector e with the multiplicity $2, 1$ and 0 respectively, these are linearly dependent only when they are identically zero.

It remains to prove now that

$$d(e \wedge Q \otimes e \wedge Q) = 0$$

implies $d(Q \otimes Q) = 0$. Let us consider a basis $e_1, \dots, e_k$ of $\operatorname{supp} Q = U$ and denote by $e^1, \dots, e^k$ the dual basis of U^*. Using definition of d, we obtain

$$\begin{aligned} 0 &= d(e \wedge Q \otimes e \wedge Q) \\ &= (e \otimes e) \otimes d_2(Q \otimes Q) - \sum_i (e \vee e_i) \otimes d_2(Q \vee (e \wedge Q_{e^i})) \\ &\quad + \sum_{i,j} (e_i \vee e_j) \otimes d_2((e \wedge Q_{e^i}) \vee (e \vee Q_{e^j})). \end{aligned}$$

Since tensors $e \otimes e$, $e \vee e_i$, $e_i \vee e_j$ are linearly independent, we have

$$\begin{aligned} 0 &= \sum_{i,j} (e_i \vee e_j) \otimes d_2((e \wedge Q_{e^i}) \vee (e \wedge Q_{e^j})) \\ &= \sum_{i,j} (e_i \vee e_j) \otimes (e \otimes e) \bar{\wedge} d_2(Q_{e^i} \vee Q_{e^j}) \\ &= (e \otimes e) \bar{\wedge} \sum_{i,j} (e_i \vee e_j) \otimes d_2(Q_{e^i} \vee Q_{e^j}) \\ &= (e \otimes e) \bar{\wedge} d(Q \otimes Q), \end{aligned}$$

where $\bar{\wedge}$ is the Kulkarni-Nomizu product in the space $\wedge V \otimes \wedge V$ of bimultivectors defined by

$$(A \otimes B) \bar{\wedge} (C \otimes D) = (A \wedge C) \otimes (B \wedge D),$$

for $A, B, C, D \in \wedge V$.

Note that $e \notin \operatorname{supp} d(Q \otimes Q) = \operatorname{supp} Q$. This implies that the operator of Kulkarni-Nomizu multiplication by $e \otimes e$ is non-degenerate and we obtain

$$d(Q \otimes Q) = 0\,. \qquad \square$$

Lemma 6. *Let $P = e_1 \wedge \dots \wedge e_m$ and $R = f_1 \wedge \dots \wedge f_m$ be decomposable non zero m-multivectors. Then*

$$d(P \vee R) = 0$$

iff R is proportional to $P : R = \lambda P$.

Proof. We can write $P = E \wedge P'$, $R = E \wedge R'$, where E, P', R' are decomposable multivectors and

$$\operatorname{supp} P' \cap \operatorname{supp} R' = 0\,. \tag{10}$$

Using the arguments as in the proof of Lemma 6, we assert $d(P \vee R) = 0$ implies $d(P' \vee R') = 0$. Suppose that deg $P' = \deg Q' = k > 0$. Then the we can write

$$P' = e'_1 \wedge \cdots \wedge e'_k, \qquad R' = f'_1 \wedge \cdots f'_k.$$

Condition (10) implies that the vectors $e'_1, \ldots, e'_k, f'_1, \ldots, f'_k$ are linearly independent. Then one can check immediately that

$$d(P' \vee R') \neq 0\,.$$

This contradiction shows $k = 0$ and $R = \lambda P$. □

4. Proof of Takhtajan's Conjecture

We want to prove the following:

Theorem 7. *A multivector $P \in \wedge^m V$, $m > 2$ satisfies the algebraic Takhtajan identity $d(P \otimes P) = 0$ iff it is decomposable, i.e. $P = e_1 \wedge \ldots \wedge e_m$, for some vectors $e_1, \ldots, e_m$.*

Proof. We will assume that supp $P = V$ and we will use method of induction on $n = \dim V$. Let $d(P \otimes P) = 0$ for $P \neq 0$ and $0 \neq e$ is a vector which belongs to supp P, choose $Q \in \wedge^{m-1} V$ and $R \in \wedge^{m-1} V$ such that

$$P = e \wedge Q + R, \qquad e \notin (\text{supp } Q + \text{supp } R) = V'$$

By Lemma 5,

$$d(Q \otimes Q) = 0, \qquad d(R \otimes R) = 0, \qquad d(e \wedge Q \vee R) = 0.$$

Since dim supp $Q < n$ and the dim supp $R < n$ by inductive conjecture we may assume that Q and R are decomposable. Then Lemma 6 shows that $R = 0$ and $P = e \wedge Q = e \wedge e_1 \wedge \ldots \wedge e_{m-1}$ is decomposable multivector. □

As a corollary we obtain the following local description of Nambu-Poisson tensors on a manifold M.

Corollary 8. *Let M be a Nambu-Poisson manifold with Nambu-Poisson tensor $P \in \Gamma(\wedge^m TM)$, $m > 2$. Assume that $P_x \neq 0$ for some point x. Then there exist a local coordinates $x_1, \ldots, x_n$ in a neighbourhood of x such that*

$$P = \partial_{x_1} \wedge \ldots \wedge \partial_{x_m}.$$

Proof. By Theorem 7, in some neighbourhood of the point x there exist a set of independant vector fields $X_1, \ldots, X_m$ such that $P = X_1 \wedge \ldots \wedge X_m$. It is sufficient to prove that m-dimensional distribution supp P generated by vector fields X_i is involutive. This follows from the facts that this distribution is generated also by all Nambu-Hamiltonian vector fields $X_{f_1,\dots,f_{m-1}}$ and that Nambu- Hamiltonian vector fields are closed under the Lie bracket. The last statement follows immediately from fundamental identity. □

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