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KNEADING THEORY FOR A FAMILY OF CIRCLE MAPS WITH ONE DISCONTINUITY

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Abstract. We apply the kneading theory techniques to a class of circle maps with one discontinuity and we characterize the rotation interval of a map in terms of the kneading sequences. As a consequence we obtain lower and upper bounds of the entropy depending on the rotation interval.

1. Introduction

We study the class $\mathcal{C}$ of maps $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ defined as follows (see Figure 1). We say that $F \in \mathcal{C}$ if:

(1) $F|_{(0,1)}$ is bounded, continuous and non-decreasing.
(2) $\lim\limits_{x\uparrow 1} F(x) > \lim\limits_{x\downarrow 1} F(x)$.
(3) $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$.

For a map $F \in \mathcal{C}$ and for each $a \in \mathbf{Z}$ we set $F(a^+) = \lim_{x\downarrow a} F(x)$ and $F(a^-) = \lim_{x\uparrow a} F(x)$. In view of (3) we have $F(a^+) = F(0^+) + a$ and $F(a^-) = F(0^-) + a$. Note that the exact value of $F(0)$ is not specified. Then in what follows we consider that $F(0)$ is either $F(0^+)$ or $F(0^-)$, or both, as necessary.

Since every map $F \in \mathcal{C}$ has a discontinuity in each integer, the class $\mathcal{C}$ can be considered as a family of liftings of circle maps with one discontinuity.

The maps of class $\mathcal{C}$ appear in a natural way in the study of many branches of dynamics. The simplest example of such maps is the family $x \to \beta x + \alpha$, which plays an important role in ergodic theory (see [**H**]). The case $\alpha = 0$ gives the famous β-transformations (see [**R**]). Also, the class $\mathcal{C}$ contains the class of the Lorenz-Like maps which has been studied by several authors (see [**ALMT**], [**G**], [**GS**], [**Gu**], [**HS**], [**S**]).

The aim of this paper is to extend the kneading theory developed in [**AM**] for continuous maps of the circle of degree one to class $\mathcal{C}$, to obtain a characterization of the rotation interval of a map in terms of its kneading sequences. From this characterization we shall obtain models with maximum and minimum entropy

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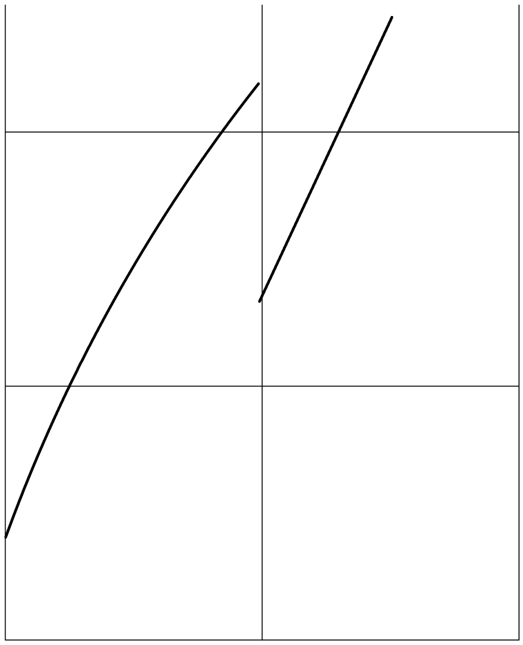

Figure 1. An example of a map of class $\mathcal{C}$.

and, hence, lower and *upper* bounds of the topological entropy depending on the rotation interval. The lower bounds of the topological entropy for this class of maps were already known (see [**ALMT**]). Here we give a different proof.

To extend the kneading theory to our class of maps we note that it is closely related to the class $\mathcal{A}'$ defined as follows. We say that $F \in \mathcal{A}'$ if (see Figure 2) :

(1) $F \in \mathcal{C}(\mathbf{R}, \mathbf{R})$ and $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$.
(2) There exists $c_F \in (0,1)$, such that F is non-decreasing in $[0, c_F]$ and non-increasing in $[c_F, 1]$.
(3) $F(c_F) > F(1)$.

To show the relation between maps from class $\mathcal{C}$ and $\mathcal{A}'$ take $F \in \mathcal{C}$ and for each $\mu > 0$ let $c_\mu \in (0,1)$ be such that $F(c_\mu) = \mu(1 - c_\mu) + F(1^+)$. Also let F_μ be the continuous map defined as follows (see Figure 3):

(1) $F_\mu|_{(0,c_\mu]} = F$,
(2) $F_\mu(x) = \mu(1-x) + F(1^+)$ for all $x \in [c_\mu, 1)$

Clearly for all $\mu > 0$, $F_\mu \in \mathcal{A}'$, $\lim_{\mu\to\infty} c_\mu = 1$ and $F(x) = \lim_{\mu\to\infty} F_\mu(x)$. In other words each map of $\mathcal{C}$ is a pointwise limit of maps from $\mathcal{A}'$.

The class $\mathcal{A}'$ contains the class $\mathcal{A}$ of those maps which satisfy the statement (2) of the definition of $\mathcal{A}'$ with strict monotonicity. In [**AM**] a kneading theory for maps from class $\mathcal{A}$ was developed. It is an easy exercise to extend this kneading

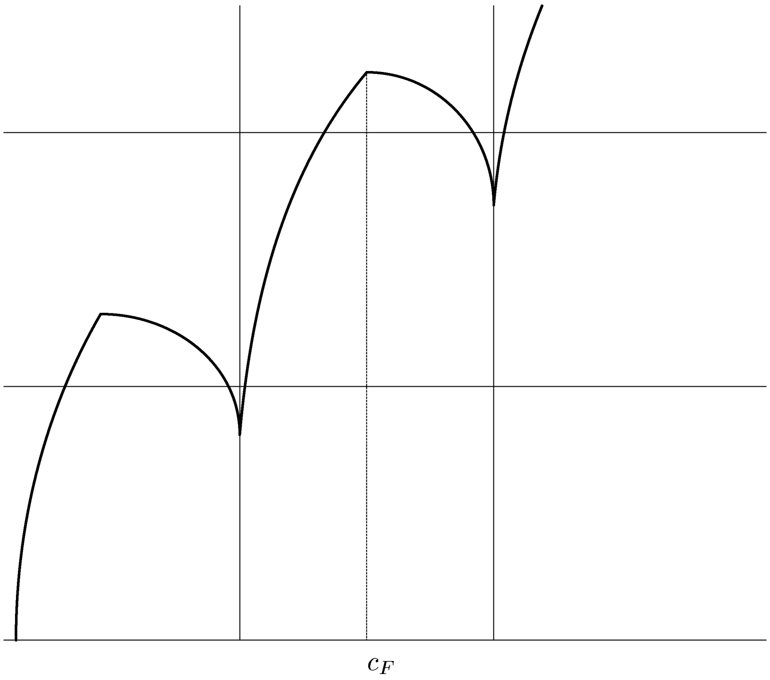

Figure 2. A map of class $\mathcal{A}'$.

theory and all the results of [**AM**] to the class $\mathcal{A}'$. To study the class $\mathcal{C}$ we shall use without proof the results from [**AM**] for class $\mathcal{A}'$. Most of the results we shall state for class $\mathcal{C}$ are also trivial extensions of the corresponding ones in the continuous case. Thus we shall also omit their proofs. However, this paper is an extension of [**AM**]. Therefore, to understand the proofs and details of this paper it is necessary to know the general theory developed in [**AM**].

The notions of periodic (mod. 1) point, rotation number, rotation interval, lap, growth number and entropy extend naturally to class $\mathcal{C}$ (see [**AM**] for a review of these notions). From [**M**] it follows that the rotation interval has the same properties as in the continuous case. We shall use the same notation as in [**AM**]. Thus, if $F \in \mathcal{C}$, L_F denotes the rotation interval of F, $s(F)$ the growth number of F and $h(F) = \log s(F)$ the topological entropy of F.

2. Kneading Theory

Let $F \in \mathcal{C}$. Given a point $x \in \mathbf{R} \setminus \mathbf{Z}$ we define its address (F-address if necessary) as $A(x) = E(F(x)) - E(x)$. If $x \in \mathbf{Z}$ we define $A(x) = E(F(x^+)) - E(x)$. The sequence $\underline{I}(x) = \underline{I}_F(x) = I_0(x)I_1(x)\dots I_n(x)\dots = A(x)A(F(x))\dots A(F^n(x))\dots$ will be called the itinerary of x. For a point $x \in \mathbf{R}$ we define $\underline{I}(x^+) = I_0(x^+)I_1(x^+)\dots$ as follows. For each $n \geq 0$ there exists δ_n such that $I_n(y)$ takes

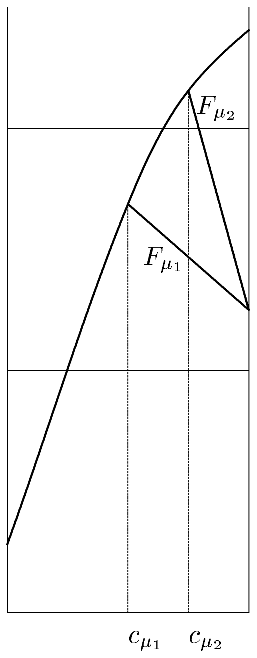

Figure 3. The maps F_μ.

a constant value in $(x, x+\delta_n)$. Denote this value by $I_n(x^+)$. This gives $\underline{I}(x^+)$. In a similar way one can define $\underline{I}(x^-)$.

Now we define an ordering in the set of itineraries. First we note that the set of addresses is naturally ordered by the order of the integers. This gives a total ordering in the set of the itineraries with the lexicographical ordering. The following lemma follows trivially.

Lemma 1. *Let $x, y \in [0,1)$ such that $x < y$. Then $\underline{I}(x) \leq \underline{I}(y)$.*

In a similar way to the continuous case, for a map $F \in \mathcal{C}$, we define the invariant coordinate, $\theta(x)$ (where for maps in $\mathcal{C}$ we define the function $\epsilon(A(x))$ to be 1 for each $x \in \mathbf{R}$), the kneading invariants and the kneading determinant $D_F(t)$, and we obtain:

Theorem 2. *For $F \in \mathcal{C}$, the function $D_F(t)$ is nonzero for $|t| < \frac{1}{s(F)}$. Moreover, if $s(F) > 1$ then the first zero of $D_F(t)$ as t varies in the interval $[0,1)$ occurs at $t = \frac{1}{s(F)}$.*

If $F \in \mathcal{C}$ and $s(F) > 1$ we can define the map ϕ_F and the twist number $T(F)$ in the same way as [**AM**] and we obtain the following result which is the analogous to Theorem 2.12 of [**AM**] for the class $\mathcal{C}$.

Theorem 3. *Let $F \in \mathcal{C}$ be such that $s(F) > 1$. Then there exists a unique map $\tilde{F}$ such that $\tilde{F} \circ \phi_F = \phi_F \circ F$. Moreover, $\tilde{F} \in \mathcal{C}$, $\tilde{F}(0) = T(F)$, $\tilde{F}$ is piecewise affine, $L_{\tilde{F}} = L_F$ and $s(\tilde{F}) = s(F)$.*

Let S be the shift operator which acts in a natural way on sequences of integers (i.e. $S(I_0 I_1 \dots) = I_1 I_2 \dots$). We say that a sequence of integers $\underline{A}$ is quasidominated by F if and only if

$$\underline{I}_F(0^+) \le \underline{A} \le \underline{I}_F(0^-).$$

We say that $\underline{A}$ is dominated by F if both of the above inequalities are strict. As in [**AM**] we obtain

Proposition 4. *Let $F \in \mathcal{C}$. Then the following hold:*

(1) *Let $x \in \mathbf{R} \setminus \mathbf{Z}$. Then $\underline{I}_F(x)$ is quasidominated by F.*
(2) *Let $\underline{A}$ be a sequence of integers dominated by F. Then there exists $x \in (0,1)$ such that $\underline{I}_F(x) = \underline{A}$.*

Corollary 5. *Let $F, G \in \mathcal{C}$ such that $\underline{I}_F(0^+) \le \underline{I}_G(0^+)$ and $\underline{I}_F(0^-) \ge \underline{I}_G(0^-)$. Then $h(F) \ge h(G)$.*

The main result of this paper is the following which is the analogous of Theorem B of [**AM**] for class $\mathcal{C}$. Its proof is similar to the proof of Theorem B of [**AM**] and hence it will be omited. To state it we need to adapt the notation used in [**AM**] to our needs.

Let $a \in \mathbf{R}$ and $i \in \mathbf{Z}$. We define $\epsilon_i(a) = E(ia) - E((i-1)a)$ and $\delta_i(a) = \tilde{E}(ia) - \tilde{E}((i-1)a)$, where $E(\cdot)$ denotes the integer part function and $\tilde{E}\colon \mathbf{R} \longrightarrow \mathbf{Z}$ is defined as follows:

$$\tilde{E}(x) = \begin{cases} E(x), & \text{if } x \notin \mathbf{Z} \\ x - 1, & \text{if } x \in \mathbf{Z}. \end{cases}$$

Set

$$\begin{aligned} \underline{I}_\epsilon(a) &= \epsilon_1(a)\epsilon_2(a)\epsilon_3(a)\dots \\ \underline{I}_\delta(a) &= \delta_1(a)\delta_2(a)\delta_3(a)\dots \\ \underline{I}^*_\epsilon(a) &= (\epsilon_1(a)+1)\epsilon_2(a)\epsilon_3(a)\dots \\ \underline{I}^*_\delta(a) &= (\delta_1(a)-1)\delta_2(a)\delta_3(a)\dots . \end{aligned}$$

Theorem 6. *For a map $F \in \mathcal{C}$ the following statements are equivalent:*

(1) $L_F = [a, b]$.
(2) $\underline{I}^*_\delta(a) \le \underline{I}(0^+) \le \underline{I}_\epsilon(a)$ *and* $\underline{I}_\delta(b) \le \underline{I}(0^-) \le \underline{I}^*_\epsilon(b)$.

3. Bounds of the Topological Entropy

First of all, for each $a, b \in \mathbf{R}$ with $a < b$ we construct maximal and minimal models with rotation interval $[a, b]$.

Lemma 7. *Let $a, b \in \mathbf{R}$ with $a < b$. Then, there exists $H^+_{a,b}$ and $H^-_{a,b} \in \mathcal{C}$ such that $\underline{I}_{H^-_{a,b}}(0^+) = \underline{I}_\epsilon(a)$, $\underline{I}_{H^-_{a,b}}(0^-) = \underline{I}_\delta(b)$, $\underline{I}_{H^+_{a,b}}(0^+) = \underline{I}^*_\delta(b)$ and $\underline{I}_{H^+_{a,b}}(0^-) = \underline{I}^*_\epsilon(a)$. Moreover $L_{H^+_{a,b}} = L_{H^-_{a,b}} = [a, b]$.*

Proof. Here we use the maps $F^+ = F^+_{a,b}$ and $F^- = F^-_{a,b}$ defined in [**AM**] (see Proposition 4.13 and Lemma 4.14). Set $c^+ = c_{F^+}$ and $c^- = c_{F^-}$. Then we define (see Figure 4)

$$H^+_{a,b}(x) = \begin{cases} F^+(x) & \text{if } x \in [0, c^+], \\ F^+(c^+) & \text{if } x \in [c^+, 1), \end{cases}$$

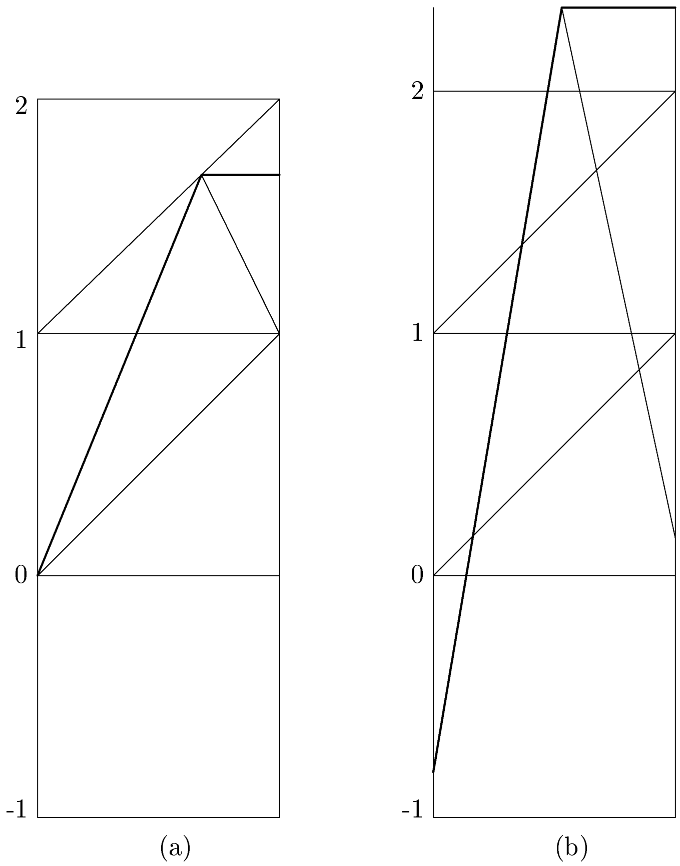

Figure 4. The maps $H^+_{0,1}$ and $H^-_{0,1}$.

$$H^-_{a,b}(x) = \begin{cases} F^-(x) & \text{if } x \in [0, c^-], \\ F^-(c^-) & \text{if } x \in [c^-, 1). \end{cases}$$

From the construction of F^+ we have that $D((F^+)^n)(0), D((F^+)^n)(c^+) \in [0, c^+]$ for all n (where $D(\cdot)$ denotes the decimal part function). Hence, $(F^+)^n(0) = (H^+_{a,b})^n(0)$ and $(H^+_{a,b})^n(0^-) = (H^+_{a,b})^n(c^+) = (F^+)^n(c^+)$. Therefore, we obtain the desired result for $H^+_{a,b}$. The assertion about $H^-_{a,b}$ follows in a similar way. $\square$

Next we compute the kneading determinants of $H^+_{a,b}$ and $H^-_{a,b}$. For $a, b \in \mathbf{R}$ with $a < b$ and $z > 1$ we set $R^-_{a,b}(z) = \sum z^{-q}$ (resp. $R^+_{a,b}(z) = \sum z^{-q}$), where the sum is taken over all pairs $(p, q) \in \mathbf{Z} \times \mathbf{N}$ for which $a < \frac{p}{q} < b$ (resp. $a \leq \frac{p}{q} \leq b$).

Proposition 8. *Let $a, b \in \mathbf{R}$ such that $a < b$. Then the kneading determinants of $H^-_{a,b}$ and $H^+_{a,b}$ are $D_{H^-_{a,b}}(t) = 1 - R^-_{a,b}(t^{-1})$ and $D_{H^+_{a,b}}(t) = 1 - R^+_{a,b}(t^{-1})$, respectively.*

Proof. First we compute $D_{H^-_{a,b}}(t)$. Set $F = H^-_{a,b}$ and $c = c_{H^-_{a,b}}$. Let $k = \tilde{E}(F(0^+)) - E(F(0^-)) + 1$ (notice that the lap number of F is $k+1$). By Lemma 7 we get $k = \tilde{E}(F(0^+)) - E(F(0^-)) + 1 = \delta_1(b) - \epsilon_1(a) = \tilde{E}(b) - E(a) + 1$.

Let $J_1, J_2, \ldots, J_{k+1}$ be the laps of F contained in the interval $[0, 1]$. Assume that for all $x \in \operatorname{int}(J_i)$, $y \in \operatorname{int}(J_j)$ we have $x < y$ if $i < j$. We note that all points in the interior of a lap have the same address. Then we can use the notion of address of a lap and hence the notation $A(J_i)$. We have

$$A(J_i) = \epsilon_1(a) + i - 1 \quad \text{for} \quad i = 1, \ldots, k+1.$$

From now on we will also denote a lap J_i by its address. Then, the invariant coordinates of 0^+ and 0^- are the following (see the definition in Section 2 of [**AM**])

$$\theta(0^+) = \sum_{i=1}^{\infty} \epsilon_i(a) t^{i-1} \quad \text{and} \quad \theta(0^-) = \sum_{i=1}^{\infty} \delta_i(a) t^{i-1}.$$

Hence $v(0) = \theta(0^+) - \theta(0^-) = \sum_{i=1}^{\infty} (\epsilon_i(a) - \delta_i(a)) t^{i-1}$.

Set $\mathcal{K} = \{i \in \mathbf{N} : \epsilon_i(a) = \epsilon_1(a)\}$. By Lemma 4.15 of [**AM**] if $i \notin \mathcal{K}$ then $\epsilon_i(a) = \epsilon_1(a) + 1 = E(a) + 1$. Thus,

$$E(a) + 1 - \epsilon_i(a) = \begin{cases} 1 & \text{for } i \in \mathcal{K}, \\ 0 & \text{if } i \in N \setminus \mathcal{K}. \end{cases}$$

Now set $\mathcal{J} = \{i \in \mathbf{N} : \delta_i(b) = \delta_1(b)\}$. If $i \notin \mathcal{J}$ then $\delta_i(b) = \delta_1(b) - 1 = \tilde{E}(b)$ and hence

$$\tilde{E}(b) + 1 - \delta_i(b) = \begin{cases} 1 & \text{for } i \in \mathcal{J}, \\ 0 & \text{for } i \in \mathbf{N} \setminus \mathcal{J}. \end{cases}$$

Therefore, writing $v(0)$ as $\sum_{i=1}^{k+1} v_i(0)J_i$ we have

$$
\begin{aligned}
v_1(0) &= \sum_{i\in\mathcal{K}} t^{i-1} = \sum_{i=1}^{\infty}(E(a)+1-\epsilon_i(a))t^{i-1},\\
v_2(0) &= \sum_{i\in\mathbf{N}-\mathcal{K},i>0} t^{i-1} = \sum_{i=1}^{\infty}(\epsilon_i(a)-E(a))t^{i-1},\\
v_j(0) &= 0 \quad \text{for} \quad j=3,\dots,k-1,\\
v_k(0) &= -\sum_{i\in\mathbf{N}-\mathcal{J},i>0} t^{i-1} = -\sum_{i=1}^{\infty}(\tilde{E}(b)+1-\delta_i(b))t^{i-1},\\
v_{k+1}(0) &= -\sum_{i\in\mathcal{J}} t^{i-1} = -\sum_{i=1}^{\infty}(\delta_i(b)-\tilde{E}(b))t^{i-1}.
\end{aligned}
$$

Denote $v_1(0)t, v_2(0)t, v_k(0)t$ and $v_{k+1}(0)t$, by φ, κ, η, ω respectively.

Now we are able to write the kneading matrix of F. Note that the turning points of F in $(0,1)$ are the elements of $\{x_1,x_2,\dots,x_k\} = \{x\in(0,1) : F(x)\in\mathbf{Z}\}$. Assume that $x_i<x_j$ if and only if $i<j$. To compute the columns of the kneading matrix we note that if $i\in\{1,\dots,k\}$ then $v(x_i) = J_{i+1}-J_i+tv(0)$.

To see more clearly the structure of the kneading matrix we make the technical assumption that $k>4$. The proof in the case $1<k\le 4$ goes in a similar way.

The kneading matrix is:

$$
\begin{pmatrix}
-1+\varphi & \varphi & \dots & \varphi & \varphi\\
1+\kappa & -1+\kappa & \dots & \kappa & \kappa\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\eta & \eta & \dots & \eta+1 & \eta-1\\
\omega & \omega & \dots & \omega & \omega+1
\end{pmatrix}.
$$

Then,

$$
D_1 = \begin{vmatrix}
1+\kappa & -1+\kappa & \dots & \kappa & \kappa\\
0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\eta & \eta & \dots & \eta+1 & \eta-1\\
\omega & \omega & \dots & \omega & \omega+1
\end{vmatrix}
= \begin{vmatrix}
1+\kappa & -2 & \dots & -1 & -1\\
0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
\eta & 0 & \dots & 1 & 1\\
\omega & 0 & \dots & 0 & 1
\end{vmatrix}
$$

$$= 1+\kappa+(-1)^{k+1}\omega \begin{vmatrix} -2 & -1 & \dots & -1 & -1 \\ 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 1 & -1 \end{vmatrix}$$

$$+ (-1)^k\eta \begin{vmatrix} -2 & -1 & \dots & -1 & -1 \\ 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$= 1+\kappa+(-1)^{k+1}(-1)^{k-2}(-k)\omega+(-1)^k(-1)^{k-3}(-k+1)\eta.$$

By substituting we obtain

$$D_1 = 1+\sum_{i=1}^{\infty}(\epsilon_i(a)-E(a))t^i - \sum_{i=1}^{\infty}(\tilde{E}(b)-E(a)+1)t^i + \sum_{i=1}^{\infty}(\tilde{E}(b)+1-\delta_i(b))t^i$$
$$= 1-\sum_{i=1}^{\infty}(\delta_i(b)-\epsilon_i(a))t^i.$$

Hence, by Lemma 4.16 of [**AM**],

$$D_F(t) = \frac{1}{1-t}\left(1-\sum_{i=1}^{\infty}(\delta_i(b)-\epsilon_i(a))t^i\right) = 1-R^-_{a,b}(t^{-1}).$$

Now we compute $D_{H^+_{a,b}}$. Set $F' = H^+_{a,b}$ and $k' = \tilde{E}(F'(0^-)) - E(F'(0^+))$ (now the lap number of F' is $k'+1$) From Lemma 7 we have that $k' = E(F'(0^-)) - E(F'(0^+)) = \epsilon_1(b)+1-(\delta_1(a)-1) = E(b)-\tilde{E}(a)+1$. We use the same notation as in the case of $H^-_{a,b}$: if $J_1, J_2, \dots, J_{k'+1}$ denote the laps of F', we have that

$$A(J_i) = \delta_1(a)-1+i-1 \quad \text{for} \quad i=1,\dots,k'+1.$$

Thus,

$$\theta(0^+) = (\delta_1(a)-1)-\delta_1(a)+\sum_{i=1}^{\infty}\delta_i(a)t^{i-1},$$
$$\theta(0^-) = (\epsilon_1(b)+1)-\epsilon_1(b)+\sum_{i=1}^{\infty}\epsilon_i(b)t^{i-1}.$$

Hence $v(0) = (\delta_1(a)-1)-\delta_1(a)+\epsilon_1(b)-(\epsilon_1(b)+1)+\sum_{i=1}^{\infty}(\delta_i(a)-\epsilon_i(b))t^{i-1}$. Set $\mathcal{K} = \{i \in \mathbf{N} : \delta_i(a) = \delta_1(a)\}$ and $\mathcal{J} = \{i \in \mathbf{N} : \epsilon_i(b) = \epsilon_1(b)$. By Lemma 4.15 of

[**AM**], if $i \notin \mathcal{K}$ then $\delta_i(a) = \tilde{E}(a)$ and if $i \notin \mathcal{J}$ then $\epsilon_i(b) = E(b) + 1$. Therefore, if we write $v(0)$ as $\sum_{i=1}^{k'+1} v_i(0) J_i$ we have

$$\begin{aligned}
v_1(0) &= 1 + \sum_{i \in \mathbf{N} - \mathcal{K}, i > 0} t^{i-1} = 1 + \sum_{i=1}^{\infty} (\tilde{E}(a) + 1 - \delta_i(a)) t^{i-1}, \\
v_2(0) &= -1 + \sum_{i \in \mathcal{K}} t^{i-1} = -1 + \sum_{i=1}^{\infty} (\delta_i(a) - \tilde{E}(a)) t^{i-1}, \\
v_j(0) &= 0 \quad \text{for} \quad j = 3, \dots, k' - 1, \\
v_{k'}(0) &= 1 - \sum_{i \in \mathcal{J}} t^{i-1} = 1 - \sum_{i=1}^{\infty} (E(b) + 1 - \epsilon_i(b)) t^{i-1}, \\
v_{k'+1}(0) &= -1 - \sum_{i \in \mathbf{N} - \mathcal{J}, i > 0} t^{i-1} = -1 - \sum_{i=1}^{\infty} (\epsilon_i(b) - E(b)) t^{i-1}.
\end{aligned}$$

As in the previous case, we set $\varphi' = t v_1(0)$, $\kappa' = t v_2(0)$, $\eta' = v_{k'}(0)$ and $\omega' = v_{k'+1}(0)$. Then, the kneading matrix of F' has the same expression as the kneading matrix of $H_{a,b}^-$ with k', φ', κ', η', ω' instead of k, φ, κ, η, ω. Hence,

$$\begin{aligned}
D_1(t) &= 1 + \kappa' + (k' - 1)\eta' + k'\omega' \\
&= 1 - 2t + \sum_{i=1}^{\infty} (\delta_i(a) - \tilde{E}(a)) t^i - \sum_{i=1}^{\infty} (E(b) - \tilde{E}(a) + 1) t^i \\
&\qquad + \sum_{i=1}^{\infty} (E(b) + 1 - \epsilon_i(b)) t^i) \\
&= 1 - 2t - \sum_{i=1}^{\infty} (\epsilon_i(b) - \delta_i(a)) t^i.
\end{aligned}$$

Thus,

$$\begin{aligned}
D_{F(t)} &= \frac{1}{1-t} \left[1 - 2t - \left(\sum_{i=1}^{\infty} (E(ib) - \tilde{E}(ia)) t^i - t \sum_{i=1}^{\infty} (E(ib) - \tilde{E}(ia)) t^i - t \right) \right] \\
&= 1 - \sum_{i=1}^{\infty} (E(ib) - \tilde{E}(ia)) t^i.
\end{aligned}$$

Hence, by Lemma 4.16 of [**AM**], we have $D_{H_{a,b}^+}(t) = 1 - R_{a,b}^+(t^{-1})$. $\square$

Lemma 9. *For $a < b$ the equations $R_{a,b}^+(t^{-1}) = 1$ and $R_{a,b}^-(t^{-1}) = 1$ have a unique solution in $(0, 1)$.*

Proof. By Lemma 4.16 of [**AM**] we know that $R_{a,b}^+(t^{-1}) = \sum_{n=1}^{\infty} (E(nb) - \tilde{E}(na)) t^n$ and $R_{a,b}^-(t^{-1}) = \sum_{n=1}^{\infty} (\tilde{E}(nb) - E(na)) t^n$ for $t \in (0, 1)$. Since $\tilde{E}(nb) -$

$E(na)$ and $E(nb)-\tilde{E}(na)$ are uniformly bounded for all $n \in \mathbf{N}$, then $R^+_{a,b}(t^{-1})$ and $R^-_{a,b}(t^{-1})$ are well defined and continuous for $t \in (0,1)$. Since the coefficients of these series are non-negative we have that $R^+_{a,b}(t^{-1})$ and $R^-_{a,b}(t^{-1})$ are increasing in $(0,1)$. We also note that since $a < b$ there exists n_0 such that $(E(nb) - \tilde{E}(na)) > 1$ and $(\tilde{E}(nb) - E(na)) > 1$ for all $n > n_0$. Hence $\lim_{t\uparrow 1} R^+_{a,b}(t^{-1}) = \lim_{t\uparrow 1} R^-_{a,b}(t^{-1}) = \infty$. Since $\lim_{t\downarrow 0} R^+_{a,b}(t^{-1}) = \lim_{t\downarrow 0} R^-_{a,b}(t^{-1}) = 0$ we obtain the desired conclusion. $\square$

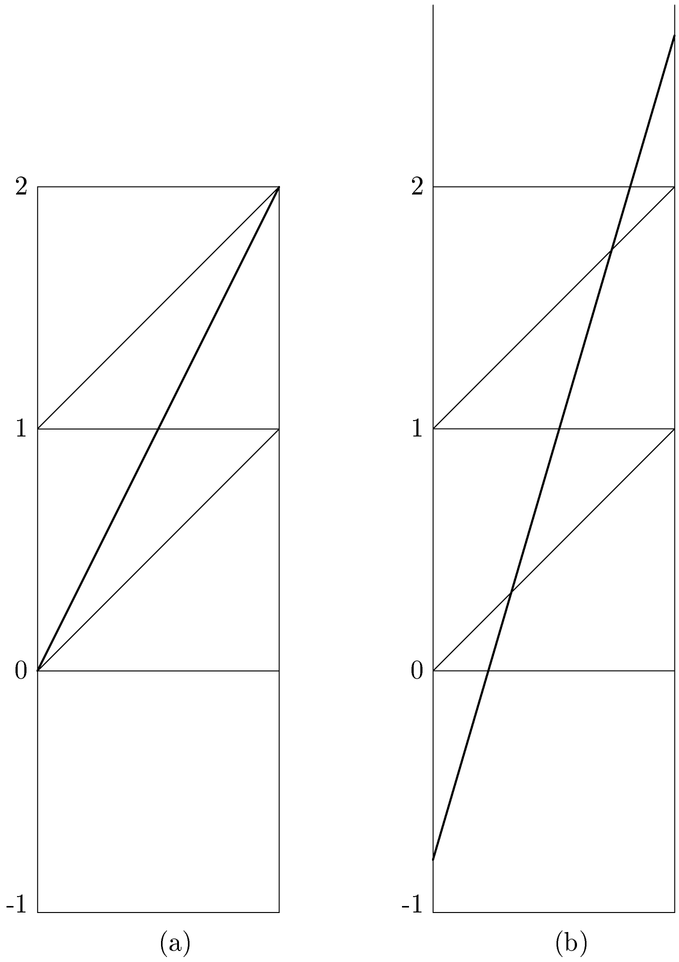

Figure 5. The maps $G^+_{0,1}$ and $G^-_{0,1}$.

From Lemma 9, Proposition 8 and Theorem 2 we obtain that the maps $H^+_{a,b}$ and $H^-_{a,b}$ have positive topological entropy.

Now let $G^+_{a,b}$ and $G^-_{a,b}$ be the piecewise linear maps given by Theorem 3 from $H^+_{a,b}$ and $H^-_{a,b}$ (see Figure 5). The following lemma follows in a similar way to Lemma 4.14 of [**AM**].

Lemma 10. *The following equalities hold:*

(1) $\underline{I}_{G^-_{a,b}}(0^+) = \underline{I}_\epsilon(a)$ *and* $\underline{I}_{G^-_{a,b}}(0^-) = \underline{I}_\delta(b)$.
(2) $\underline{I}_{G^+_{a,b}}(0^+) = \underline{I}^*_\delta(a)$ *and* $\underline{I}_{G^+_{a,b}}(0^-) = \underline{I}^*_\epsilon(b)$.

In what follows we denote the inverses of the solutions of the equations $R^+_{a,b}(t^{-1}) = 1$ and $R^-_{a,b}(t^{-1}) = 1$ in $(0,1)$ by $\alpha^+_{a,b}$ and $\alpha^-_{a,b}$ respectively (in view of Lemma 9 these numbers are well defined).

The next result is the analogous of Corollary C of [**AM**] for class $\mathcal{C}$ and gives lower and upper bounds of the topological entropy for maps from $\mathcal{C}$ depending on the rotation interval. The statement $\log \alpha^-_{a,b} = h(G^-_{a,b}) \leq h(F)$ was already known (see [**ALMT**]). Here we give a different proof.

Corollary 11. *Let $F \in \mathcal{C}$ such that $L_F = [a,b]$ with $a < b$. Then*

$$\log \alpha^-_{a,b} = h(G^-_{a,b}) \leq h(F) \leq h(G^+_{a,b}) = \log \alpha^+_{a,b}.$$

Proof. It follows by Lemma 10, Corollary 5 and Theorem 6. □

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