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PERMUTABILITY OF TOLERANCES WITH FACTOR AND DECOMPOSING CONGRUENCES

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Abstract. A variety $\mathcal{V}$ has tolerances permutable with factor congruences if for any A_1, A_2 of $\mathcal{V}$ and every tolerance T on $A_1 \times A_2$ we have $T \circ \Pi_1 = \Pi_1 \circ T$ and $T \circ \Pi_2 = \Pi_2 \circ T$, where Π_1, Π_2 are factor congruences. If B is a subalgebra of $A_1 \times A_2$, the congruences $\Theta_i = \Pi_i \cap B^2$ are called decomposing congruences. $\mathcal{V}$ has tolerances permutable with decomposing congruences if $T \circ \Theta_i = \Theta_i \circ T$ $(i = 1, 2)$ for each $A_1, A_2 \in \mathcal{V}$, every subalgebra B of $A_1 \times A_2$ and any tolerance T on B. The paper contains Mal'cev type condition characterizing these varieties.

Permutability of congruences with factor congruences was introduced by J. Hageman [**5**]: If A_1, A_2 are algebras of the same type, denote by Π_1, Π_2 the so called **factor congruences** on $A_1 \times A_2$, i.e. Π_i is a congruence induced by the i-th projection of $A_1 \times A_2$ onto A_i $(i = 1, 2)$. A variety $\mathcal{V}$ has **congruences permutable with factor congruences** if for any A_1, A_2 of $\mathcal{V}$ and each $\Theta \in \operatorname{Con} A_1 \times A_2$,

$$\Theta \circ \Pi_1 = \Pi_1 \circ \Theta \qquad \text{and} \qquad \Theta \circ \Pi_2 = \Pi_2 \circ \Theta\,.$$

The paper [**4**] contains a Mal'cev type characterization of varieties satisfying this condition, see also [**4**] for some details. This property was studied by J. Duda [**3**] for 3-permutability. In this paper we generalize the original property for tolerances and we give Mal'cev conditions characterizing these varieties.

By a **tolerance** on an algebra (A, F) is meant a reflexive and symmetric binary relation on A satisfying the substitution property with respect to all operations of F. The set of all tolerances on A forms a complete (even algebraic) lattice $\operatorname{Tol} A$ with respect to set inclusion. Clearly, every congruence is a tolerance on A. The least element of $\operatorname{Tol} A$ is the identity relation ω_A, the greatest one is $\iota_A = A \times A$. Hence, for every two elements a, b of A there exists the least tolerance on A containing the pair $\langle a, b\rangle$; it will be denoted by $T(a, b)$ or $T(\langle a, b\rangle)$ and called the **principal tolerance generated by** $\langle a, b\rangle$.

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Definition 1. A variety $\mathcal{V}$ has **tolerances permutable with factor congruences** if for every A_1, A_2 of $\mathcal{V}$ and each $T \in \operatorname{Tol} A_1 \times A_2$,

$$T \circ \Pi_1 = \Pi_1 \circ T \qquad \text{and} \qquad T \circ \Pi_2 = \Pi_2 \circ T\,.$$

Recall that a variety $\mathcal{V}$ is called **tolerance trivial** if for each $A \in \mathcal{V}$, every tolerance on A is a congruence on A. The following result contains Lemma 1.7 and Theorem 4.11 of [**2**]:

Proposition.

(1) *Let A be an algebra and a, b, c, d be elements of A. Then $\langle a, b\rangle \in T(c, d)$ if and only if there exists a binary algebraic function φ over A with $a = \varphi(c, d)$ and $b = \varphi(d, c)$.*

(2) *A variety $\mathcal{V}$ is tolerance trivial if and only if $\mathcal{V}$ is congruence-permutable.*

If $\mathcal{V}$ is a variety, denote by $F_{\mathcal{V}}(x_1, \dots, x_n)$ the free algebra of $\mathcal{V}$ with free generators $x_1, \dots, x_n$.

Theorem 1. *For a variety $\mathcal{V}$, the following are equivalent:*

(1) *$\mathcal{V}$ has tolerances permutable with factor congruences;*

(2) *there exists $n \geq 1$, a $(2+n)$-ary term q, binary terms $e_1, \dots, e_n$ and ternary terms $f_1, \dots, f_n$ such that:*

$$\begin{aligned} x &= q\left(x, y, e_1(x, y), \dots, e_n(x, y)\right)\\ y &= q\left(y, x, e_1(x, y), \dots, e_n(x, y)\right)\\ z &= q\left(y, x, f_1(x, y, z), \dots, f_n(x, y, z)\right)\,. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2): Let $A_1 = F_{\mathcal{V}}(x, y)$, $A_2 = F_{\mathcal{V}}(x, y, z)$ and $T \in \operatorname{Tol} A_1 \times A_2$ be the principal tolerance generated by the pair $\langle (x, x), (y, y)\rangle$. Then

$$(x, x) T (y, y) \Pi_1 (y, z)$$

and, by (1), also $(x, x)\Pi_1 \circ T(y, z)$, i.e. there exists an element $d \in A_2$ with

$$(x, x)\Pi_1(x, d) T (y, z)\,.$$

Hence, by the Proposition, there exists a binary algebraic function φ over $A_1 \times A_2$ with

$$\begin{aligned} (x, d) &= \varphi\left((x, x), (y, y)\right)\\ (y, z) &= \varphi\left((y, y), (x, x)\right)\,. \end{aligned} \tag{$*$}$$

Hence, there exists a $(2+n)$-ary term q and elements $(e_1, f_1), \dots, (e_n, f_n) \in A_1 \times A_2$ such that

$$\varphi(v, w) = q\left(v, w, (e_1, f_1), \dots, (e_n, f_n)\right) .$$

Since $e_i \in F_{\mathcal{V}}(x, y)$ and $f_i \in F_{\mathcal{V}}(x, y, z)$, each e_i is a binary and f_i a ternary term. If we substitute q and $e_i(x, y), f_i(x, y, z)$ into $(*)$ and we read it coordinatewise, then the first, second and fourth equation of $(*)$ form (2).

(2) $\Rightarrow$ (1): Suppose $A_1, A_2 \in \mathcal{V}$, $T \in \operatorname{Tol} A_1 \times A_2$ and $(a_1, a_2), (c_1, c_2) \in A_1 \times A_2$. If $(a_1, a_2) T \circ \Pi_1 (c_1, c_2)$ then $(a_1, a_2) T (c_1, b_2) \Pi_1 (c_1, c_2)$ for some $b_2 \in A_2$. Put $d_2 = q\left(a_2, b_2, f_1(a_2, b_2, c_2), \dots, f_n(a_2, b_2, c_2)\right)$. Then, by (2), we have

$$\begin{aligned}
(a_1, d_2) = q\big(&(a_1, a_2), (c_1, b_2), (e_1(a_1, c_1), f_1(a_2, b_2, c_2)), \dots, \\
&(e_n(a_1, c_1), f_n(a_2, b_2, c_2))\big) \\
(c_1, c_2) = q\big(&(c_1, b_2), (a_1, a_2), (e_1(a_1, c_1), f_1(a_2, b_2, c_2)), \dots, \\
&(e_n(a_1, c_1), f_n(a_2, b_2, c_2))\big) .
\end{aligned}$$

By the Proposition, it gives $(a_1, d_2) T (c_1, c_2)$, i.e. $(a_1, a_2) \Pi_1 (a_1, d_2) T (c_1, c_2)$ proving

$$T \circ \Pi_1 \subseteq \Pi_1 \circ T .$$

Conversely, if $(a_1, a_2) \Pi_1 (a_1, b_2) T (c_1, c_2)$, we can take $d_2 = q\big(c_2, b_2, f_1(c_2, b_2, a_2), \dots, f_n(c_2, b_2, a_2)\big)$ and prove $(a_1, a_2) T (c_1, d_2) \Pi_1 (c_1, c_2)$, i.e. also $T \circ \Pi_1 \supseteq \Pi_1 \circ T$. The identity $T \circ \Pi_2 = \Pi_2 \circ T$ can be shown analogously if we interchange the role of the first and second coordinate. □

Remark. If $\mathcal{V}$ has tolerances permutable with factor congruences then $\mathcal{V}$ has clearly also congruences permutable with factor congruences since every $\Theta \in \operatorname{Con} A$ for $A \in \mathcal{V}$ is also a tolerance on A. The converse implication does not hold, see the following.

Example 1. An **implication algebra** (see [**1**]) is a groupoid satisfying the following identities

$$(xy)x = x, \quad (xy)y = (yx)x, \quad x(yz) = y(xz) .$$

As it was shown in [**1**], in every implication algebra it holds $xx = yy$, hence we can put $xx = 1$ which is an algebraic constant. Moreover, every implication algebra is a $\vee$-semilattice with the greatest element 1 with respect to the term operation

$$x \vee y = (xy)y .$$

Denote by $\mathcal{V}$ the variety of all implication algebras. It is well-known that $\mathcal{V}$ is congruence-distributive, i.e. $\mathcal{V}$ has the Fraser-Horn property (alias directly decomposable congruences) and hence $\mathcal{V}$ has also congruences permutable with factor congruences. On the other hand, let A be a three element implication algebra whose table is the following:

	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Introduce a relation T on $A \times A$ by its blocks, where we set xTy iff x and y are from the same block:

$$B_1 = \{(a,a),(b,a),(1,a),(a,1),(b,1),(1,1)\}$$
$$B_2 = \{(a,b),(a,1),(b,1),(1,1)\}$$
$$B_3 = \{(b,b),(1,b),(b,1),(1,1)\}$$

see Fig. 1, where the implication algebra $A \times A$ is visualized with respect to its semilattice order.

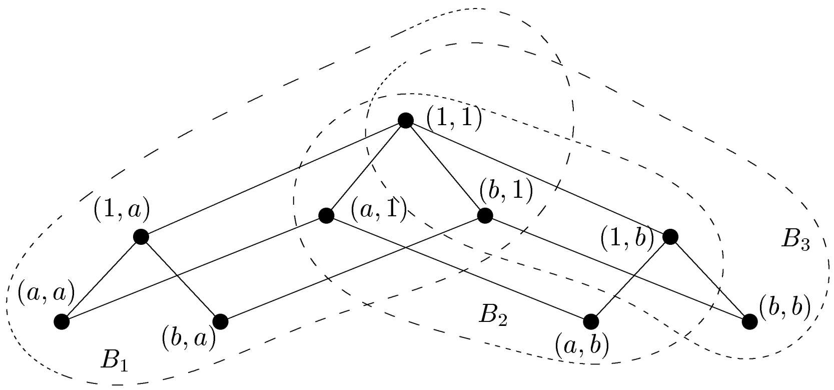

Figure 1.

It is easy to show that $T \in \operatorname{Tol} A \times A$. Moreover, we have

$$(a,a)T(b,1)\Pi_1(b,b)$$

but there does not exist $d \in \{a,b,1\}$ with

$$(a,a)\Pi_1(a,d)T(b,b)$$

whence $T \circ \Pi_1 \neq \Pi_1 \circ T$.

Example 2. Every variety of lattices has tolerances permutable with factor congruences (by the Proposition, lattice varieties are not tolerance trivial): we can take $n = 2$, $e_1(x,y) = x \wedge y$, $e_2(x,y) = x \vee y$, $f_1(x,y,z) = y \vee z$, $f_2(x,y,z) = z$ and $q(x_1,x_2,x_3,x_4) = (x_1 \vee x_3) \wedge x_4$. Then

$$\begin{aligned} q(x,y,e_1(x,y),e_2(x,y)) &= (x \vee (x \wedge y)) \wedge (x \vee y) = x \\ q(y,x,e_1(x,y),e_2(x,y)) &= (y \vee (x \wedge y)) \wedge (x \vee y) = y \\ q(y,x,f_1(x,y,z),f_2(x,y,z)) &= (y \vee (y \vee z)) \wedge z = z\,. \end{aligned}$$

There exist varieties whose similarity type contains two nullary operations, say 0 and 1, and the greatest tolerance (and hence also congruence) ι_A on $A \in \mathcal{V}$ is equal to the principal tolerance $T(0,1)$. Such a variety will be called a **$T(0,1)$-variety**. Typical examples of $T(0,1)$-varieties are varieties of bounded lattices or unitary rings. For $T(0,1)$-varieties, the Mal'cev condition of Theorem 1 can be replaced by a strong Mal'cev condition:

Theorem 2. *Let $\mathcal{V}$ be a variety. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is a $T(0,1)$-variety;*

(2) *there exists a 4-ary term q such that*

$$x = q(0,1,x,y) \qquad \text{and} \qquad y = q(1,0,x,y)\,.$$

Moreover, every $T(0,1)$-variety has tolerances (thus also congruences) permutable with factor congruences.

Proof. (1) $\Rightarrow$ (2): Let $A_1 = A_2 = F_{\mathcal{V}}(x,y)$. Since $\mathcal{V}$ is a $T(0,1)$-variety, clearly $\langle x,y\rangle \in T(0,1)$. Applying the Proposition, there exists a 4-ary term q with

$$x = q(0,1,x,y) \qquad \text{and} \qquad y = q(1,0,x,y)\,.$$

(2) $\Rightarrow$ (1): For each $A \in \mathcal{V}$ and $a,b \in A$ we have

$$\langle a,b\rangle = \langle q(0,1,a,b), q(1,0,a,b)\rangle \in T(0,1)$$

whence $T(0,1) = \iota_A$.

Suppose now that $\mathcal{V}$ is a $T(0,1)$-variety and $A_1, A_2 \in \mathcal{V}$ and $T \in \operatorname{Tol} A_1 \times A_2$. Let

$$(a_1,a_2) T (c_1,b_2) \Pi_1 (c_1,c_2).$$

Put $d = q(1,0,a_2,c_2)$. Then

$$\begin{aligned} &(a_1,a_2)\Pi_1(a_1,d) \\ &\qquad = (q(0,1,a_1,c_1), q(1,0,a_2,c_2)) \\ &\qquad = q\left((0,1),(1,0),(a_1,a_2),(c_1,c_2)\right) T q\left((0,1),(1,0),(c_1,b_2),(c_1,c_2)\right) \\ &\qquad = (q(0,1,c_1,c_1), q(1,0,b_2,c_2)) = (c_1,c_2) \end{aligned}$$

proving $T \circ \Pi_1 \subseteq \Pi_1 \circ T$. The converse inclusion can be shown analogously, the equality $T \circ \Pi_2 = \Pi_2 \circ T$ can be reached by interchanging roles of the first and second coordinate. □

The condition (2) of Theorem 2 does not imply congruence-permutability and hence tolerance triviality as one can see in the following:

Example 3. Let $\mathcal{V}$ be the variety of all bounded lattices. Then $\mathcal{V}$ is not congruence-permutable. However, we can put $q(a,b,x,y) = (b \wedge x) \vee (a \wedge y)$. Then

$$q(0,1,x,y) = (1 \wedge x) \vee (0 \wedge y) = x$$
$$q(1,0,x,y) = (0 \wedge x) \vee (1 \wedge y) = y$$

proving (2) of Theorem 2.

Now, we turn our attention to subalgebras of direct products:

Definition 2. Let B be a subalgebra of a direct product $A_1 \times A_2$. By **decomposing congruences** on B we mean the following $\Theta_1, \Theta_2 \in \operatorname{Con} B$:

$$\Theta_1 = \{\langle (b_1,b_2),(c_1,c_2)\rangle \in B^2; b_1 = c_1\}$$
$$\Theta_1 = \{\langle (b_1,b_2),(c_1,c_2)\rangle \in B^2; b_2 = c_2\}\,.$$

A variety $\mathcal{V}$ is said to have **tolerances permutable with decomposing congruences** if for every A_1, A_2 of $\mathcal{V}$, each subalgebra B of $A_1 \times A_2$ and each $T \in \operatorname{Tol} B$,

$$T \circ \Theta_1 = \Theta_1 \circ T \qquad \text{and} \qquad T \circ \Theta_2 = \Theta_2 \circ T\,.$$

Theorem 3. *For a variety $\mathcal{V}$, the following are equivalent:*

(1) *$\mathcal{V}$ has tolerances permutable with decomposing congruences;*
(2) *$\mathcal{V}$ is congruence-permutable.*

Proof. If $\mathcal{V}$ is congruence-permutable then it has evidently congruences permutable with factor congruences and, by the Proposition, $\mathcal{V}$ is tolerance trivial, i.e. it satisfies (1).

Prove (1) $\Rightarrow$ (2): Let again $A_1 = F_{\mathcal{V}}(x,y)$, $A_2 = F_{\mathcal{V}}(x,y,z)$ and let B be a subalgebra of $A_1 \times A_2$ generated by three elements: $(x,x),(y,y),(y,z)$. Let $T \in \operatorname{Tol} A_1 \times A_2$ be the principal tolerance generated by the pair $\langle (x,x),(y,y)\rangle$. Then $(x,x) T (y,y) \Theta_1 (y,z)$. By (1), there exists an element $(x,d) \in B$ with

$$\langle (x,d),(y,z)\rangle \in T\left(\langle (x,x),(y,y)\rangle\right)\,.$$

By the Proposition, there is a binary algebraic function φ over B such that

$$(x,d) = \varphi((x,x),(y,y)),$$
$$(y,z) = \varphi((y,y),(x,x))\,.$$

However, B has three generators, thus

$$\varphi(v,w) = q(v,w,(x,x),(y,y),(y,z))$$

for some 5-ary term q. If we write it coordinatewise and apply the first and last equation, we obtain

$$x = q(x,y,x,y,y) \qquad \text{and} \qquad z = q(y,x,x,y,z)\,.$$

Thus for $t(x,y,z) = q(x,y,x,y,z)$ we have

$$\begin{aligned} t(x,y,y) &= q(x,y,x,y,y) = x \\ t(x,x,z) &= q(x,x,x,x,z) = z\,, \end{aligned}$$

i.e. t is a Mal'cev term and $\mathcal{V}$ is congruence-permutable. $\square$

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