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ON MEASURE ZERO SETS IN TOPOLOGICAL VECTOR SPACES

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Abstract. We present short proofs of the well known facts that there exists a probability measure vanishing on all the Aronszajn's zero sets and that nonempty open sets in separable **F**-spaces are not Aronszajn's zero sets.

1. Let X be a real vector space. Following N. Aronszajn [**1**] we accept the following definitions.

If $x \in X$ and $a \in X \setminus \{0\}$ then we say that a set $A \subset x + \mathbb{R}a$ is of (Lebesgue) measure zero iff the set $\{t \in \mathbb{R} : x + ta \in A\}$ has one dimensional Lebesgue measure l_1 zero. For $a \in X \setminus \{0\}$ we put

$$\mathcal{N}_a := \{A \subset X : \ \ A \cap (x + \mathbb{R}a) \text{ is of measure zero for every } x \in X\},$$

and if (a_n) is a sequence of nonzero vectors of X then

$$\mathcal{N}(a_n) := \left\{A \subset X : \ \ A = \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \in \mathcal{N}_{a_n} \text{ for every } n \in \mathbb{N}\right\}.$$

Let us note a simple consequence of the above definitions.

Proposition 1. *For every $a \in X \setminus \{0\}$ the family $\mathcal{N}_a$ is a σ-ideal invariant under translations and homoteties, and for every sequence (a_n) of nonzero vectors of X the family $\mathcal{N}(a_n)$ is a σ-ideal invariant under translations and homoteties as well.*

2. Assume now that X is an **F**-space (in the sense of W. Rudin [**4**, 1.8]) and let $\mathcal{B}$ denote the family of all Borel subsets of X.

We start with the following simple fact (in which it is enough to assume that X is a topological vector space).

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Proposition 2. *For every $a \in X \setminus \{0\}$ the σ-ideal $\mathcal{N}_a$ does not contain any nonempty open subsets of X and there exists a probability measure μ on $\mathcal{B}$ such that*

$$\mu(B) = 0 \ \textit{for every}\ B \in \mathcal{B} \cap \mathcal{N}_a$$

Proof. If $U \subset X$ is a neighbourhood of the origin then so is the set $\{t \in \mathbb{R} : ta \in U\}$ on the real line and, consequently, $U \notin \mathcal{N}_a$. Hence and from the fact that the family $\mathcal{N}_a$ is invariant under translations it follows that $\mathcal{N}_a$ contains no nonempty open subset of X. Moreover, it follows from the continuity of the function $\varphi\colon [0,1] \longrightarrow X$ given by $\varphi(t) = ta$ that the formula

$$\mu(B) = l_1(\varphi^{-1}(B))$$

defines a probability measure on $\mathcal{B}$. If $B \in \mathcal{B} \cap \mathcal{N}_a$ then, in particular, $l_1(\varphi^{-1}(B)) = 0$. Therefore $\mu(B) = 0$. $\square$

If (a_n) is a sequence of nonzero vectors of X then we put

$$\mathcal{N}_{\mathcal{B}}(a_n) := \left\{ B \subset X :\ B = \bigcup_{n=1}^{\infty} B_n \text{ and } B_n \in \mathcal{B} \cap \mathcal{N}_{a_n} \text{ for every } n \in \mathbb{N} \right\}.$$

Using an idea from the proof of [**3**, Fact 3] we shall present now our proof of the following theorem (cf. [**1**, Chapter IV] by N. Aronszajn and [**2**] by V. I. Bogachev).

Theorem. *If (a_n) is a sequence of nonzero vectors of X then there exists a probability measure μ on $\mathcal{B}$ such that*

$$\mu(B) = 0 \ \textit{for every}\ B \in \mathcal{N}_{\mathcal{B}}(a_n).$$

Proof. For every positive integer n let us fix a closed and nondegenerate interval $I_n \subset \mathbb{R}$ containing zero and such that the set Z_n defined by

$$Z_n := \{ta_n :\ t \in I_n\}$$

has the diameter less than $\frac{1}{2^n}$, consider the function $\varphi_n\colon I_n \longrightarrow X$ given by

$$\varphi_n(t) = ta_n$$

and a probability measure μ_n on $\mathcal{B}$ defined by

$$\mu_n(B) = \frac{l_1(\varphi_n^{-1}(B))}{l_1(I_n)}\,;$$

moreover, let ν_n denote the restriction of μ_n to the σ-algebra of all Borel subsets of the (compact) space Z_n. Of course,

$$\mu_n(B) = \nu_n(B \cap Z_n) \text{ for every } B \in \mathcal{B};$$

in particular, ν_n is a probability measure for every $n \in \mathbb{N}$. Let ν be the product of the sequence of measures (ν_n). Since for every $z \in \prod_{n=1}^{\infty} Z_n$ the series $\sum_{n=1}^{\infty} z_n$ converges and the function $S\colon \prod_{n=1}^{\infty} Z_n \longrightarrow X$ defined by

$$S(z) = \sum_{n=1}^{\infty} z_n$$

is continuous, we see that the formula

$$\mu(B) = \nu(S^{-1}(B))$$

defines a probability measure on $\mathcal{B}$. For every positive integer n let $\underset{n}{\nu}$ denote the product of the sequence of measures $(\nu_1, \dots, \nu_{n-1}, \nu_{n+1}, \dots)$, consider the function $\underset{n}{S}\colon \prod_{\nu=1,\ \nu\neq n}^{\infty} Z_\nu \longrightarrow X$ given by

$$\underset{n}{S}(z_1, \dots, z_{n-1}, z_{n+1}, z_{n+2}, \dots) = \sum_{\nu=1,\ \nu\neq n}^{\infty} z_\nu$$

and a probability measure $\underset{n}{\mu}$ defined on $\mathcal{B}$ by

$$\underset{n}{\mu}(B) = \underset{n}{\nu}(\underset{n}{S}^{-1}(B)).$$

We shall prove that

$$\underset{n}{\mu} * \mu_n = \mu \text{ for every } n \in \mathbb{N}.$$

In fact, if $B \in \mathcal{B}$, then using the theorem on integrating by substitution and the theorem of Fubini we have:

$$\begin{aligned}
&(\underset{n}{\mu} * \mu_n)(B)\\
&\quad = \int_X \mu_n(B - x)\underset{n}{\mu}(dx) = \int_{\underset{n}{S}^{-1}(X)} \mu_n(B - \underset{n}{S}(\underset{n}{z}))\underset{n}{\nu}(d\underset{n}{z})\\
&\quad = \int_{\prod_{\nu=1,\nu\neq n}^{\infty} Z_\nu} \nu_n((B - \underset{n}{S}(\underset{n}{z})) \cap Z_n)\underset{n}{\nu}(d\underset{n}{z})\\
&\quad = \int_{\prod_{\nu=1,\nu\neq n}^{\infty} Z_\nu} \left[\int_{Z_n} \mathbf{1}_{(B - \underset{n}{S}(\underset{n}{z}))\cap Z_n}(z_n)\nu_n(dz_n)\right] \underset{n}{\nu}(d\underset{n}{z})\\
&\quad = \int_{\prod_{\nu=1,\nu\neq n}^{\infty} Z_\nu} \left[\int_{Z_n} \mathbf{1}_{S^{-1}(B)}(z_1, z_2, \dots)\nu_n(dz_n)\right] \underset{n}{\nu}(d(z_1, \dots, z_{n-1}, z_{n+1}, \dots))\\
&\quad = \int_{\prod_{\nu=1}^{\infty} Z_\nu} \mathbf{1}_{S^{-1}(B)}(z)\nu(dz) = \nu(S^{-1}(B)) = \mu(B).
\end{aligned}$$

Now, if $n \in \mathbb{N}$ and $B \in \mathcal{B} \cap \mathcal{N}_{a_n}$ then $B - x \in \mathcal{B} \cap \mathcal{N}_{a_n}$ for every $x \in X$, whence $\mu_n(B - x) = 0$ for every $x \in X$ and, consequently,

$$\mu(B) = (\underset{n}{\mu} * \mu_n)(B) = \int_X \mu_n(B - x)\underset{n}{\mu}(dx) = 0.$$

This ends the proof. □

The above theorem allows us to give a simple proof of the Aronszajn's theorem [**1**, Theorem 3.1].

Corollary. *Let X be a separable space. If (a_n) is a sequence of nonzero vectors of X then the family $\mathcal{N}_{\mathcal{B}}(a_n)$ does not contain any nonempty open subset of X.*

Proof. Suppose that a nonempty open set $U \subset X$ belongs to $\mathcal{N}_{\mathcal{B}}(a_n)$. If Q is a countable and dense subset of X then

$$X = U + Q \in \mathcal{N}_{\mathcal{B}}(a_n),$$

which contradicts with the Theorem. □

3. The following sets considered in [**3**] by B. R. Hunt, T. Sauer and J. A. Yorke are examples of **measure zero sets**.

Example 1, (cf. [**3**, Fact 8]). If X is an infinite dimensional **F**-space then for every compact set $Z \subset X$ there exists a first category set $P \subset X$ such that $Z \in \mathcal{N}_a$ for every $a \in X \setminus P$.

Example 2, (cf. [**3**, Proposition 2]). The set

$$\left\{(a_n) \in \boldsymbol{l}^2 : \text{ the series} \sum_{n=1}^{\infty} a_n \text{ converges}\right\}$$

belongs to $\mathcal{N}_{(\frac{1}{n})}$.

Example 3, (cf. [**3**, Proposition 1]). The set

$$\left\{f \in \mathbf{L}^1(0,1) : \int_0^1 f(x)\,dx = 0\right\}$$

belongs to $\mathcal{N}_{\mathcal{B}}(a_n)$ for every linearly dense sequence (a_n) of elements of $\mathbf{L}^1(0,1)$.

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