Acta Math. Univ. Comenianae
Vol. LXV, 1(1996), pp. 65–85

ERROR ESTIMATES OF A FULLY DISCRETE LINEAR APPROXIMATION SCHEME FOR STEFAN PROBLEM

A. HANDLOVIČOVÁ

1. Introduction

In this paper we analyze the accuracy of a fully discrete linear approximation scheme, which can be used in solving nonlinear Stefan-like parabolic problems with nonlinear boundary condition

$$\begin{aligned}\partial_t u(t,x) - \Delta\beta(u(t,x))) &= f(t,x,\beta(u)) \quad \text{on } Q := [0,T]\times\Omega\\ u(0,x) &= u_p(x) \quad \text{on } \Omega\\ -\partial_\nu \beta(u(t,x)) &= c_g\beta(u(t,x)) + a_c(t,x) \quad \text{for } x\in\Gamma, t\in(0,T)\end{aligned}$$

where $u\colon (0,T)\times\Omega \to \Re$ is unknown function, $\Omega\subset\Re^d$ is a polygonal convex domain with the boundary Γ, $0<T<\infty$, ν is the outward normal to Γ, $f(t,x,s)$ and $a_c(t,x)$ are Lipschitz continuous functions and $\beta\colon\Re\to\Re$ is a nondecreasing Lipschitz continuous function. Finally c_g is a real number and $c_g\ge 0$.

There are several linear approximation schemes, deal with the Stefan like problems or with the problems concerning non linear diffusion. Among them linear approximation scheme based on so-called nonlinear Chernoff's formula with constant relaxation parameter μ have been studied especially in [**1**], [**12**], [**14**], [**15**], where also some energy error estimates have been investigated.

Another linear approximation schemes have been proposed in [**5**], [**6**], [**7**] and [**3**]. [**8**] investigates problems with elliptic operator and a nonlinearity also on the boundary (function $g(t,x,s)$). Jäger-Kačur approximation scheme [**5**] is of the type

$$\begin{gathered}\mu_i(\theta_i-\beta(u_{i-1}))-\tau\Delta\theta_i=\tau f(t_i,x,\beta(u_{i-1})) \qquad \text{in } \Omega,\\ -\partial_\nu\theta_i=g(t_i,x,\theta_{i-1}) \qquad \text{on } \Gamma, \qquad \text{or} \qquad u_i=0 \qquad \text{on } \Gamma,\\ |\beta(u_{i-1}+\mu_i(\theta_i-\beta(u_{i-1})))-\beta(u_{i-1})| \;\le\; \alpha|\theta_i-\beta(u_{i-1})|+o\left(\frac{1}{\sqrt{n}}\right),\\ u_i:=u_{i-1}+\mu_i(\theta_i-\beta(u_{i-1})), \quad i=1,\dots,n, \quad u_0=u_0(x)\end{gathered}$$

Received February 18, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 65N15.

where θ_i is the approximation of the function $\beta(u)$ at time t_i, $\mu_i \in L_\infty(\Omega)$, $0 < \delta \le \mu_i \le K$ and K^{-1}, $1-\alpha$, δ are sufficiently small positive constants. By $o(\frac{1}{n})$ (Landau's symbol), we denote sequence c_n, $n = 1, 2, \dots$ such that $nc_n \to 0$ for $n \to \infty$.

This scheme has the disadvantage of not being explicit with respect to θ_i and μ_i and therefore some iterative method in order to determine them must be used. In papers mentioned above the convergence and energy error estimates for this semi-discretization scheme have been investigated for both strictly monotone [**5**] and nondecreasing [**7**] function β. In [**4**] linear approximation scheme based on that from [**5**] is used but spreads it of two aspects:

— a fully discretization scheme, that means not only discretization in time, but also finite element method for space discretization is involved,

— linearization is used not only for nonlinear function β but also for nonlinearities in right hand side of the equation and the nonlinearity in the boundary condition as well.

If V_h^0 and V_h^1 are function spaces defined by (2.1) and P_h^0 and P_h^1 are defined by (2.6) and (2.8) then this full discrete scheme is as follows:
For $1 \le i \le n$ find $\{u_i(x), \theta_i(x)\}$ such that $u_i \in V_h^0$ and $\theta_i \in V_h^1$, and for the functions μ_i, $\rho_i \in V_h^0$ it holds

$$u_0 := P_h^0 u_p(x), \quad \theta_0 := P_h^1(\beta(u_p(x))) \tag{1.1}$$

(1.2)
$$\begin{aligned}\left\langle \mu_i P_h^0 \theta_i, \psi \right\rangle &+ \tau \left\langle \nabla\theta_i, \nabla\psi \right\rangle + \tau < c_g(\theta_i) + a_{ci}, \psi >_{h,\Gamma} \\ &= \left\langle \mu_i \beta(u_{i-1}), \psi \right\rangle + \tau \left\langle f(\beta(u_{i-1})), \psi \right\rangle + \tau \left\langle \rho_i (P_h^0\theta_i - \beta(u_{i-1})), \psi \right\rangle \\ &\quad + \left\langle p_i, \psi \right\rangle \ \text{ for all } \psi \in V_h^1,\end{aligned}$$

$$\begin{aligned}&||\beta(u_{i-1} + \mu_i(P_h^0\theta_i - \beta(u_{i-1}))) - \beta(u_{i-1})||_{L_2,\mu_i} \\ &\qquad \le \alpha ||P_h^0\theta_i - \beta(u_{i-1})||_{L_2,\mu_i} + o\left(\frac{1}{\sqrt{n}}\right),\end{aligned} \tag{1.3}$$

$$u_i := u_{i-1} + \mu_i(P_h^0\theta_i - \beta(u_{i-1})), \tag{1.4}$$

for $i = 1, \dots, n$, where functionals $p_i \in H^*$ satisfy

$$\begin{aligned}&|\left\langle p_i, \psi \right\rangle| \le ||p_i||_{H^*(\Omega)} ||\psi||_H \ \text{ for all } \psi \in H^1(\Omega) \quad \text{and} \\ &||p_i||_{H^*} = O\left(\frac{1}{n^\sigma}\right), \quad \text{where } \sigma > 1 \\ &\qquad a_{ci} = a_c(t_i, x), \\ &\qquad 0 < \delta \le \mu_i \le K, \quad |\rho_i| \le K.\end{aligned} \tag{1.5}$$

In this scheme function θ_i is used for linearization of the nonlinearity β, functions μ_i ρ_i represent aproximations of $\frac{1}{\beta'(u)}$ and $f'(u)$ respectively. $K^{-1}, 1-2\alpha, \alpha, \delta$ are sufficiently small positive constants and $||\cdot||_{L_2,\mu_i}$ denotes weight function space L_2 with the weight function μ_i (e.g. [**10**]).

The values of functions θ_i, μ_i, ρ_i can be determined by an iterative process (see [**4**]). In paper [**4**] the convergence of this fully discrete method is proved. The aim of this paper is to prove some error estimates for this method.

Let $e_\theta(t,x)$ and $e_u(t,x)$ have the same meaning as in (2.20). The main result of this paper are the following error estimates for the scheme (1.2)–(1.4):
Let the relation between time and space discretization be

$$\tau \le C_* h^{\frac{4}{3}}. \tag{1.6}$$

Then (under assumptions and notations stated in Section 2*)*

$$||e_u||_{L_\infty(0,T;L_2(\Omega))} + ||e_\theta||_{L_2(I,L_2(\Omega))} + ess \sup_{0\le t\le T} \left\| \nabla \int_0^t e_\theta \, dt \right\|_{L_2^2(\Omega)} + ess \sup_{0\le t\le T} \left\| \int_0^t e_\theta \, dt \right\|_{L_2(\Gamma)} \le C h^{\frac{1}{3}}.$$

In Section 2 we give some notations, basic assumptions and summarize previous results. In Section 3 we prove the error estimates.

2. Basic Notations, Assumptions and Preliminary Results

Let us assume

(H_Ω) $\Omega \subset \Re^d$ $(d \ge 1)$ is polygonal convex domain with the boundary Γ, $Q := I \times \Omega$, where $I = (0,T)$, $0 < T < \infty$ and T is fixed.

Further we shall use function spaces and their symbols as in [**10**]: $L_2(\Omega)$, $L_2^2(\Omega)$ with norm $||\cdot||$, $L_2(I, L_2(\Omega)) = L_2(Q)$ with norm $||\cdot||_{L_2(I,L_2(\Omega))}$, $H^1(\Omega) := W^{1,2}(\Omega)$ with norm $||\cdot||_H$. Dual space to $H^1(\Omega)$ we denote $H^*(\Omega)$. Finally we shall use functional space $L_2(\Gamma)$ and its norm we denote $||\cdot||_\Gamma$.

We use notation $\langle\cdot,\cdot\rangle$ both for inner product in $L_2(\Omega)$ and duality pairing between $H^1(\Omega)$ and $H^*(\Omega)$,

(H_β) $\beta\colon \Re \to \Re$, $\beta(0) = 0$ is nondecreasing Lipschitz continuous function satisfying $0 \le l_\beta \le \beta^{'}(s) \le L_\beta < \infty$ for almost all $s \in \Re$ and there exist $c_1, c_2 > 0$, that for all $s \in \Re$ it holds $|\beta(s)| \ge c_1|s| - c_2$,

(H_f) $f\colon I\times\Omega\times\Re \to \Re$ is uniformly Lipschitz continuous function with Lipschitz constant L_f, and there exists constant C such that $||f(t,x,0)|| \le C$ for a.a. $t \in I$, $x \in \Omega$.

(H_g) $c_g \ge 0$ and $a_c \in L_\infty(I, H(\Omega))$ and a_c is Lipschitz function with Lipschitz constant L_a.

Let us denote

$$\bar{g}^i = \frac{1}{\tau}\int_{I_i}(c_g\beta(u(t,x)) + a_c(t,x))\,dt,$$
$$g_i = c_g\theta_i(x) + a_c(t_i,x) = c_g\theta_i + a_{ci}$$

(H_{u_p}) $u_p \in H^1(\Omega),\ \beta(u_p) \in H^1(\Omega)$.

Let $\{S_h\}_h$ be a family of triangulations

$$S_h = \{S_k\}_{k=1}^{K_h}$$

of Ω consisting of d-simplices $\bar{\Omega} = \cup_{k=1}^{K_h} S_k$ and h stands for the mesh size $h = \sup_{k=1,\dots,K_h} \mathrm{diam} S_k$.

(H_{S_h}) The family $\{S_h\}$ is regular and also the assumption for inverse inequality holds in the sense of Ciarlet ([**2**, Chapter 3]).

Further we denote

$$\begin{aligned} V_h^0 &:= \{\psi : \psi/_{S_k} \text{ is constant for all } k = 1,\dots,K_h\},\\ V_h^1 &:= \{\psi \in C^0(\bar{\Omega}); \psi/_{S_k} \in P_1(S_k) \text{ for all } k = 1,\dots,K_h\}, \end{aligned} \tag{2.1}$$
$$\text{where } P_1(S) = \{q : S \to R | q \text{ is polynomial of degree at most one}\}$$

On the boundary Γ is defined discrete inner product by

$$\langle \psi, \phi\rangle_{h,\Gamma} := \sum_{k=1}^{K_h}\int_{S_k\cap\Gamma}\Pi_h(\psi\phi)\,ds \tag{2.2}$$

for any piecewise continuous functions $\psi, \phi \in \bar{\Omega}$ where Π_h stands for the local linear interpolation operator ([**2**]).

Notice that the integral in (2.2) can be evaluated easily by means of the vertex quadrature rule which is exact for functions from V_h^1 ([**2**, p. 182]).

It holds ([**20**], [**15**]) for any $\psi, \varphi \in V_h^1$

$$|\langle \psi, \varphi\rangle_{h,\Gamma}| \le C_3 ||\psi||_H ||\varphi||_H \tag{2.3}$$

It is also well known the error bound takes into account the effect of numerical integration ([**15**]) for all $\psi, \varphi \in V_h^1$

$$|\langle \psi, \varphi\rangle_{\Gamma} - \langle \psi, \varphi\rangle_{h,\Gamma}| \le C_5 h ||\psi||_H ||\varphi||_H \tag{2.4}$$

The estimate (2.4) holds also for functions $\varphi \in V_h^1$ and $\psi = b\cdot v$, where $v \in V_h^1$ and $b \in V_h^0$; $||b||_{L_\infty} \le K$, $K > 0$.

If we denote by $a(\cdot,\cdot)$ the inner product in $H^1(\Omega)$

$$a(w,z) := \langle \nabla w, \nabla z\rangle + \langle w, z\rangle$$

and by $G\colon H^*(\Omega) \to H^1(\Omega)$ the Green's operator defined by

$$a(G\psi, z) = \langle \psi, z\rangle \text{ for all } z \in H^1(\Omega),\ \psi \in H^*(\Omega),$$

then the associated discrete operator $G_h\colon H^*(\Omega) \to V_h^1$ is defined by

$$a(G_h\psi, \xi) = a(G\psi, \xi) \text{ for all } \xi \in V_h^1, \psi \in H^*(\Omega).$$

Because of assumptions above the operator G is regular in the sense of Ciarlet ([**2**]), namely: for any $\psi \in L_2(\Omega)$ we have $G\psi \in H^2(\Omega)$ and $||G\psi||_{H^2(\Omega)} \leq C||\psi||_{L_2(\Omega)}$.

If $\{S_h\}_h$ and G are regular, then it holds

$$||[G - G_h]\psi||_{H^s} \leq C_6 h^{2-(r+s)}||\psi||_{B^{-r}} \text{ for } 0 \leq s, r \leq 1, \tag{2.5}$$

where H^s and B^{-s} are the following function spaces $0 \leq s \leq 1$: $H^s := [H^1(\Omega), L_2(\Omega)]_{1-s}$, $B^{-s} := [L_2(\Omega), H^*(\Omega)]_s$, (see [**11**]).

Let us now introduce the L_2-projection operator P_h^0 onto V_h^0, which for any $z \in L_2(\Omega)$ is defined by

$$\left\langle P_h^0 z, \psi\right\rangle = \langle z, \psi\rangle \text{ for any } \psi \in V_h^0 \tag{2.6}$$

and satisfies

$$||z - P_h^0 z||_{H^{-s}(\Omega)} \leq C_8 h^{r+s}||z||_{H^r(\Omega)}, \quad 0 \leq s, r \leq 1. \tag{2.7}$$

Let us also introduce the discrete H^1-projection operator P_h^1, which for any $z \in H^1(\Omega)$, $P_h^1 z \in V_h^1$ is defined by

$$a(z - P_h^1 z, \psi) = 0 \text{ for any } \psi \in V_h^1. \tag{2.8}$$

Moreover from approximation property of the operator G_h it holds ([**15**])

$$||z - P_h^1 z||_{H^s(\Omega)} \leq C_9 h^{2-(r+s)}||z||_{H^{2-r}(\Omega)}, \quad 0 \leq s, r \leq 1, \tag{2.9}$$

We shall use the following well-known inequalities

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \text{ for any } a, b \in \Re \text{ and } \varepsilon > 0, \tag{2.10}$$

$$2a(a-b) = a^2 - b^2 + (a-b)^2 \text{ for any } a, b, \in \Re. \tag{2.11}$$

$$||v||_\Gamma^2 \leq C_{10}\left(\varepsilon||\nabla v||^2 + \frac{1}{\varepsilon}||v||^2\right) \tag{2.12}$$

for any $v \in W^{1,2}(\Omega)$ and sufficiently small $\varepsilon > 0$.

The inequality (2.12) see e.g. [**13**, p. 15].

We conclude with some notations concerning the time discretization. Let $\tau = \frac{T}{n}$ be the time step and $t_i = i\tau$, $I_i = (t_{i-1}, t_i]$ for $1 \le i \le n$.

We also set $z^i := z(\cdot, t_i)$, $\bar{z}^i := \frac{1}{\tau}\int_{I_i} z(\cdot, t)\, dt$ for any continuous (resp. integrable) function in time defined in Q and $\partial z^i := (z^i - z^{i-1})/\tau$, $1 \le i \le n$ for any given family $\{z^i\}_{i=0}^n$. Similarly we denote $\delta u_i = (u_i - u_{i-1})/\tau$ for $i = 1, \dots, n$ and $u_i,\ u_{i-1} \in V_h^0$.

For simplicity we shall denote $f(x,t,s)$ by $f(s)$ The constant C will be generic constant independent on time and space discretization parameters τ and h.

Further we denote $\gamma := (\tau, h)$ a pair of discretization parameters, where τ and h have the same meaning as before. Then Rothe's function is

$$\theta^{(\gamma)}(t,x) := \theta_{i-1}(x) + \frac{t - t_{i-1}}{\tau}(\theta_i - \theta_{i-1}) \tag{2.13}$$

for $t \in \langle t_{i-1}, t_i\rangle$, $\theta_i \in V_h^1$ a $i = 1, \dots, n$.

Step function we define as

$$\begin{aligned} \bar{\theta}^{(\gamma)}(t,x) &= \theta_i(x) \quad \text{for } t \in (t_{i-1}, t_i\rangle\,,\ i = 1, \dots, n, \\ \bar{\theta}^{(\gamma)}(0,x) &= \beta(u_0(x)). \end{aligned} \tag{2.14}$$

We define also the function

$$\begin{aligned} P_h^0\bar{\theta}^{(\gamma)}(t,x) &= P_h^0\theta_i \quad \text{for } t \in (t_{i-1}, t_i\rangle\,,\ i = 1, \dots, n, \\ P_h^0\bar{\theta}^{(\gamma)}(0,x) &= \beta(u_0(x)). \end{aligned} \tag{2.15}$$

Analogously we can define also Rothe's and step function $u^{(\gamma)}(t,x)$ and $\bar{u}^{(\gamma)}(t,x)$.

Time derivative for Rothe's function $u^{(\gamma)}$ we denote $u_t^{(\gamma)}$ and for $t \in (t_{i-1}, t_i)$ we have

$$u_t^{(\gamma)} = \frac{u_i - u_{i-1}}{\tau}.$$

The dual space to the space V_h^1 we denote $V_h^{1,*}$.

All functions defined on Q can be prolongated by zero value outside Q.

Variational formulation of our problem is:

Problem (P). *Find* $\{u(t,x), \theta(t,x)\}$ *such that*

$$\begin{aligned} &u(t,x) \in L_2(I; L_2(\Omega)),\ \partial_t u(t,x) \in L_2(I; H^*(\Omega)) \\ &\qquad \textit{and } u(0,x) = u_p(x) \end{aligned} \tag{2.16}$$

$$\theta(t,x) \in L_2(I; H^1(\Omega)) \tag{2.17}$$

$$\theta(t,x) = \beta(u(t,x)) \quad \textit{for a.e.} (t,x) \in Q \tag{2.18}$$

and for all $\varphi \in L_2(I, H^1(\Omega))$ *the following equation holds (arguments are ommitted)*

$$\int_I \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt + \int_I \langle \nabla\theta, \nabla\varphi\rangle \, dt + \int_I \langle g(\theta), \varphi\rangle_\Gamma \, dt = \int_I \langle f(\theta), \varphi\rangle \, dt. \tag{2.19}$$

Finally we denote

$$\begin{aligned} e_\theta(t,x) &= \beta(u(t,x)) - \bar{\theta}^{(\gamma)}(t,x) \\ e_u(t,x) &= u(t,x) - \bar{u}^{(\gamma)}(t,x) \quad \text{for } x \in \Omega \quad t \in I \end{aligned} \tag{2.20}$$

We conclude this section with the basic results of [**4**].

Theorem. *Assume that* (H_Ω), (H_β), (H_{u_p}), (H_f), (H_g), (H_{S_h}) *hold. Let* θ_i, $\beta(u_i), \mu_i$ *fulfill* (1)–(1.5). *Then for* $\tau \le \tau_0$ *and* $h \le h_0$ *such that* (1.6) *is fulfilled and* τ_0, h_0 *are sufficiently small, there exists a constant* C, *independent of the discretization parameters such that*

$$\max_{1\le i\le n} ||\beta(u_i)|| + \sum_{i=1}^n ||u_i - u_{i-1}||^2 + \sum_{i=1}^n \tau ||\nabla\theta_i||^2 \le C, \tag{2.21}$$

$$\max_{i=1,\dots,n} ||P_h^0\theta_i|| + \max_{i=1,\dots,n} ||u_i|| \le C, \tag{2.22}$$

$$\sum_{i=1}^m \tau ||\theta_i||^2 \le C, \tag{2.23}$$

$$\sum_{i=1}^m \tau ||\theta_i||_H^2 \le C. \tag{2.24}$$

3. Error Estimates

We choose $\varphi\chi_{[t_{i-1},t_i]}$ for arbitrary $\varphi \in H^1(\Omega)$ and $i = 1, \dots, n$ as a test function in (2.19). In the sense of previous notations we obtain

$$\left\langle \partial u^i, \varphi \right\rangle + \left\langle \nabla\bar{\beta}^i, \nabla\varphi \right\rangle + \left\langle \bar{g}^i, \varphi \right\rangle_\Gamma = \left\langle \bar{f}^i, \varphi \right\rangle \tag{3.1}$$

Variational formulation of the fully discretized problem (1.1)–(1.4) is

$$\begin{aligned} &\langle \delta u_i, \varphi\rangle + \langle \nabla\theta_i, \nabla\varphi\rangle + \langle g_i, \varphi\rangle_{h,\Gamma} \\ &\qquad = \langle f(\beta(u_{i-1}), \varphi\rangle + \left\langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \varphi\right\rangle + \langle p_i, \varphi\rangle , \end{aligned} \tag{3.2}$$

for arbitrary $\varphi \in V_h^1$.

Now we prove the following lemma

Lemma 1. *Let the assumptions* (H_Ω), (H_β), (H_g), (H_f),(H_{u_p}), (H_{S_h}) *and the relation between time and space discretization fulfil* (1.6). *Than it holds*

$$||e_\theta||^2_{L_2(I,L_2(\Omega))} + \operatorname{ess}\sup_{0\le t\le T}\left\|\nabla\int_0^t e_\theta\,dt\right\|^2 + \operatorname{ess}\sup_{0\le t\le T}\left\|\int_0^t e_\theta\,dt\right\|^2_\Gamma \le C\left(\tau^{\frac{1}{2}} + h + \frac{h^2}{\tau}\right) := C\psi(h,\tau).$$

Proof. We substract variational identity (3.2) from (3.1), multiply by τ and sum over $i=1,\dots,k$. We have

(3.3)
$$\begin{aligned}
\langle u^k - u_k, \varphi\rangle &+ \tau\left\langle\nabla\sum_{i=1}^k(\bar\beta^i - \theta_i), \nabla\varphi\right\rangle + \tau\left\langle\sum_{i=1}^k(\bar g^i - g_i),\varphi\right\rangle_\Gamma\\
&= \tau\left\langle\sum_{i=1}^k(\bar f^i - f(t_i,\beta(u_{i-1}))),\varphi\right\rangle - \tau\left\langle\sum_{i=1}^k p_i,\varphi\right\rangle\\
&\quad - \tau\left(\left\langle\sum_{i=1}^k g_i,\varphi\right\rangle_\Gamma - \left\langle\sum_{i=1}^k g_i,\varphi\right\rangle_{h,\Gamma}\right)\\
&\quad - \tau\left\langle\sum_{i=1}^k \rho_i(P_h^0\theta_i - \beta(u_{i-1})),\varphi\right\rangle + \langle u_p - P_h^0 u_p,\varphi\rangle.
\end{aligned}$$

Now we choose as a test function $\varphi = \tau(P_h^1\bar\beta^k - \theta_k) \in V_h^1$, sum again over $k=1,\dots,m$, and multiply by τ. We obtain

(3.4)
$$\begin{aligned}
\sum_{k=1}^m\int_{I_k}\langle e_u, e_\theta\rangle\,dt &+ \sum_{k=1}^m\tau^2\left\langle\nabla\sum_{i=1}^k(\bar\beta^i - \theta_i), \nabla(P_h^1\bar\beta^k - \theta_k)\right\rangle\\
&+ \sum_{k=1}^m\tau^2\left\langle\sum_{i=1}^k(\bar g^i - g_i), P_h^1\bar\beta^k - \theta_k\right\rangle_\Gamma\\
= -\tau^2\sum_{k=1}^m&\left\langle\sum_{i=1}^k p_i, P_h^1\bar\beta^k - \theta_k\right\rangle - -\sum_{k=1}^m\tau\left\langle u^k - u_k, P_h^1\bar\beta^k - \bar\beta^k\right\rangle dt\\
&- \sum_{k=1}^m\int_{I_k}\langle u^k - u, e_\theta\rangle\,dt\\
&- \tau^2\sum_{k=1}^m\left(\left\langle\sum_{i=1}^k g_i, P_h^1\bar\beta^k - \theta_k\right\rangle_\Gamma - \left\langle\sum_{i=1}^k g_i, P_h^1\bar\beta^k - \theta_k\right\rangle_{h,\Gamma}\right)
\end{aligned}$$

$$- \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} \rho_i (P_h^0 \theta_i - \beta(u_{i-1})), P_h^1 \bar{\beta}^k - \theta_k \right\rangle$$
$$+ \sum_{k=1}^{m} \tau^2 \left\langle \sum_{i=1}^{k} (\bar{f}^i - f(t_i, \beta(u_{i-1}))), P_h^1 \bar{\beta}^k - \theta_k \right\rangle$$
$$+ \tau \sum_{k=1}^{m} \left\langle u_p - P_h^0 u_p, P_h^1 \bar{\beta}^k - \theta_k \right\rangle .$$

We will estimate each term of this identity, so we denote (3.4) formally as follows

$$I + II + III = IV + \cdots + X$$

First we define a function

$$h(s) = s - \frac{\beta(s)}{H} ,$$

where

$$H = \max \left\{ \frac{1}{\delta} \ ; L_\beta \right\}$$

(L_β is Lipschitz constant for β and for δ it holds $0 < \delta \le \mu_i \le K$ for all $i = 1, \dots, n$. For function $h(s)$ it holds

$$0 \le h'(s) \le 1 \text{ for a.a. } s \in \Re. \tag{3.5}$$

If we use (1.4) for $t \in I_k$ we get

$$e_\theta = \beta(u) - \theta_k = \beta(u) - \beta(u_{k-1}) - \frac{u_k - u_{k-1}}{\mu_k} + (P_h^0 \theta_k - \theta_k)$$

and also

$$e_u = \frac{e_\theta}{H} + h(u) - h(u_{k-1}) + \left(\frac{1}{H\mu_k} - 1 \right) (u_k - u_{k-1}) + \frac{1}{H} (\theta_k - P_h^0 \theta_k)$$

for all $k = 1, \dots, n$.

We will estimate first term as follows

$$I = \frac{1}{H} \sum_{k=1}^{m} \int_{I_k} ||e_\theta||^2 \, dt$$
$$+ \sum_{k=1}^{m} \int_{I_k} \left\langle h(u) - h(u_{k-1}) + (\frac{1}{H\mu_k} - 1)(u_k - u_{k-1}) + \frac{1}{H} (\theta_k - P_h^0 \theta_k), e_\theta \right\rangle dt$$
$$= \frac{1}{H} I_1^m + I_{21},$$

where $\mathrm{I}_1^m = \|e_\theta\|^2_{L_2(0,t_m;L_2(\Omega))}$ and

$$\begin{aligned} I_{21} &= \sum_{k=1}^m \int_{I_k} \langle h(u) - h(u_{k-1}), \beta(u) - \beta(u_{k-1})\rangle \, dt \\ &- \sum_{k=1}^m \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \\ &+ \sum_{k=1}^m \int_{I_k} \langle h(u) - h(u_{k-1}), P_h^0\theta_k - \theta_k \rangle \, dt \\ &- \sum_{k=1}^m \int_{I_k} \left\langle \left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta \right\rangle dt + \frac{1}{H} \sum_{k=1}^m \int_{I_k} \langle \theta_k - P_h^0\theta_k, e_\theta \rangle \, dt \end{aligned}$$

Finally we have

$$\begin{aligned} I &\geq \frac{\mathrm{I}_1^m}{H} + l_\beta \|h(u) - h(u_{k-1})\|^2_{L_2(0,t_m,L_2(\Omega))} \\ &- \sum_{k=1}^m \int_{I_k} \left\langle h(u) - h(u_{k-1}), \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \\ &+ \sum_{k=1}^m \int_{I_k} \langle h(u) - h(u_{k-1}), P_h^0\theta_k - \theta_k \rangle \, dt \\ &- \sum_{k=1}^m \int_{I_k} \left\langle \left(1 - \frac{1}{H\mu_k}\right)(u_k - u_{k-1}), e_\theta \right\rangle dt \\ &+ \frac{1}{H} \sum_{k=1}^m \int_{I_k} \langle \theta_k - P_h^0\theta_k, e_\theta \rangle \, dt \\ &= \frac{1}{H} I_1^m + I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

The terms I_2–I_5 we give on the right hand side of identity (3.4) and then we estimate them. First we realize that

$$h(u) - h(u_{k-1}) = u - u_{k-1} - \frac{e_\theta}{H} - \frac{u_k - u_{k-1}}{H\mu_k} + \frac{P_h^0\theta_k - \theta_k}{H}.$$

So we have

$$\begin{aligned} I_2 = &- \sum_{k=1}^m \int_{I_k} \left\langle u, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt + \tau \sum_{k=1}^m \left\langle u_{k-1}, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle \\ &+ \frac{1}{H} \sum_{k=1}^m \int_{I_k} \left\langle e_\theta, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle dt \end{aligned}$$

$$+ \tau \sum_{k=1}^{m} \left\langle \frac{u_k - u_{k-1}}{H\mu_k}, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle - \frac{\tau}{H} \sum_{k=1}^{m} \left\langle P_h^0 \theta_k - \theta_k, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle$$

If we now use (2.21), (2.24) and (2.7), we obtain

$$\begin{aligned} |I_2| &\leq \frac{1}{\delta} \|u\|_{L_2(Q)} \left[\tau \sum_{k=1}^{m} \|u_k - u_{k-1}\|^2 \right]^{\frac{1}{2}} + \tau \left| \sum_{k=1}^{m} \left\langle u_{k-1}, \frac{u_k - u_{k-1}}{\mu_k} \right\rangle \right| \\ &\quad + \frac{1}{16H} I_1^m + C\tau + \frac{C_8 h^2 \tau}{2H} \sum_{k=1}^{m} ||\theta_k||_H^2 \\ &\leq \frac{1}{16H} I_1^m + C\tau^{\frac{1}{2}} + Ch^2 + C \left| \sum_{k=1}^{m} \left\langle \tau^{\frac{3}{4}} u_{k-1}, \tau^{\frac{1}{4}} \frac{u_k - u_{k-1}}{\mu_k} \right\rangle \right| \\ &\leq \frac{1}{16H} I_1^m + C\tau^{\frac{1}{2}} + Ch^2 + C\tau^{\frac{3}{2}} \sum_{k=1}^{m} ||u_{k-1}||^2 + C\tau^{\frac{1}{2}} \sum_{k=1}^{m} ||\frac{u_k - u_{k-1}}{\mu_k}||^2 \\ &\leq \frac{1}{16H} I_1^m + C\tau^{\frac{1}{2}} + Ch^2 \end{aligned}$$

Analogously we estimate the term I_3

$$\begin{aligned} |I_3| &\leq ||u||_{L_2(Q)} \left(C_8 h^2 \tau \sum_{k=1}^{m} ||\theta_k||_H^2 \right)^{\frac{1}{2}} + C_8 h \tau \sum_{k=1}^{m} ||u_{k-1}|| ||\theta_k||_H \\ &\quad + \frac{1}{16H} I_1^m + \frac{4}{H} C_8 h^2 \tau \sum_{k=1}^{m} ||\theta_k||_H^2 + \frac{3}{2H} C_8 h^2 \tau \sum_{k=1}^{m} ||\theta_k||_H^2 + C\tau \\ &\leq Ch + Ch^2 + C\tau + \frac{1}{16H} I_1^m. \end{aligned}$$

For term IV we use similarly (2.21)–(2.24) and (2.10)

$$\begin{aligned} |I_4| &\leq \sum_{k=1}^{m} \int_{I_k} ||u_k - u_{k-1}|| ||e_\theta|| \, dt \\ &\leq \tau 4H \sum_{k=1}^{m} ||u_k - u_{k-1}||^2 + \frac{1}{16H} I_1^m \leq C\tau + \frac{1}{16H} I_1^m. \end{aligned}$$

We estimate the last term

$$\begin{aligned} |I_5| &\leq \frac{1}{16H} I_1^m + 4HC_8 h^2 \tau \sum_{k=1}^{m} ||\theta_k||_H^2 \\ &\leq \frac{1}{16H} I_1^m + 4HC_8 C h^2. \end{aligned}$$

For the term II of identity (3.4) we use the definition of operator P_h^1. We get

$$\begin{aligned} II &= \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} \nabla(P_h^1 \bar{\beta}^i - \theta_i), \nabla(P_h^1 \bar{\beta}^k - \theta_k) \right\rangle \\ &\quad + \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (P_h^1 \bar{\beta}^i - \bar{\beta}^i), P_h^1 \bar{\beta}^k - \theta_k \right\rangle \\ &= II_1 + II_2. \end{aligned}$$

For the first term we use well known identity

$$2\sum_{k=1}^{m} a_k \left(\sum_{i=1}^{k} a_i \right) = \left(\sum_{k=1}^{m} a_k \right)^2 + \sum_{k=1}^{m} a_k^2 \quad \text{for } a_k \in \Re. \tag{3.6}$$

Further we use also well known regularity $\int_0^t \beta(u) \in L_\infty(0,T;H^2(\Omega))$, that we have from (2.19) for $\varphi\chi_{[0,t]}$ ([**14**]).

$$\begin{aligned} II_1 &= \frac{\tau^2}{2} \left\| \sum_{k=1}^{m} \nabla(P_h^1 \bar{\beta}^k - \theta_k) \right\|^2 + \frac{\tau^2}{2} \sum_{k=1}^{m} ||\nabla(P_h^1 \bar{\beta}^k - \theta_k)||^2 \\ &\geq \frac{1}{2} \left\| \nabla \int_0^{t_m} P_h^1 e_\theta(t)\, dt \right\|^2 \geq \frac{1}{2} \left\| \nabla \int_0^{t_m} e_\theta(t)\, dt \right\|^2 - Ch^2 =: II^m - Ch^2. \end{aligned}$$

Second term we estimate as follows

$$\begin{aligned} |II_2| &\leq \tau^3 4H \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} (P_h^1 \bar{\beta}^i - \bar{\beta}^i) \right\|^2 + \frac{\tau}{16H} \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k - \theta_k||^2 \\ &\leq \tau 4H \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \int_{I_i} (P_h^1 \beta(u(t)) - \beta(u(t))) \right\|^2 \\ &\quad + \frac{1}{8H} \int_0^{t_m} \left(||P_h^1 \beta(u(t)) - \beta(u(t))||^2 + ||e_\theta||^2 \right) dt \\ &\leq \tau 4HT \sum_{k=1}^{m} \sum_{i=1}^{k} \int_{I_i} ||P_h^1 \beta(u(t)) - \beta(u(t))||^2 \\ &\quad + \frac{1}{8H} C_9^2 h^2 ||\beta(u)||^2_{L_2(I,H^1(\Omega))} + \frac{1}{8H} I_1^m \\ &\leq 4HT^2 C_9^2 h^2 ||\beta||^2_{L_2(I,H^1(\Omega))} + \frac{1}{8H} C_9^2 h^2 ||\beta(u)||^2_{L_2(I,H^1(\Omega))} + \frac{1}{8H} I_1^m \\ &\leq Ch^2 + \frac{1}{8H} I_1^m, \end{aligned}$$

where we have used the properties of function $\beta(u)$, relations (2.9), (2.10) and well-known inequality

$$\left(\sum_{k=1}^{m} a_k\right)^2 \le m \sum_{k=1}^{m} a_k^2 \quad \text{for arbitrary real numbers } a_k,\ k = 1,\dots,m.$$

We estimate third term

$$\begin{aligned} III &= \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (\bar{g}^i - g_i), P_h^1 \bar{\beta}^k - \bar{\beta}^k \right\rangle_\Gamma + \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (\bar{g}^i - g_i), \bar{\beta}^k - \theta_k \right\rangle_\Gamma \\ &= III_1 + III_2, \end{aligned}$$

where III_1 we estimate using (H_g)

$$|III_1| \le \frac{1}{2} \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k - \bar{\beta}^k||_\Gamma^2 + \frac{\tau^4}{2} \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} (\bar{g}^i - g_i) \right\|_\Gamma^2 = III_{11} + III_{12},$$

where

$$III_{11} \le \frac{1}{2\tau} \int_0^{t_m} ||P_h^1 \beta(u(t)) - \beta(u(t))||_\Gamma^2 \, dt \le \frac{C_{10} C_9^2}{2} \frac{h^2}{\tau} \int_0^{t_m} ||\beta||_{H^2(\Omega)}^2.$$

If we now use regularity $\int_0^t \beta(u) \in L_\infty(I, H^2(\Omega))$, again, we have

$$III_{11} \le C \frac{h^2}{\tau}$$

$$\begin{aligned} III_{12} &\le c_g^2 \tau^4 \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} (\bar{\beta}^i - \theta_i) \right\|_\Gamma^2 + \tau^4 \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \bar{a}_c^i - a_c(t_i) \right\|_\Gamma^2 \\ &= III_A + III_B \end{aligned}$$

$$\begin{aligned} III_A &\le c_g^2 \tau^2 \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \int_{I_i} (\beta(u(t)) - \theta_i) \, dt \right\|_\Gamma^2 \\ &\le c_g^2 T \tau^2 \sum_{k=1}^{m} \sum_{i=1}^{k} \int_{I_i} ||\beta(u(t)) - \theta_i||_\Gamma^2 \\ &\le 2 c_g^2 T^2 \tau \left(||\beta(u)||_{L_2(I, L_2(\Gamma))}^2 + ||\bar{\theta}^{(\gamma)}||_{L_2(I, L_2(\Gamma))}^2 \right) \le C\tau, \end{aligned}$$

where we have used again the properties of $\beta(u)$ and $\bar{\theta}^{(\gamma)}$.

$$III_B \leq 2TL_a^2\tau^2 \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \int_{I_i} (t-t_i)^2\, dt \right\|_\Gamma^2 \leq C\tau^2.$$

Now we estimate III_2

$$\begin{aligned}
III_2 &= \sum_{k=1}^{m} \tau^2 \left\langle \sum_{i=1}^{k} (\bar{g}^i - g_i), \bar{\beta}^k - \theta_k \right\rangle_\Gamma \\
&= c_g\tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (\bar{\beta}^i - \theta_i), \bar{\beta}^k - \theta_k \right\rangle_\Gamma + \tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (\bar{a^i}_c - a_c^i), \bar{\beta^k} - \theta_k \right\rangle_\Gamma \\
&= III_{21} + III_{22}.
\end{aligned}$$

We use the property H_g

$$\begin{aligned}
III_{21} &= c_g\tau^2 \sum_{k=1}^{m} \left\langle \sum_{i=1}^{k} (\bar{\beta}^i - \theta_i), \bar{\beta}^k - \theta_k \right\rangle_\Gamma \\
&= \frac{c_g}{2}\tau^2 \sum_{k=1}^{m} ||\bar{\beta}^k - \theta_k||_\Gamma^2 + \frac{c_g}{2} \left\| \int_0^{t_m} e_\theta\, dt \right\|_\Gamma^2 \\
&= \frac{c_g\tau^2}{2} III_{211} + \frac{c_g}{2} \left\| \int_0^{t_m} e_\theta\, dt \right\|_\Gamma^2.
\end{aligned}$$

Next term we estimate as follows

$$\begin{aligned}
|III_{22}| &\leq \frac{\tau^2 c_g}{8} \sum_{k=1}^{m} ||\bar{\beta}_k - \theta_k||_\Gamma^2 + \frac{4\tau^2}{c_g} \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \bar{a}_c^i - a_c^i \right\|_\Gamma^2 \\
&\leq \frac{\tau^2 c_g}{8} III_{211} + C\tau^2 \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} \frac{L_a}{\tau} \int_{I_i} |t - t_i|\, dt \right\|_\Gamma^2 \\
&\leq \frac{\tau^2 c_g}{8} III_{211} + C\tau.
\end{aligned}$$

Whole third term of (3.4) is now estimated.

We can estimate the term IV

$$|IV| \leq \frac{\tau^2}{2} \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} p_i \right\|_{H^*(\Omega)}^2 + \frac{\tau^2}{2} \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k - \theta_k||_H^2 = IV_1 + IV_2,$$

Using the properties of p_i we have

$$IV_1 \le \tau T \sum_{k=1}^{m} \sum_{i=1}^{k} ||p_i||^2_{H^*(\Omega)} \le T^2 \sum_{k=1}^{m} ||p_k||^2_{H^*(\Omega)} \le C\tau$$

and using the properties of β and $\bar{\theta}^{(\gamma)}$ together with (2.9) it holds

$$IV_2 \le \tau^2 \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k||^2_H + \tau^2 \sum_{k=1}^{m} ||\theta_k||^2_H \le \tau(2 + C_9)||\beta||^2_{L_2(I,H^1(\Omega))} + \tau C \le C\tau.$$

The fifth term of (3.4) we estimate following the similar estimate of [**14**]

$$|V| \le C \left[||u||_{L_2(0,t_m;L_2(\Omega)} + \left(\sum_{k=1}^{m} \tau ||u_k||^2 \right)^{\frac{1}{2}} \right] \left[\sum_{k=1}^{m} \tau ||P_h^1 \bar{\beta}^k - \bar{\beta}^k||^2 \right]^{\frac{1}{2}} \le Ch.$$

The term VI we can estimate as in [**3**]

$$\begin{aligned} |VI| &= \left| \sum_{k=1}^{m} \int_{I_k} \langle u^k - u, e_\theta \rangle \, dt \right| \\ &= \left| \sum_{k=1}^{m} \int_{I_k} \left\langle \int_t^{t_k} \partial_s u \, ds, e_\theta(t) \right\rangle \, dt \right| \\ &\le \tau \|\partial_t u \,\|_{L_2(0,t_m,H^*(\Omega))} \|e_\theta\|_{L_2(0,t_m,H(\Omega))} \le C\tau, \end{aligned}$$

that holds due to the properties of $\partial_t u, \beta(u)$and a result of (2.24).

For the effect of a numerical integration we use (2.4) and obtain

$$\begin{aligned} |VII| &\le C_5 \tau^2 h \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} g_i \right\|_H ||P_h^1 \bar{\beta}^k - \theta_k||_H \\ &\le \tau^2 h^2 \frac{C_5}{2} \sum_{k=1}^{m} \left\| \sum_{i=1}^{k} g_i \right\|_H^2 + \tau^2 \frac{C_5}{2} \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k - \theta_k||^2_H \\ &= VII_1 + VII_2, \end{aligned}$$

where VII_2 we estimate in the same way as in IV_2 and in VII_1 we use (H_g)

$$\begin{aligned} VII_1 &\le 2C_5 \tau^2 h^2 \sum_{k=1}^{m} \left(c_g^2 \left\| \sum_{i=1}^{k} \theta_i \right\|_H^2 + \left\| \sum_{k=1}^{m} a_{ci} \right\|_H^2 \right) \\ &\le Ch^2 \sum_{k=1}^{m} \left(||\theta_k||^2_H + ||a_{ck}||^2_H \right) \le C \frac{h^2}{\tau} \end{aligned}$$

For the estimation of eighth term we use (2.10), (1.4) and properties of ρ_i too. We have

$$|VIII| \leq \frac{4H\tau^3}{2}\sum_{k=1}^{m}\left\|\sum_{i=1}^{k}\frac{\rho_i}{\mu_i}(u_i-u_{i-1})\right\|^2+\frac{\tau}{16H}\sum_{k=1}^{m}||P_h^1\bar{\beta}^k-\theta_k||^2,$$

where second term we have already estimated by a value $\frac{1}{8H}I_1^m+Ch^2$ in the way as in II_2. The first term we easily estimate using (2.21)

$$\frac{4H\tau^3}{2}\sum_{k=1}^{m}\left\|\sum_{i=1}^{k}\frac{\rho_i}{\mu_i}(u_i-u_{i-1})\right\|^2 \leq \frac{4HK^2T^2}{\delta^2}\tau\sum_{k=1}^{m}||u_k-u_{k-1}||^2 \leq C\tau.$$

If we use (2.10) in the ninth term, it holds

$$\begin{aligned}|IX| &\leq 4H\tau^3\sum_{k=1}^{m}\left\|\sum_{i=1}^{k}(\bar{f}^i-f(t_i,\beta(u_{i-1}))\right|^2+\frac{1}{16H}\tau\sum_{k=1}^{m}||P_h^1\bar{\beta}^k-\theta_k||^2\\ &= IX_1+IX_2,\end{aligned}$$

where the second term is again less then $\frac{1}{8H}I_1^m+Ch^2$. For the first term of this inequality we derive

$$\begin{aligned}|\bar{f}^i-f(t_i,\beta(u_{i-1}))| &\leq \frac{1}{\tau}\int_{I_i}|f(t,\beta(u))-f(t_i,\beta(u_{i-1}))|\,dt\\ &\leq \frac{L_f}{\tau}\int_{I_i}|\beta(u)-\beta(u_{i-1}))\,dt+L_f\tau\\ &\leq \frac{L_f}{\tau}\int_{I_i}|e_\theta|\,dt+L_f\tau+\frac{L_f}{\tau}\int_{I_i}\left|\frac{u_i-u_{i-1}}{\mu_i}\right|\,dt\\ &\quad+L_f|\theta_i-P_h^0\theta_i|\\ &= \frac{L_f}{\tau}\int_{I_i}|e_\theta|\,dt+L_f\tau+L_f\left|\frac{u_i-u_{i-1}}{\mu_i}\right|+L_f|\theta_i-P_h^0\theta_i|,\end{aligned}$$

where we have used Lipschitz continuity of the function f, (1.4) and the definition of e_θ. Finally we have

$$\begin{aligned}|\bar{f}^i-f(t_i,\beta(u_{i-1}))|^2 &\leq \frac{4L_f^2}{\tau^2}\left(\int_{Ii}|e_\theta|\,dt\right)^2+4L_f^2\tau^2\\ &\quad+4L_f^2\frac{|u_i-u_{i-1}|^2}{\mu_i^2}+4L_f^2|\theta_i-P_h^0\theta_i|^2\\ &\leq \frac{4L_f^2}{\tau}\int_{Ii}e_\theta^2\,dt+4L_f^2\tau^2+4L_f^2\frac{|u_i-u_{i-1}|^2}{\mu_i^2}\end{aligned}$$

$$+ 4L_f^2|\theta_i - P_h^0\theta_i|^2.$$

For the first term of the inequality in estimate of the ninth term we have

$$\begin{aligned}
IX_1 &= 4H \sum_{k=1}^{m} \tau \left\| \sum_{i=1}^{k} \tau(\bar{f}^i - f(t_i, \beta(u_{i-1}))) \right|^2 \\
&\leq 4HT \sum_{k=1}^{m} \tau^2 \sum_{i=1}^{k} ||\bar{f}^i - f(t_i, \beta(u_{i-1})||^2 \\
&\leq 16HL_f^2 T \sum_{k=1}^{m} \tau \sum_{i=1}^{k} \left\| \int_{I_i} e_\theta^2 \, dt \right\|^2 \\
&\quad + 16HL_f^2 T \sum_{k=1}^{m} \tau^2 \sum_{i=1}^{k} \left(|| \frac{u_i - u_{i-1}}{\mu_i} ||^2 + ||\theta_i - P_h^0\theta_i||^2 \right) + C\tau^2 \\
&\leq C\tau \sum_{k=1}^{m} I_1^k + C(\tau + h^2),
\end{aligned}$$

where we have again used properties (2.21) and (2.7). Finally for this term we have

$$|IX| \leq \frac{1}{8H} I_1^m + C\tau \sum_{k=1}^{m} I_1^k + C(\tau + h^2)$$

The last term of (3.4), we estimate again using (2.10) and we have

$$|X| \leq \tau 4H \sum_{k=1}^{m} ||u_p - P_h^0 u_p||^2 + \frac{1}{16H} \tau \sum_{k=1}^{m} ||P_h^1 \bar{\beta}^k - \theta_k||^2,$$

The estimate of the second term is well-known: $\frac{1}{8H} I_1^m + Ch^2$. First term we estimate easily using (2.7) and a property $\mathrm{H}(_{u_p})$

$$|X| \leq \frac{1}{8H} I_1^m + Ch^2.$$

For now on we have estimated all terms of our identity (3.4). This result we can write formally

$$\begin{aligned}
\frac{1}{H} I_1^m + I_1 + II_2^m + III_{21} &\leq -I_2 - I_3 - I_4 - I_5 + Ch^2 - II_2 - III_1 \\
&\quad - III_{22} + IV + V + VI + VII + VIII + IX + X
\end{aligned}$$

thus

$$\begin{aligned}
\frac{1}{H} I_1^m + II_2^m + III_{21} &\leq |I_2| + |I_3| + |I_4| + |I_5| + Ch^2 |II_2| + |III_1| \\
&\quad + |III_{22}| + |IV| + |V| + |VI| + |VII| + |VIII| + |IX| + |X|
\end{aligned}$$

Now substituting estimates for all terms we obtain inequality

$$\frac{3}{8H} I_1^m + \frac{1}{2} \left\| \nabla \int_0^{t_m} e_\theta \, dt \right\|^2 + \frac{c_g}{2} \left\| \int_0^{t_m} e_\theta \, dt \right|_\Gamma^2$$
$$\leq C(h + h^2 + \frac{h^2}{\tau} + \tau^2 + \tau) + C\tau \sum_{k=1}^{m} I_1^k .$$

Using Gronwall's lemma assuming without lost of generality that $h < 1$ and $\tau < 1$, we get the final result

$$||e_\theta||^2_{L_2(I,L_2(\Omega))} + \text{ess} \sup_{0 \leq t \leq T} \left\| \nabla \int_0^t e_\theta \, dt \right|^2 + \text{ess} \sup_{0 \leq t \leq T} \left\| \int_0^t e_\theta \, dt \right|_\Gamma^2$$
$$\leq C \left(\tau^{\frac{1}{2}} + h + \frac{h^2}{\tau} \right) .$$

This proved the assertion of lemma. □

Now the following lemma results

Lemma 2. *Under the assumptions of Lemma* 1 *it holds*

$$||e_u||^2_{L_\infty(0,T;H^*(\Omega))} \leq C\psi(h,\tau).$$

Proof. We use not only result of Lemma 1 but also the properties of Green's operator G and a discretization operator G_h from (2.5). We easily have

$$||\varphi||_{H^*(\Omega)} = \langle \varphi, G\varphi \rangle^{\frac{1}{2}} \quad \text{for any } \varphi \in H^*(\Omega)$$

Chosen now in (3.3) as a test function $G_h(u^k - u_k) \in V_h^1$, after some rearrangement we have

(3.7)
$$\begin{aligned} ||e_u^k||^2_{H^*(\Omega)} &= ||u^k - u_k||^2_{H^*(\Omega)} = \langle u^k - u_k, G(u^k - u_k) \rangle \\ &= \langle e_u^k, [G - G_h] e_u^k \rangle - \tau \left\langle \nabla \sum_{i=1}^{k} (\bar{\beta}^i - \theta_i), \nabla G_h(u^k - u_k) \right\rangle \\ &\quad - \tau \left\langle \sum_{i=1}^{k} (\bar{g}^i - g_i), G_h(u^k - u_k) \right\rangle_\Gamma \\ &\quad + \tau \left\langle \sum_{i=1}^{k} (\bar{f}^i - f(t_i, \beta(u_{i-1}))), G_h(u^k - u_k) \right\rangle \\ &\quad - \tau \left\langle \sum_{i=1}^{k} p_i, G_h(u^k - u_k) \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+ -\tau \left(\left\langle \sum_{i=1}^{k} g_i, G_h(u^k - u_k) \right\rangle_\Gamma - \left\langle \sum_{i=1}^{k} g_i, G_h(u^k - u_k) \right\rangle_{h,\Gamma} \right) \\
&- \tau \left\langle \sum_{i=1}^{k} \rho_i (P_h^0 \theta_i - \beta(u_{i-1})), G_h(u^k - u_k) \right\rangle \\
&+ \left\langle u_p - P_h^0 u_p, G_h(u^k - u_k) \right\rangle ,
\end{aligned}
$$

Formally we denote it as

$$||e_u^k||^2_{H^*(\Omega)} = XI + XII + XIII + \cdots + XVIII$$

Now the idea is as in the previous lemma; we will estimate all members of this equation. XI we easily estimate using (2.5)

$$XI \le C_6 h^2 ||e_u^k||^2 \le Ch^2,$$

because both functions u^k and u_k too are bounded in $L_2(\Omega)$.

$$
\begin{aligned}
XII \le ||\nabla G_h e_u^k|| \left\| \nabla \int_0^{t_k} e_\theta \right\| &\le \frac{1}{10C_7} a(G_h e_u^k, G_h e_u^k) + \frac{5C_7}{2} \left\| \nabla \int_0^{t_k} e_\theta \right|^2 \\
&\le \frac{1}{10} ||e_u^k||^2_{H^*(\Omega)} + \psi(h,\tau).
\end{aligned}
$$

Here the estimate of G_h and Lemma 1 were used.

$$
\begin{aligned}
XIII &\le \frac{5C_{10}\tau^2}{2} \left\| \sum_{i=1}^{k} (\bar{g}^i - g_i) \right\|^2_\Gamma + \frac{1}{10} ||G_h e_u^k||^2_H \\
&\le 5C_{10}^2 c_g^2 \left\| \int_0^{t_k} e_\theta \, dt \right|^2_\Gamma + \frac{5C_{10}^2 \tau^2}{2} \left| \sum_{i=1}^{k} \bar{a}_c^i - a_c^i \right|^2_\Gamma + \frac{1}{10} ||e_u^k||^2_{H^*(\Omega)} \\
&\le C\psi(h,\tau) + 5C_{10}^2 L_a^2 T^2 meas(\Gamma)^2 \tau^2 + \frac{1}{10} ||e_u^k||^2_{H^*(\Omega)}.
\end{aligned}
$$

Now we again use the estimation as in IX

$$
\begin{aligned}
|XIV| &\le \frac{5\tau^2}{2} \left\| \sum_{i=1}^{k} (\bar{f}^i - f(\beta(u_{i-1}))) \right|^2 + \frac{1}{10} ||e_u^k||^2_{H^*(\Omega)} \\
&\le \frac{15L_f^2}{2} \sum_{i=1}^{k} \left\| \int_{I_i} e_\theta^2 \, dt \right\|^2 + C\tau^3 + \frac{15 L_f^2 \tau}{2\delta^2} \sum_{i=1}^{k} ||u_i - u_{i-1}||^2 \\
&\quad + \frac{15 L_f^2 C_8^2 h^2}{2} \sum_{i=1}^{k} \tau ||\theta_i||^2_H + \frac{1}{10} ||e_u^k||^2_{H^*(\Omega)}
\end{aligned}
$$

$$\leq C(||e_\theta||^2_{L_2(Q)} + \tau^3 + \tau + h^2) + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}$$
$$\leq C(\psi(h,\tau) + \tau^3 + \tau + h^2) + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}.$$

For XV we use (2.10) and properties of p_i

$$XV \leq C\tau^2 + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}.$$

Numerical integration error bounds effect, (2.11), (2.10) and (H$_g$) we use for XVI

$$\begin{aligned} XVI &\leq \tau h C_5 \left\| \sum_{i=1}^k g_i \right\|_H ||G_h e_u^k||_H \\ &\leq \frac{5C_5^2 \tau h^2}{2} \left(\sum_{i=1}^k c_g^2 ||\theta_i||^2_H + \sum_{i=1}^k ||a_c^i||^2_H \right) + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)} \\ &\leq Ch^2 + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}. \end{aligned}$$

Analogously we can estimate in $XVII$, using namely (1.4), (2.10), (1.5), (2.21)

$$\begin{aligned} XVII &\leq \frac{5K^2\tau}{2\delta^2} \sum_{i=1}^k ||u_i - u_{i-1}||^2 + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)} \\ &\leq C\tau + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}. \end{aligned}$$

Finally from the properties P_h^0 (2.7) we conclude

$$XVIII \leq \frac{5C_8^2 h^2}{2}||u_p||^2_H + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)} \leq Ch^2 + \frac{1}{10}||e_u^k||^2_{H^*(\Omega)}.$$

As a result we obtain

$$||e_u^k||^2_{H^*(\Omega)} \leq C(\psi(h,\tau) + h^2 + \tau + \tau^2 + \tau^3).$$

If we assume that, $h < 1$ a $\tau < 1$ we immediately have the result of lemma. □

Theorem 1. *Let the assumptions* (H$_\Omega$), (H$_\beta$), (H$_{up}$), (H$_g$), (H$_f$), (H$_{S_h}$) *hold and for space and time step the following relation holds*

$$\tau \leq C_* h^{\frac{4}{3}},$$

where constant C_* is independent on τ, h. Then

$$\begin{aligned} &||e_u||_{L_\infty(0,T;L_2(\Omega))} + ||e_\theta||_{L_2(I,L_2(\Omega))} \\ &\qquad + \operatorname{ess}\sup_{0\leq t\leq T} \left\| \nabla \int_0^t e_\theta \, dt \right|_{L_2^2(\Omega)} + \operatorname{ess}\sup_{0\leq t\leq T} \left\| \int_0^t e_\theta \, dt \right\|_\Gamma \leq Ch^{\frac{1}{3}} \end{aligned}$$

Proof. The proof results immediately from Lemmas 1 and 2. □

References

1. Berger A. E., Brezis H. and Rogers J. C. W., *A numerical method for solving the problem* $u_t - \Delta f(u) = 0$, R. A. I. R. O Anal. Numer **13** (1979), 297–312.

2. Ciarlet P. F., *The Finite Element Method For Elliptic Problems*, Publishing Company, North-Holland, 1978.

3. Handlovičová A., *Error estimates of a linear approximation scheme for nonlinear diffusion problems*, Acta Math. Univ. Comenianae **LXI** (1992), 27–39.

4. ______, *Solution of Stefan problem by fully discrete linear approximation scheme*, submitted for publication.

5. Jäger W. and Kačur J., *Solution of porous medium type systems by linear approximation schemes*, Numer. Math. **60** (1991), 407–427.

6. ______, *Approximation of porous medium type systems by nondegenerate elliptic systems*, Preprint 582, Universität Heidelberg, SFB 123.

7. Kačur J., Handlovičová A. and Kačurová M., *Solution of nonli diffusion problems by linear approximation schemes*, SIAM J. Numer. Anal. **30** (1993), 1703–1722.

8. Kačurová M., *Solution of porous medium type problems with nonlinear conditions by linear approximation schemes*, Phd. Thesis, Comenius University, 1989.

9. Kačur J., *Method of Rothe in Evolution Equations*, Teubner Verlag, 1985.

10. Kufner A., John V. and Fučík S., *Function spaces*, Academia, Prague, 1977.

11. Lions J. L. and Magenes E., *Non Homogeneous Boundary Value Problems and Aplications, Vol. 1*, Springer-Verlag, Berlin, New York, Heidelberg, 1972.

12. Magenes E., Nochetto R. H. and Verdi C., *Energy error estimates for a lin to approximate nonlinear parabolic problems*, Mathematical modelling and numerical analysis **21** (1987), 655–678.

13. Nečas J., *Les methodes directes en theorie des equationes elliptiques*, Academia, Prague, 1967.

14. Nochetto R. H. and Verdi C., *An efficient scheme to approximate parabolic free boundary problems: error estimates and implementation*, American Mathematical Society (1988), 27–53.

15. ______, *Approximation of degenerate parabolic problems using numerical integrati*, SIAM J. Numer. Anal **25(4)** (1988).

16. Nochetto R. H., *Error estimates for multidimensional Stefan problems with general boundary conditions, A. Bossavit, A. Damilamian, M. Fremond Edsy Free b problems: Applications and Theory, vol. III*, Research notes in Math. **1** (1985), 50–60, Pitman, Boston.

17. ______, *Error estimates for Multidimensional Singular Parabolic Problems.*, Japan J. Appl. Math. **4** (1987), 111–138.

18. ______, *A class of non-degenerate two- phase Stefan problems in several space variables*, Comun. in partial differential equations **12** (1987), 21–45.

19. ______, *Numerical methods for free boundary problems, Free Boundary Problems: Theory and Applications II*, Res. Notes Math. **186** (1990), 555–566 (K. H. Hoffmann and J. Sprekels, eds.), Pitman, London.

20. Raviart P. A., *The use of numerical integration in finite elemente methods solving parabolic equations*, Topics in Numerical Analysis (1973), 233–264, Academic Press, London.

21. Verdi C., *On the numerical approach to a two-phase Stefan problem with nonlinear flux*, Calcolo **22** (1985), 351–381.

A. Handlovičová, Slovak Technical University, Dept. of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia