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SOME LIMIT PROPERTIES OF AN APPROXIMATE LEAST SQUARES ESTIMATOR IN A NONLINEAR REGRESSION MODEL WITH CORRELATED NONZERO MEAN ERRORS

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Abstract. A nonlinear regression model with correlated, normally distributed errors with non zero means is investigated. The limit properties of bias and the mean square error matrix of the approximate least squares estimator of regression parameters are studied.

1. Introduction

Let us consider a linear regression model

$$\mathbf{X}_{n\times 1} = \mathbf{F}_{n\times k}\beta_{k\times 1} + \varepsilon_{n\times 1}, \quad E(\varepsilon) = 0, \quad \mathrm{Var}\,(\varepsilon) = \Sigma \tag{1.1}$$

where the $n \times k$ matrix $\mathbf{F}$ is known, $\beta \in R^k$ (k-dimensional Euclidean space) is an unknown vector parameter and $\varepsilon_{n\times 1}$ is $n \times 1$ vector of the errors.

Under the condition of stationarity of covariance functions:

$$\Sigma = \sum_{i=1}^{n} R(i-1)\mathbf{U}_i \tag{1.2}$$

where

$$(\mathbf{U}_i)_{kl} = \begin{cases} 1, & \text{for} \quad k-l = i-1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

We will consider that $R(\cdot)$ is a nonlinear function of $p \times 1$ parameter θ ($\theta \in R^p$) and therefore we mark $R(\cdot) \cong R_\theta(\cdot)$.

Its estimator is given by

$$\hat{R}_\theta(t) = \frac{1}{n-t}\sum_{i=1}^{n-t}(X(i+t) - \hat{\mathbf{F}}\beta(i+t))(X(i) - \hat{\mathbf{F}}\beta(i)) \tag{1.4}$$

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for $t = 0, 1, \ldots, n-1$ where $\hat{\mathbf{F}}\beta = (\hat{\mathbf{F}}\beta(1), \ldots, \hat{\mathbf{F}}\beta(n))$, (see [**6**]), and $\hat{\beta}$ given by $(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'X$ is LSE of R_θ. Note that the nonparametric estimator of R_θ given in (1.4) is asymptotically unbiased only (see [**6**]).

We consider further

$$\hat{R}_\theta(t) = R_\theta(t) + (\hat{R}_\theta(t) - R_\theta(t)) \tag{1.5}$$

for $t = 0, 1, \ldots, m-1$ where $m < n - k + 1$. The function $R_\theta(\cdot)$ is assumed to be known, continuous and twice continuously differentiable in θ.

Let $(\hat{R}_\theta(t) - R_\theta(t)) = \zeta_\theta(t)$, $\mathbb{E}(\zeta_\theta(t)) = \mu_n(t)$ for $t = 0, 1, \ldots, m-1$ and $m < n - k + 1$.

We will investigate a nonlinear model

$$Y_n(t) = f_n(x_t, \theta) + \zeta_\theta(t) \quad \text{for} \quad t = 0, 1, \ldots, m-1 \text{ and } m < n - k + 1 \tag{1.6}$$

for $n \to \infty$ and fixed m, where f_n is a nonlinear function of parameter $\theta = (\theta_1, \ldots, \theta_p)$ continuous and twice continuously differentiable in θ. Further $\zeta_\theta = (\zeta_\theta(0), \ldots, \zeta_\theta(m-1))$ is $m \times 1$ vector of errors with $\mathbb{E}(\zeta_\theta) = \mu_n$, $\operatorname{Var}(\zeta_0) = \Sigma_n$ and we will consider that $\lim_{n\to\infty} \mu_n = 0$. This condition is fulfilled for $\zeta(t) = \hat{R}(t) - R(t)$ and $t = 0, 1, \ldots, m-1$, $m < n - k + 1$.

2. An Approximate Least Squares Estimator

Let us consider a model described by (1.5). The approximate least squares estimator $\tilde{\theta}$ is based on a method due to Box (see [**2**]) for derivation of an approximate bias of $\hat{\theta}$. Let us denote by $f_n(\theta)$ the $m \times 1$ vector $(f(x_0, \theta), \ldots, f(x_{m-1}, \theta))'$ and let $j_t(\theta)$ be the $p \times 1$ vector with components

$$j_t(\theta) = \left(\frac{\partial f(x_t, \theta)}{\partial \theta_i}\right)_{i=1,\ldots,p} \qquad t = 0, \ldots, m-1, \quad m < n - k + 1.$$

Let $\mathbf{J}(\theta) = \begin{pmatrix} j_0'(\theta) \\ \vdots \\ j_{m-1}'(\theta) \end{pmatrix}$ be $m \times p$ matrix of the first derivatives of $f_n(\theta)$. Let $\mathbf{H}_t$, $t = 0, \ldots, m-1$, be the $p \times p$ matrix of second derivatives, $(\mathbf{H}_t)_{ij} = \dfrac{\partial^2 f_n(x_t, \theta)}{\partial \theta_i \partial \theta_j}$ for $i, j = 1, \ldots, p$.

Since $\hat{\theta}$ is the least squares estimator of θ, the following matrix equality should hold:

$$\mathbf{J}'(\theta) \cdot (Y - f_n(\hat{\theta})) = 0 \tag{2.1}$$

where $Y = (Y_0, \ldots, Y_{m-1})'$. By [**2**] (2.1) and using the Taylor expansion of $\mathbf{J}(\theta)$ and $f_n(\theta)$ it follows that LSE $\hat{\theta}$ of θ can be approximated by the estimator given by:

$$\tilde{\theta}_m = \theta + (\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'\zeta_0 + (\mathbf{J}'\mathbf{J})^{-1}\left[\mathbf{U}'(\zeta_0)\mathbf{M}\zeta_0 - \frac{1}{2}\mathbf{J}'\mathbf{H}(\zeta_0)\right] \tag{2.2}$$

where $\mathbf{M} = \mathbf{I} - \mathbf{J}(\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'$, $\mathbf{A} = \mathbf{J}(\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'$, $\mathbf{U}(\zeta_\theta)$ is a $m \times p$ matrix of the form $\mathbf{U}(\zeta_\theta) = \begin{bmatrix} \zeta_\theta'\mathbf{A}'\mathbf{H}_0 \\ \vdots \\ \zeta_\theta'\mathbf{A}_n'\mathbf{H}_{m-1} \end{bmatrix}$ and $H_n(\zeta_\theta)$ is the $m \times 1$ random vector with components $\zeta_\theta'\mathbf{A}'H_t\mathbf{A}\zeta_\theta$ for $t = 0, 1, \ldots, m-1$. For the j-th component of the random vector $(\mathbf{U}'(\zeta_\theta)\mathbf{M}\zeta_\theta)_j$ we get:

$$(\mathbf{U}'(\zeta_\theta)\mathbf{M}\zeta_\theta)_j = \sum_{i=0}^{m-1}(\mathbf{U}'(\zeta_\theta))_{ji}(\mathbf{M}\zeta_\theta)_i = \sum_k\sum_l\left(\sum_i(\mathbf{H}_i\mathbf{A})_{jk}(\mathbf{M})_{il}\right).$$

$\zeta_\theta(k)\zeta_\theta(l) = \zeta_\theta'\mathbf{N}_j\zeta_\theta$ for $j = 1, \ldots, p$ where $(\mathbf{N}_j)_{kl} = \sum_{i=0}^{m-1}(H_iA)_{jk} \cdot (\mathbf{M})_{il}$, $k, l = 0, \ldots, m-1$ and $(\mathbf{U}'(\zeta_\theta)\mathbf{M}\zeta_\theta)_j = \zeta_\theta'\left(\frac{\mathbf{N}_j + \mathbf{N}_j'}{2}\right)\zeta_\theta$ is a quadratic form with symetric matrices.

3. The Mean and the Mean Square Error Matrix of an Approximate Least Squares Estimator

Let $\tilde{\theta}_m$ be in the form (2.2) and let $\tilde{\theta}_m$ be an approximate LSE of θ in (1.5). We can write:

$$\begin{aligned}\mathbb{E}_\theta(\hat{\theta}_m) = \theta + \mathbf{A}\mu_n + (\mathbf{J}'\mathbf{J})^{-1} \cdot \Big[&\operatorname{tr}(\mathbf{N}\Sigma_n) + \mu_n'\mathbf{N}\mu_n \\ &- \frac{1}{2}J(\operatorname{tr}(\mathbf{A}'\mathbf{H}\mathbf{A}\Sigma_n) + \mu_n'\mathbf{A}'N\mathbf{A}\mu_n)\Big] \cdot (\mathbf{J}'\mathbf{J})^{-1}.\end{aligned} \tag{2.3}$$

We will try to bound this term.

(2.4) $(\mathbf{A}\mu_n)_j = \sum\limits_{i=0}^{m-1} a_{jk}\mu_n(k)$, where $a_{jk} = ((\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}')_{jk}$. As far as m is fixed number $m < n-k+1$ and $\lim\limits_{n\to\infty}\mu_n(k) = 0$ for $k = 0, \ldots, m-1$ this term tends to zero for every fixed $m < n-k+1$ and $n \to \infty$.

In what follows we use the relations:

$$\operatorname{tr}(\mathbf{A}\mathbf{B}') = \sum_{i,j} A_{ij}B_{ij}, \quad |\operatorname{tr}(\mathbf{A}\mathbf{B}')| \le \|\mathbf{A}\| \cdot \|\mathbf{B}\| \text{ where } \|\mathbf{A}\| = \left(\sum_{i,j}^{n}\mathbf{A}_{ij}^2\right)^{1/2}$$

is the Euclidean norm of matrix $\mathbf{A}$, for which the inequality $\|\mathbf{AB}\|^2 \leq \|\mathbf{A}\|^2\|\mathbf{B}\|^2$ holds.

(2.5) $$\operatorname{tr}(\mathbf{N}\Sigma_n) = \begin{pmatrix} \operatorname{tr}(N_0\Sigma_n) \\ \vdots \\ \operatorname{tr}(N_{m-1}\Sigma_n) \end{pmatrix} \implies \operatorname{tr}(N_j\Sigma_n) \leq \|N_j\| \cdot \|\Sigma_n\| \text{ and}$$
for every fixed m, $m < n-k+1$ and for $\lim\limits_{n\to\infty} \|\Sigma_n\| = 0$ this term tends to zero for $n \to \infty$.

(2.6) $$\mu_n'\mathbf{N}\mu_n = \begin{pmatrix} \operatorname{tr}(\mu_n' N_0 \mu_n) \\ \vdots \\ \operatorname{tr}(\mu_n' N_{m-1}\mu_n) \end{pmatrix} \implies \operatorname{tr}(\mu_n'\mathbf{N}_j\mu_n) =$$
$\operatorname{tr}(\mathbf{N}_j\mu_n\mu_n') = \operatorname{tr}(\mathbf{N}_j\mathbf{W}_n)$, where $\mathbf{W}_n = \mu_n\mu_n'$ and $j = 0,\dots,m-1$ then $|\operatorname{tr}(\mathbf{N}_j\mathbf{W}_n)| \leq \|\mathbf{N}_j\| \cdot \|\mathbf{W}_n\|$, what for every fixed $m < n-k+1$ and for $n \to \infty$ tend to zero.

Now

(2.7) $$\operatorname{tr}(\mu_n'\mathbf{A}'N_j\mathbf{A}\mu_n) = \operatorname{tr}(\mu_n\mu_n'\mathbf{A}'N_j\mathbf{A}) = \operatorname{tr}(\mathbf{W}_n\mathbf{A}'N_j\mathbf{A}) \leq \|\mathbf{W}\|\,\|\mathbf{A}\|^2\|N_j\|.$$

This term tends to zero for every fixed $m < n-k+1$ and for $n \to \infty$. We can easily see that for the last term of (2.3) we have

(2.8) $$\left[\mathbf{J}'\begin{pmatrix} \operatorname{tr}(\mu_n'\mathbf{A}'\mathbf{H}_0\mathbf{A}\mu_n) \\ \vdots \\ \operatorname{tr}(\mu_n'\mathbf{A}'\mathbf{H}_{m-1}\mathbf{A}\mu_n) \end{pmatrix}\right]_j = \sum_{i=0}^{m-1} (j_i)_j \operatorname{tr}(\mathbf{A}'\mathbf{H}_i\mathbf{A}\mathbf{W}_n)$$
$= (\operatorname{tr}(\mathbf{A}'\sum_{i=0}^{m-1}(j_i)_j\mathbf{H}_i\mathbf{A}\mathbf{W}_n)$ and hence it is sufficient to bound $|\operatorname{tr}(\mathbf{A}'\mathbf{B}_j\mathbf{A}\mathbf{W}_n)| \cdot |\operatorname{tr}(A'B_jA\mathbf{W}_n)| \leq \|\mathbf{A}'\mathbf{B}_j\mathbf{A}\mathbf{W}_n\|$, where $\mathbf{B}_j = \sum\limits_{i=0}^{m-1}(\mathbf{J}_i)_j\mathbf{H}_i$.

Now we can state

Theorem 1.1. *Let the following conditions be fulfilled in model* (1.6)*:*

(i) *The limit* $\lim_{n\to\infty}\mu_n(t) = 0$ *for* $t = 0,1,\dots,m-1$ *and* $m < n-k+1$,

(ii) $\lim_{n\to\infty}\|\Sigma_n\| = 0$ *for* $m < n-k+1$.

Then, for every fixed $m < n-k+1$*, the approximate least squares estimator* $\tilde{\theta}_m$ *is asymptotically unbiased, i.e.* $\lim_{n\to\infty}\mathbb{E}_\theta(\tilde{\theta}_m) = \theta$.

Proof. The results follow directly from (2.4)–(2.8) and from (i), (ii). □

Now, let $\zeta_\theta \sim \mathbf{N}(\mu_n, \Sigma_n)$. We can express $\mathbb{E}_\theta\,[(\tilde{\theta}_m - \theta)(\tilde{\theta}_m - \theta)']$ as follows:

$$\begin{aligned}
&\mathbb{E}_\theta\,[(\tilde{\theta}_m - \theta)(\tilde{\theta}_m - \theta)'] \\
&\quad = \mathbf{A}\mathbb{E}_\theta\,(\zeta_\theta'\zeta_\theta)\mathbf{A}' + (\mathbf{J}'\mathbf{J})^{-1}\mathbb{E}_\theta\left[(\mathbf{N}(\zeta_\theta) - \frac{1}{2}\mathbf{J}'\mathbf{H}(\zeta_\theta)\right]\cdot\left[N(\zeta_\theta) - \frac{1}{2}\mathbf{J}'\mathbf{H}(\zeta_\theta)\right]' \\
&\quad = \mathbf{A}\mathbb{E}_\theta\,(\zeta_\theta'\zeta_\theta)\mathbf{A}' + (\mathbf{J}'\mathbf{J})^{-1}\Big[\mathbb{E}_\theta\,(\mathbf{N}(\zeta_\theta)\mathbf{N}(\zeta_\theta)') - \frac{1}{2}\mathbb{E}_\theta\,(N(\zeta_\theta)\mathbf{H}(\zeta_\theta)')\mathbf{J} \\
&\qquad - \frac{1}{2}\mathbf{J}'\mathbb{E}_\theta\,(\mathbf{H}(\zeta_\theta)\mathbf{N}(\zeta_\theta)') + \frac{1}{4}\mathbf{J}'\mathbb{E}_\theta\,(\mathbf{H}(\zeta_\theta)\mathbf{H}(\zeta_\theta)')\mathbf{J}\Big](\mathbf{J}'\mathbf{J})^{-1}.
\end{aligned}$$

We will delimit the terms in the last expression member by member

1. $\mathbf{A}\mathbb{E}_\theta\,(\zeta_\theta\zeta_\theta')\mathbf{A} = \mathbf{A}(\Sigma_n + \mu_n\mu_n')\mathbf{A}' = \mathbf{A}\Sigma_n\mathbf{A}' + \mathbf{A}\mathbf{W}_n\mathbf{A}'$ in the norm:

$$\begin{aligned}
\|\mathbf{A}\Sigma_n\mathbf{A}'\| &\le \|\Sigma\|\cdot\|\mathbf{A}\mathbf{A}'\| = \|\Sigma_n\|\cdot\|(\mathbf{J}\mathbf{J}')^{-1}\|, \\
\|\mathbf{A}\mathbf{W}_n\mathbf{A}'\| &\le \|\mathbf{W}_n\|\cdot\|(\mathbf{J}'\mathbf{J})^{-1}\|.
\end{aligned} \tag{2.9}$$

This term tends to zero as $n \to \infty$ and for every fixed m, $m < n - k + 1$.

2. We express the (i,j)-element for the term $\mathbb{E}_\theta\,(\mathbf{N}(\zeta)(\mathbf{N}(\zeta)')$.

$$\begin{aligned}
&[\mathbb{E}_\theta\,(\mathbf{N}(\zeta_\theta)(\mathbf{N}(\zeta_\theta)')]_{i,j} \\
&\quad = 2\,\mathrm{tr}\,(\mathbf{N}_i\Sigma_n\mathbf{N}_j\Sigma) + \mathrm{tr}\,(\mathbf{N}_i\Sigma_n)\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n) + \mu_n'\mathbf{N}_i\mu_n\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n) \\
&\qquad + \mu_n'\mathbf{N}_j\mu_n\,\mathrm{tr}\,(\mathbf{N}_i\Sigma_n) + 4\mu_n'\mathbf{N}_i\Sigma_n\mathbf{N}_j\mu_n + \mu_n'\mathbf{N}_i\mu_n\mu_n'\mathbf{N}_j\mu_n.
\end{aligned}$$

Now we have

$$\begin{aligned}
|\,\mathrm{tr}\,(\mathbf{N}_i\Sigma_n\mathbf{N}_j\Sigma| &\le \|\Sigma_n\|^2\cdot\|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|, \\
\mathrm{tr}\,(\mathbf{N}_i\Sigma_n)\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n) &\le \|\Sigma_n\|^2\cdot\|\mathbf{N}_i\|\cdot\|N_j\|, \\
\mu_n'\mathbf{N}_i\mu_n\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n) &= \mathrm{tr}\,(\mu_n'\mathbf{N}_i\mu_n)\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n)
\end{aligned}$$

and consequently,

$$\begin{aligned}
|\,\mathrm{tr}\,(\mathbf{W}_n\mathbf{N}_i)\,\mathrm{tr}\,(\mathbf{N}_j\Sigma_n)| &\le \|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|\cdot\|\Sigma_n\|^2, \\
\mu_n'\mathbf{N}_i\Sigma_n\mathbf{N}_j\mu_n &= \mathrm{tr}\,(\mu_n'\mathbf{N}_i\Sigma_n\mathbf{N}_j\mu_n) \le \|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|\cdot\|\Sigma_n\|\cdot\|\mathbf{W}_n\|, \\
\mu_n'\mathbf{N}_i\mu_n\mu_n'\mathbf{N}_j\mu_n &= \mathrm{tr}\,(\mu_n'\mathbf{N}_i\mu_n\mu_n'\mathbf{N}_j\mu) \\
&= \mathrm{tr}\,(\mathbf{W}_n\mathbf{N}_i\mathbf{W}_n\mathbf{N}_j) \le \|\mathbf{W}_n\|^2\cdot\|\mathbf{N}_i\|\,\|\mathbf{N}_j\|
\end{aligned}$$

and

$$\begin{aligned}
[\mathbb{E}_\theta\,(\mathbf{N}(\zeta_\theta)(\mathbf{N}_n(\zeta_\theta)']_{i,j} &\le 3\|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|\cdot\|\Sigma_n\|^2 \\
&\quad + 6\|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|\cdot\|\mathbf{W}_n\|\|\Sigma_n\| + \|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|\cdot\|\mathbf{W}_n\|^2 \\
&= \|\mathbf{N}_i\|\cdot\|\mathbf{N}_j\|(3\|\Sigma_n\|^2 + 6\|\Sigma_n\|\cdot\|\mathbf{W}_n\| + \|\mathbf{W}_n\|^2)
\end{aligned} \tag{2.10}$$

This term tends to zero as $n \to \infty$ for every fixed $m < n - k + 1$.

3. Step by step, we express $\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)_{k,p}$ and $((\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta) \cdot \mathbf{J})_{k,p}$:

$$\begin{aligned}\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)_{k,p} &= 2\operatorname{tr}(\mathbf{N}_k\Sigma_n\mathbf{A}'\mathbf{H}_p\mathbf{A}\Sigma_n) + \operatorname{tr}(\mathbf{N}_k\Sigma_n)\operatorname{tr}(\mathbf{A}'\mathbf{H}_p\mathbf{A}\Sigma_n)\\ &\quad + \mu_n'\mathbf{N}_k\mu_n\operatorname{tr}(\mathbf{A}'\mathbf{H}_p\mathbf{A}\Sigma_n) + \mu_n'\mathbf{A}'\mathbf{H}_p\mathbf{A}\operatorname{tr}(\mathbf{N}_k\Sigma_n)\\ &\quad + 4\mu_n'\mathbf{N}_k\Sigma_n\mathbf{A}_n'\mathbf{H}_p\mathbf{A}\mu_n + \mu_n'\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\mathbf{H}_p\mathbf{A}\mu_n.\end{aligned}$$

The first two members of $((\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta) \cdot \mathbf{J})_{k,p}$ are bounded (see [**7**]):

$$\begin{aligned}&\left|\sum_{j=0}^{m-1}(2\operatorname{tr}(N_k\Sigma_n\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n) + \operatorname{tr}(\mathbf{N}_k\Sigma_n)\cdot\operatorname{tr}(\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n)(j_j)_p\right|\\ &\qquad \le 3\|N_i\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\Sigma_n\|^2\end{aligned}\tag{2.11}$$

From this we calculate only last four members of $((\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta) \cdot \mathbf{J})_{k,p}$.

$$\begin{aligned}&(\mathbb{E}_\theta(\zeta_\theta' \mathbf{N}\zeta_\theta\zeta_\theta' \mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta) \cdot j)_{k,p}\\ &= \sum_{j=0}^{m-1}(\mu_n'\mathbf{N}_k\mu_n\operatorname{tr}(\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n) + \mu_n'\mathbf{A}_n'\mathbf{H}_j\mathbf{A}_n\mu_n\operatorname{tr}(\mathbf{N}_k\Sigma_n)\\ &\quad + 4\mu_n'\mathbf{N}_k\Sigma_n\mathbf{A}'\mathbf{H}_j\mathbf{A}\mu_n + \mu_n'\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\mathbf{H}_j\mathbf{A}\mu_n)(j_j)_p\\ &= \sum_{j=0}^{m-1}(\operatorname{tr}(\mathbf{W}_n\mathbf{N}_k)\operatorname{tr}(\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n) + \operatorname{tr}(\mathbf{W}_n\mathbf{A}'\mathbf{H}_j\mathbf{A})\operatorname{tr}(\mathbf{N}_k\Sigma_n)\\ &\quad + 4\operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\Sigma_n\mathbf{A}'\mathbf{H}_j\mathbf{A}_n)\\ &\quad + \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\mathbf{H}_j\mathbf{A}_n) + \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\mathbf{H}_j\mathbf{A}))(j_j)_p\\ &= \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k)\cdot\operatorname{tr}\left(\mathbf{A}'\sum_{j=0}^{m-1}\mathbf{H}_j(j_j)_p\mathbf{A}\Sigma_n\right)\\ &\quad + \operatorname{tr}\left(\mathbf{W}_n\mathbf{N}_k\mathbf{W}_n\mathbf{A}\sum_{j=0}^{m-1}\mathbf{H}_j(j_j)_p\mathbf{A}\right)\operatorname{tr}(\mathbf{N}_k\Sigma_n)\\ &\quad + 4\operatorname{tr}\left(\mathbf{W}_n\mathbf{N}_k\Sigma_n\mathbf{A}'\sum_{j=0}^{m-1}\mathbf{H}_j(j_j)_p\mathbf{A}\Sigma_n\right) + \operatorname{tr}\left(\mathbf{W}_n\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\sum_{j=0}^{m-1}\mathbf{H}_j(j_j)_p\mathbf{A}\right)\\ &= \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k)\cdot\operatorname{tr}(\mathbf{A}'\mathbf{B}_p\mathbf{A}\Sigma_n) + \operatorname{tr}(\mathbf{W}_n\mathbf{A}'\mathbf{B}_p\mathbf{A})\cdot\operatorname{tr}(\mathbf{N}_k\Sigma_n)\\ &\quad + 4\operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\Sigma_n\mathbf{A}'\mathbf{B}_p\mathbf{A}) + \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\mathbf{W}_n\mathbf{A}'\mathbf{B}_p\Sigma_n)\end{aligned}$$

where $\mathbf{B}_p = \sum_{j=0}^{m-1}(j_j)_p\mathbf{H}_j$. Now, we have

$$\begin{aligned}\operatorname{tr}(\mathbf{W}_n\mathbf{N}_k)\operatorname{tr}(\mathbf{A}'\mathbf{B}_p\mathbf{A}\Sigma_n) &\le \|\mathbf{W}_n\|\cdot\|\mathbf{N}_k\|\cdot\|\Sigma_n\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\\ \operatorname{tr}(\mathbf{W}_n\mathbf{A}'\mathbf{B}_p\mathbf{A})\operatorname{tr}(\mathbf{N}_k\Sigma_n) &\le \|\mathbf{W}_n\|\cdot\|\mathbf{N}_k\|\cdot\|\Sigma_n\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\\ \operatorname{tr}(\mathbf{W}_n\mathbf{N}_k\Sigma_n\mathbf{A}'\mathbf{B}_p\mathbf{A}) &\le \|\mathbf{W}_n\|\cdot\|\mathbf{N}_k\|\cdot\|\Sigma_n\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\\ \operatorname{tr}(\mathbf{W}\mathbf{N}_k\mathbf{W}\mathbf{A}'\mathbf{B}_p\mathbf{A}) &\le \|\mathbf{W}_n\|^2\cdot\|\mathbf{N}_k\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\end{aligned}$$

and consequently

$$
\begin{aligned}
(2.12)\qquad & ((\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{N}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)\cdot\mathbf{J})_{k,p} \\
&\quad \le 6\|\mathbf{W}_n\|\cdot\|\mathbf{N}_k\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\Sigma_n\| + \|\mathbf{W}_n\|^2\cdot\|\mathbf{N}_k\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\| \\
&\quad = \|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\mathbf{N}_k\|\cdot\|\mathbf{W}_n\|\cdot(6\|\Sigma_n\|+\|\mathbf{W}_n\|)
\end{aligned}
$$

This term tends to zero for every fixed $m < n-k+1$ and for $n\to\infty$.

4. The case of the third member $\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{N}\zeta_\theta)_{k,p}$ is analogous. For $(\mathbf{J}'(\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{N}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)))_{k,p}$ we have:

$$
\begin{aligned}
(2.13)\qquad & (\mathbf{J}'(\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{N}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)))_{k,p} \\
&\quad \le \|\mathbf{W}_n\|\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{N}_p\|\cdot(6\|\Sigma_n\|+\|\mathbf{W}_n\|).
\end{aligned}
$$

5. By expressing the last member in the form $\mathbf{J}'\mathbb{E}_\theta\,((\mathbf{H}(\zeta_\theta)\mathbf{H}(\zeta_\theta))\cdot\mathbf{J}$, we calculate in the first place the members $\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)_{i,j}$ and $(\mathbf{J}'(\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta))\mathbf{J})_{i,j}$.

$$
\begin{aligned}
&\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta)_{i,j} \\
&\quad = 2\,\mathrm{tr}\,(\mathbf{A}'\mathbf{H}_i\mathbf{A}\Sigma_n\mathbf{A}'\mathbf{H}_j\Sigma_n) + \mathrm{tr}\,(\mathbf{A}'\mathbf{H}_i\mathbf{A}\Sigma_n)\cdot\mathrm{tr}\,(\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n) \\
&\qquad + \mu_n'\mathbf{A}'\mathbf{H}_i\mathbf{A}\mu_n\,\mathrm{tr}\,(\mathbf{A}'\mathbf{H}_j\mathbf{A}\Sigma_n) + \mu_n'\mathbf{A}'\mathbf{H}_j\mathbf{A}\mu_n\,\mathrm{tr}\,(\mathbf{A}'\mathbf{H}_i\mathbf{A}\Sigma_n) \\
&\qquad + 4\mu_n'\mathbf{A}'\mathbf{H}_i\mathbf{A}\Sigma_n\mathbf{A}'\mathbf{H}_j\mathbf{A}\mu_n + \mu_n'\mathbf{A}'\mathbf{H}_i\mathbf{A}\mathbf{W}_n\mathbf{A}'\mathbf{H}_j\mathbf{A}\mu_n.
\end{aligned}
$$

For the first two members in this formula we can write (see Štulajter [**7**])

$$
\begin{aligned}
&\Big|[\mathbf{J}'(2\,\mathrm{tr}\,(\mathbf{A}'\mathbf{H}\mathbf{A}\Sigma_n\mathbf{A}'\mathbf{H}\mathbf{A}\Sigma_n) + \mathrm{tr}\,(\mathbf{A}'\mathbf{H}\mathbf{A})\,\mathrm{tr}\,(\mathbf{A}'\mathbf{H}\mathbf{A}\Sigma_n))\mathbf{J}]_{k,p}\Big| \\
&\quad \le 3\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}'\mathbf{B}_l\mathbf{A}\|\cdot\|\Sigma_n\|^2.
\end{aligned}
$$

We denote the last four members of this formula as $\mathbf{Z}_{i,j}$

$$
\begin{aligned}
(\mathbf{J}'\mathbf{Z}\mathbf{J})_{k,p} &= \left[(\mathbf{J}_0,\dots,\mathbf{J}_{m-1})\cdot\mathbf{Z}\cdot\begin{pmatrix}\mathbf{J}_0\\ \vdots\\ \mathbf{J}_{m-1}\end{pmatrix}\right]_{k,p} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(j_i)_k(\mathbf{Z}_{i,j})(j_j)_p \\
&= \mathrm{tr}\,(\mathbf{W}_n\mathbf{A}'\mathbf{B}_k\mathbf{A})\,\mathrm{tr}\,(\mathbf{W}_n\mathbf{A}'\mathbf{B}_p\mathbf{A}\Sigma_n) + \mathrm{tr}\,(\mathbf{W}_n\mathbf{A}'\mathbf{B}_k\mathbf{A})\,\mathrm{tr}\,(\mathbf{A}\mathbf{B}_p\mathbf{A}\Sigma_n) \\
&\quad + 4\,\mathrm{tr}\,(\mathbf{W}_n\mathbf{A}'\mathbf{B}_k\mathbf{A}\Sigma_n\mathbf{A}'\mathbf{B}_p\mathbf{A}) + \mathrm{tr}\,(\mathbf{W}_n\mathbf{A}'\mathbf{H}_i\mathbf{A}\mathbf{W}_n\mathbf{A}'\mathbf{B}_p\mathbf{A}) \\
&= (\mathbf{J}'(\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta))\mathbf{J})_{i,j}
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
(2.14)\qquad & (\mathbf{J}'(\mathbb{E}_\theta\,(\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta\zeta_\theta'\mathbf{A}'\mathbf{H}\mathbf{A}\zeta_\theta))\mathbf{J})_{i,j} \\
&\quad \le \|\mathbf{W}_n\|\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\Sigma_n\| \\
&\qquad + \|\mathbf{W}_n\|\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\Sigma_n\| \\
&\qquad + 4\|\mathbf{W}_n\|\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot\|\Sigma_n\| \\
&\qquad + \|\mathbf{W}_n\|^2\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}\mathbf{B}_p\mathbf{A}\| \\
&\quad \le \|\mathbf{W}_n\|\cdot\|\mathbf{A}'\mathbf{B}_k\mathbf{A}\|\cdot\|\mathbf{A}'\mathbf{B}_p\mathbf{A}\|\cdot(6\|\Sigma_n\|+\|\mathbf{W}_n\|)
\end{aligned}
$$

Now, we are ready to prove the following result.

Theorem 1.2. *Let $\zeta_\theta \sim \mathbf{N}(\mu_n, \Sigma_n)$ and let the assumptions of Theorem* 1.1 *be fulfilled. Then the approximate LSE $\tilde{\theta}_m$ of parameter θ fulfils*

$$\lim_{n\to\infty} \mathbb{E}_\theta\,[(\tilde{\theta}_m - \theta)(\tilde{\theta}_m - \theta)'] = 0\,.$$

Proof. Based on (2.9)–(2.14) and conditions (i), (ii) of Theorem 1.1 we can easily see that every member of the mean square error matrix of the approximate estimator $\tilde{\theta}_m$ converges to zero for every fixed $m < n-k+1$ if n tends to infinity.□

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