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MINIMAL AND MAXIMAL SETS OF BELL–TYPE INEQUALITIES HOLDING IN A LOGIC

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Dedicated to the memory of Prof. Milan Kolibiar

Abstract. It is shown that for every integer $n > 1$ the poset $(\{\{f : 2^{\{1,\dots,n\}} \to \mathbf{Z} \mid \sum_{I\subseteq\{1,\dots,n\}} f(I)p(\bigwedge_{i\in I} a_i) \in [0,1]$ for all states p on L and all $a_1,\dots,a_n \in L\} \mid L$: ortholattice$\}, \subseteq)$ possesses a smallest and a greatest element. The functions in this poset are interpreted as Bell-type inequalities holding in L.

1. Introduction

Consider a quantum mechanical (physical) system whose event space (the so-called logic) is described by an ortholattice L. By performing experiments and by measuring relative frequencies of events and relative frequencies of intersections of certain events (so-called correlations) one obtains some informations concerning the structure of L. For instance, if there exist a state p on L and events a, b of L such that $0 \le p(a) + p(b) - p(a \wedge b) \le 1$ is not satisfied (as it is the case e.g. for $L = MO2$) then L cannot be a Boolean algebra, i.e., the corresponding physical system cannot be a classical one. Inequalities of the above type are called Bell-type inequalities. A Bell-type inequality is said to hold in L if it holds for every state p on L and for every elements $a_1, \dots, a_n$ of L. It is shown that there exists a smallest and a greatest (both with respect to set-theoretical inclusion) set of Bell-type inequalities (with integer coefficients) holding in an ortholattice. Sufficient and necessary conditions in order that for a given L these sets are smallest or greatest are discussed. The cardinalities of these sets are estimated.

2. Basic Notions and Results

By a logic we mean an ortholattice, that is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type (2,2,1,0,0) satisfying the following conditions:

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(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
(ii) $(a \vee b)' = a' \wedge b'$ for all $a, b \in L$,
(iii) $a \vee a' = 1$ for all $a \in L$,
(iv) $(a')' = a$ for all $a \in L$.

In the following let L, L_1, L_2 denote fixed logics.

L is called orthomodular if $b = a \vee (b \wedge a')$ for all $a, b \in L$ with $a \leq b$. $a, b \in L$ are called mutually orthogonal (in signs $a \perp b$) if $a \leq b'$.

By a state on L we mean a mapping $p \colon L \to [0, 1]$ satisfying the following conditions:
(i') $p(1) = 1$,
(ii') if $a, b \in L$ and $a \perp b$ then $p(a \vee b) = p(a) + p(b)$.

Let $S(L)$ denote the set of all states on L.

Remark 1. By using an induction argument it can be shown that (ii') implies finite additivity of p.

Remark 2. If $a, b \in L$ and $a \leq b$ then $a \perp b'$ and hence $p(a \vee b') = p(a) + p(b')$ whence $p(b \wedge a') = 1 - p(a \vee b') = 1 - p(a) - p(b') = p(b) - p(a)$.

We say that a state p_1 on L_1 can be derived from a state p_2 on L_2 if there exists a homomorphism φ from L_1 to L_2 with $p_2 \circ \varphi = p_1$.

L is called nearly Boolean if for every $p \in S(L)$ there exists a Boolean algebra B and a $q \in S(B)$ such that p can be derived from q.

Obviously, every Boolean algebra is nearly Boolean.

- $p \in L$ is called subadditive if $p(a \vee b) \leq p(a) + p(b)$ for all $a, b \in L$.
- $p \in S(L)$ is said to have the Jauch-Piron property if $p(a \wedge b) = 1$ whenever a, b are elements of L with $p(a) = p(b) = 1$.
- $S \subseteq S(L)$ is called separating if for all $a, b \in L$ with $a \neq b$ there exists a $p \in S$ with $p(a) \neq p(b)$.
- $S \subseteq S(L)$ is called full if $a \leq b$ whenever a, b are elements of L with $p(a) \leq p(b)$ for all $p \in S(L)$.

Obviously, every full set of states is separating.

In the following let n denote an arbitrary fixed positive integer and put $N := \{1, \ldots, n\}$.

For every $a \in L^n$ and every $i \in N$ let a_i denote the i-th component of a. For every $I \subseteq N$ let a_I denote the element $b \in L^n$ with $b_i = 1$ for $i \in I$ and $b_i = 0$ otherwise. $\overline{p}_a(I) := p(\bigwedge_{i \in I} a_i)$ for all $p \in S(L)$, $a \in L^n$ and $I \subseteq N$ (where $\bigwedge_{i \in \emptyset} a_i := 1$ as usual). $f^*(I) := \sum_{K \subseteq I} f(K)$ for all $f \colon 2^N \to \mathbf{Z}$ and $I \subseteq N$.

By a Bell-type inequality (with integer coefficients) of order n we mean an expression of the form

$$0 \leq \langle f, \overline{p}_a \rangle \leq 1 \tag{1}$$

where $f\colon 2^N \to \mathbf{Z}$, $p \in S(L)$ and $a \in L^n$ and where $\langle f, \overline{p}_a\rangle$ denotes the inner product

$$\sum_{I \subseteq N} f(I)\overline{p}_a(I)$$

of f and $\overline{p}_a$ in $\mathbf{R}^{2^N}$. (1) is said to hold in L if (1) holds for all $p \in S(L)$ and all $a \in L^n$. Let $A(L)$ denote the set of all $f\colon 2^N \to \mathbf{Z}$ such that (1) holds in L.

If $S(L) = \emptyset$ (it is well-known that such logics L exist; the classical example of such a logic can be found in [**5**]) then obviously every Bell-type inequality holds in L. To avoid this trivial case let us assume that $S(L) \neq \emptyset$.

$f\colon 2^N \to \mathbf{Z}$ is called L-representable if there exists an n-ary term t such that $\langle f, \overline{p}_\bullet\rangle = p \circ t_L$ for all $p \in S(L)$ where here and in the following t_L denotes the term function on L corresponding to t. Let $R(L)$ denote the set of all L-representable functions from 2^N to $\mathbf{Z}$.

Put

$$\begin{aligned} A_0 := \{f : 2^N \to \mathbf{Z} \mid\ & \text{there exist a non-negative integer } m \text{ and subsets} \\ & I_1, \dots, I_m \text{ of } N \text{ with } I_1 \subset \cdots \subset I_m \text{ such that } f(I_j) = (-1)^{j+1} \\ & \text{for all } j = 1, \dots, m \text{ and } f(I) = 0 \text{ otherwise}\}, \\ A_1 := \{f : 2^N \to \mathbf{Z} \mid\ & f^*(2^N) \subseteq \{0, 1\}\}. \end{aligned}$$

(Here and in the following $\subset$ denotes proper inclusion.)

Theorem 1. *Assume that every state on L_1 can be derived from some state on L_2. Then $R(L_2) \subseteq R(L_1)$.*

Proof. Let $f \in R(L_2)$. Then there exists an n-ary term t such that $\langle f, \overline{p}_\bullet\rangle = p \circ t_{L_2}$ for all $p \in S(L_2)$. Now let $q \in S(L_1)$. Since q can be derived from some state on L_2 there exists a state r on L_2 and a homomorphism φ from L_1 to L_2 with $r \circ \varphi = q$. Now let $a \in L_1^n$ and put $b := (\varphi(a_1), \dots, \varphi(a_n))$. Then

$$\langle f, \overline{q}_a\rangle = \langle f, \overline{r \circ \varphi}_a\rangle = \langle f, \overline{r}_b\rangle = r(t_{L_2}(b)) = r(\varphi(t_{L_1}(a))) = (q \circ t_{L_1})(a).$$

Hence $f \in R(L_1)$. □

The following lemma shows that states on ortholattices are monotone.

Lemma 2. *Let $p \in S(L)$ and $a, b \in L$ with $a \leq b$. Then $p(b) = p(a) + p(b \wedge a')$.*

Proof. Because of $a \perp b'$ we have $p(a) + p(b \wedge a') = p(a) + 1 - p(a \vee b') = p(a) + 1 - p(a) - p(b') = p(b)$. □

Theorem 3. $A_0 \subseteq R(L) \subseteq A(L) \subseteq A_1$

Proof. Let $p \in S(L)$ and $b \in L^n$.

First assume $f \in A_0$. Then there exist a non-negative integer m and subsets $I_1, \ldots, I_m$ of N with $I_1 \subset \cdots \subset I_m$ such that $f(I_j) = (-1)^{j+1}$ for all $j = 1, \ldots, m$ and $f(I) = 0$ otherwise. Put

$$c_j := \bigwedge_{i \in I_j} b_i$$

for all $j = 1, \ldots, m$. Then

$$\langle f, \overline{p}_b \rangle = \sum_{j=1}^{m} (-1)^{j+1} p(c_j).$$

Using Lemma 1 and the additivity of p we obtain

$$\langle f, \overline{p}_b \rangle = \sum_{k=1}^{\frac{m}{2}} (p(c_{2k-1}) - p(c_{2k})) = p(\bigvee_{k=1}^{\frac{m}{2}} (c_{2k-1} \wedge c'_{2k}))$$

if m is even, and

$$\langle f, \overline{p}_b \rangle = \sum_{k=1}^{\frac{m-1}{2}} (p(c_{2k-1}) - p(c_{2k})) + p(c_m) = p(\bigvee_{k=1}^{\frac{m-1}{2}} (c_{2k-1} \wedge c'_{2k}) \vee c_m)$$

if m is odd. (Observe that in the first case the elements $c_1 \wedge c'_2, \ldots, c_{m-1} \wedge c'_m$ are mutually orthogonal since $c_1 \geq \ldots \geq c_m$ and since for $1 \leq k < l \leq \frac{m}{2}$ we have $2k < 2l - 1$ and therefore

$$c_{2l-1} \wedge c'_{2l} \leq c_{2l-1} \leq c_{2k} \leq c'_{2k-1} \vee c_{2k} = (c_{2k-1} \wedge c'_{2k})',$$

i.e., $c_{2l-1} \wedge c'_{2l} \perp c_{2k-1} \wedge c'_{2k}$. In the second case it follows analogously that $c_1 \wedge c'_2, \ldots, c_{m-2} \wedge c'_{m-1}$ are mutually orthogonal. Moreover, we have in this case $c_m \leq c_{2k} \leq c'_{2k-1} \vee c_{2k} = (c_{2k-1} \wedge c'_{2k})'$ for all $k = 1, \ldots, \frac{m-1}{2}$ and hence the elements $c_1 \wedge c'_2, \ldots, c_{m-2} \wedge c'_{m-1}, c_m$ are mutually orthogonal in this case.) Therefore $f \in R(L)$.

The inclusion $R(L) \subseteq A(L)$ is obvious.

Finally, assume $f \in A(L)$. Then $f^*(I) = \langle f, \overline{p}_{a_I} \rangle \in \{0, 1\}$ for all $I \subseteq N$. Hence $f \in A_1$. □

Lemma 4. *For $I, K \subseteq N$ define $a_{IK} := 1$ and $b_{IK} := (-1)^{|I \setminus K|}$ if $K \subseteq I$ and $a_{IK} = b_{IK} := 0$ otherwise. Then the matrices $(a_{IK})_{I,K \subseteq N}$ and $(b_{IK})_{I,K \subseteq N}$ are mutually inverse.*

Proof. The proof uses the method of interchanging of sum signs and the special case

$$\sum_{i=0}^{m} \binom{m}{i} (-1)^i = (1 + (-1))^m = \delta_{m0} \quad (m \geq 0)$$

of the binomial theorem. □

Lemma 5. *For every $f, g\colon 2^N \to \mathbf{R}$ the following are equivalent:*
(i) $\sum_{K\subseteq I} f(K) = g(I)$ *for all* $I \subseteq N$,
(ii) $\sum_{K\subseteq I} (-1)^{|I\setminus K|} g(K) = f(I)$ *for all* $I \subseteq N$.

Proof. With the notation of the previous lemma we have

$$\sum_{K\subseteq I} f(K) = \sum_{K\subseteq I} a_{IK} f(K) \text{ and } \sum_{K\subseteq I} (-1)^{|I\setminus K|} g(K) = \sum_{K\subseteq I} b_{IK} g(K).$$

□

Remark. According to Lemma 5 the elements of A_1 correspond in a natural bijective way to the mappings from 2^N to $\{0, 1\}$.

3. Minimal Sets of Valid Bell-type Inequalities

Theorem 6. *Assume $n \in \{2, 3\}$. Then the following are equivalent:*
(i) $A(L) = A_0$,
(ii) *there exist $p \in S(L)$ and $a, b \in L$ with $p(a \wedge b) < \frac{p(a)+p(b)-1}{2}$.*

Proof. First we remark that for $p \in S(L)$ and $a, b \in L$ the inequalities

$$p(a \wedge b) < \frac{p(a) + p(b) - 1}{2}$$

and

$$p(a) + p(b) - 2p(a \wedge b) > 1$$

are equivalent. If (i) holds then the inequality

$$0 \le p(a) + p(b) - 2p(a \wedge b) \le 1$$

does not hold in L and hence because of the monotonicity of states there exist $p \in S(L)$ and $a, b \in L$ with $p(a) + p(b) - 2p(a \wedge b) > 1$, i.e. (ii) holds.

Conversely, assume (ii) holds. In the following we only consider inequalities of the form

$$0 \le \sum_{I\subseteq N} f(I) p(\bigwedge_{i\in I} a_i) \le 1$$

with $f(\emptyset) = 0$ since for all $f \in A_1$ it holds $f(\emptyset) \in \{0, 1\}$ and since the inequalities $0 \le x \le 1$ and $0 \le 1 - x \le 1$ are equivalent.

Case 1. $n = 2$

The only Bell-type inequalities corresponding to the elements f of $A_1 \setminus A_0$ with $f(\emptyset) = 0$ are the inequalities

$$0 \le p(a) + p(b) - p(a \wedge b) \le 1,$$
$$0 \le p(a) + p(b) - 2p(a \wedge b) \le 1.$$

Since there exist $p \in S(L)$ and $a, b \in L$ with $p(a) + p(b) - 2p(a \wedge b) > 1$ and since $p(a) + p(b) - p(a \wedge b) \geq p(a) + p(b) - 2p(a \wedge b)$ we have $p(a) + p(b) - p(a \wedge b) > 1$. Hence (i) holds.

Case 2. $n = 3$

Let $f \in A_1 \setminus A_0$ with $f(\emptyset) = 0$. If there exist $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $f(\{i\}) = f(\{j\}) = 1$ then

$$\sum_{I \subseteq \{1,2,3\}} f(I) p(\bigwedge_{i \in I} a_i) = p(a) + p(b) - \left(2 - \sum_{I \subseteq \{1,2,3\}} f(I)\right) p(a \wedge b) > 1$$

if $(a_i, a_j, a_k) = (a, b, a \wedge b)$ and hence the inequality

$$0 \leq \sum_{I \subseteq \{1,2,3\}} f(I) p(\bigwedge_{i \in I} a_i) \leq 1$$

does not hold in L. Now, up to symmetry, the following twelve inequalities remain to be considered:

$$\begin{aligned}
&0 \leq p(a \wedge b) + p(a \wedge c) - p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a \wedge b) + p(a \wedge c) - 2p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a \wedge b) + p(a \wedge c) + p(b \wedge c) - 2p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a \wedge b) + p(a \wedge c) + p(b \wedge c) - 3p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) + p(b \wedge c) - p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) + p(b \wedge c) - 2p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) - p(a \wedge c) + p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) - p(a \wedge c) + 2p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) + p(b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) + p(b \wedge c) - p(a \wedge b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) - p(a \wedge c) + p(b \wedge c) \leq 1, \\
&0 \leq p(a) - p(a \wedge b) - p(a \wedge c) + p(b \wedge c) + p(a \wedge b \wedge c) \leq 1.
\end{aligned}$$

By setting in each single of these inequalities an appropriate one of the three variables a, b, c equal to 1 one sees that these inequalities are not satisfied in L. This shows that (i) holds. □

Remark. The idea of the last part of the proof of Theorem 6 can be used in order to obtain the following general result: If the inequality

$$0 \leq p(a) + p(b) - 2p(a \wedge b) \leq 1$$

does not hold in L and if $f \in A_1$ is such that there exist $i, j \in N$ with $i \neq j$ and $(f(\emptyset), f(\{i\}), f(\{j\})) \in \{(0,1,1),(1,-1,-1)\}$ then the inequality

$$0 \leq \sum_{I \subseteq N} f(I) p(\bigwedge_{i \in I} a_i) \leq 1$$

does not hold in L. (For the proof put $a_i := a$, $a_j := b$ and $a_k := a \wedge b$ for all $k \in N \setminus \{i,j\}$.)

Lemma 7. *Exactly half of the subsets of a finite non-empty set are of even cardinality (and therefore exactly half of the subsets are of odd cardinality).*

Proof. Induction on the cardinality of the base set. □

Lemma 8. $|f(I)| \leq 2^{|I|-1}$ *for all* $f \in A_1$ *and all* $I \in 2^N \setminus \{\emptyset\}$.

Proof. For $f \in A_1$ and $\emptyset \neq I \subseteq N$ we have according to Lemma 5

$$-2^{|I|-1} = -|\{K \subseteq I \,|\, |I \setminus K| \text{ odd}\}| \leq f(I) \leq |\{K \subseteq I \,|\, |I \setminus K| \text{ even}\}| = 2^{|I|-1}.$$

□

The following theorem gives a sufficient condition for the minimality of $A(L)$:

Theorem 9. *If there exist a* $p \in S(L)$ *and* $a, b \in L$ *with* $p(a) + p(b) > 1$ *and* $p(a \wedge b) \leq \frac{2(p(a)+p(b)-1)}{3^n}$ *then* $A(L) = A_0$.

Proof. Assume there exist $p \in S(L)$ and $a, b \in L$ with $p(a) + p(b) > 1$ and $p(a \wedge b) \leq \frac{2(p(a)+p(b)-1)}{3^n}$. Let $f \in A(L)$. Then $f \in A_1$. Put $M := \{I \subseteq N \,|\, f(I) \neq 0\}$. Assume $(M, \subseteq)$ not to be a chain. Then there exist two elements of M which are not comparable w.r.t. $\subseteq$. Let B, C be two such elements of M with the additional property that $|B| + |C|$ is minimal. If $D \in (M \cap 2^B) \setminus \{B\}$ then because of $D, C \in M$ and $|D| + |C| < |B| + |C|$, D and C are comparable with respect to $\subseteq$. Now $C \subseteq D$ would imply $C \subseteq B$ which contradicts the choice of B and C. Hence $C \not\subseteq D$ and therefore $D \subset C$, i. e. $D \in (M \cap 2^C) \setminus \{C\}$. This shows $(M \cap 2^B) \setminus \{B\} \subseteq (M \cap 2^C) \setminus \{C\}$. By a symmetry argument it follows $(M \cap 2^C) \setminus \{C\} \subseteq (M \cap 2^B) \setminus \{B\}$. Hence $(M \cap 2^B) \setminus \{B\} = (M \cap 2^C) \setminus \{C\} =: E$. Because of the choice of B and C, E is a chain and we have $M \cap 2^B = E \cup \{B\}$ and $M \cap 2^C = E \cup \{C\}$. We put

$$S := \sum_{I \in E} f(I).$$

If $E = \emptyset$ then $S = 0$ and if $E \neq \emptyset$ then $(E, \subseteq)$ has a greatest element, say F, and $S = f^*(F) \in \{0,1\}$. Moreover, we have $S + f(B) = f^*(B) \in \{0,1\}$ and $S + f(C) = f^*(C) \in \{0,1\}$. Since $f(B), f(C) \in \mathbf{Z} \setminus \{0\}$, we conclude that

$$(S, f(B), f(C)) \in \{(0,1,1),(1,-1,-1)\}. \tag{2}$$

Now define $a \in L^n$ as follows:

$$a_i := \begin{Bmatrix} a \\ b \\ 0 \\ 1 \end{Bmatrix} \text{ if } i \in \begin{cases} B \setminus C \\ C \setminus B \\ N \setminus (B \cup C) \\ B \cap C \end{cases}$$

Put $T := \langle f, \tilde{p}_a \rangle$ and

$$U := \sum_{I \subseteq B \cup C,\ I \not\subseteq B,\ I \not\subseteq C} f(I).$$

Then $T = S + f(B)p(a) + f(C)p(b) + Up(a \wedge b)$ and hence because of (2)

$$T \in \{p(a) + p(b) + Up(a \wedge b), 1 - p(a) - p(b) + Up(a \wedge b)\}. \tag{3}$$

But because of

$$|U| \leq \sum_{I \subseteq B \cup C,\ I \not\subseteq B,\ I \not\subseteq C} |f(I)| < \sum_{\emptyset \neq I \subseteq N} |f(I)| \leq \sum_{i=1}^{n} \binom{n}{i} 2^{i-1} < \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} 2^i = \frac{3^n}{2}$$

we have $|Up(a \wedge b)| < p(a) + p(b) - 1$ which together with (3) contradicts $T \in \{0, 1\}$. Hence $(M, \subseteq)$ is a chain and $A(L) = A_0$ now easily follows from $f \in A_1$. □

A weaker sufficient condition for the minimality of $A(L)$, which is often satisfied, is the following:

Theorem 10. *If there exist a $p \in S(L)$ and $a, b \in L$ with $p(a) = p(b) = 1$ and $p(a \wedge b) = 0$ then $A(L) = A_0$.*

Proof. Theorem 9. □

Remark. The conditions of Theorem 10 are often satisfied in so-called Greechie logics when one takes for a and b atoms lying in different maximal Boolean subalgebras. (Greechie logics are logics which are built up from finitely many finite Boolean algebras (containing at least three atoms) by "pasting them together" in a certain way.) So, one can say that in Greechie logics often only those Bell-type inequalities are valid which follow directly from the fact that every state is monotone. This phenomenon also appeared in another context in [**1**].

Next we want to show that in non-trivial "Hilbert logics" $L(H)$ there hold only "a few" Bell-type inequalities.

Lemma 11. *Let H be a complex Hilbert space of dimension > 1 and let $\varepsilon \in (0, 1]$. Then there exist $a_1, a_2 \in H$ and $p \in S(L(H))$ such that $p(\langle\{a_1\}\rangle), p(\langle\{a_2\}\rangle) > 1 - \varepsilon$ and $\langle\{a_1\}\rangle \cap \langle\{a_2\}\rangle = \{0\}$.*

Proof. Since $\dim H > 1$ there exist two linearly independent elements a, b of H. Clearly, $a, b \neq 0$. Without loss of generality, $|a| = 1$. Because of the continuity of polynomials there exists a $\delta > 0$ such that $a^2 + 2\mathrm{Re}(ab)\lambda + b^2\lambda^2 > 0$ for all $\lambda \in [0, \delta)$. Because of the continuity of rational functions on their domain there exist $\delta_1, \delta_2 \in (0, \delta)$ with $\delta_1 \neq \delta_2$ and

$$\frac{|a^2 + ab\delta_i|}{a^2 + 2\mathrm{Re}(ab)\delta_i + b^2\delta_i^2} > \sqrt{1 - \varepsilon}$$

for $i = 1, 2$. Now put $a_i := a + \delta_i b$ for $i = 1, 2$ and $p(M) := |P_M a|^2$ for all $M \in L(H)$ where for each $M \in L(H)$ P_M denotes the orthogonal projection of H onto M. Then $a_1, a_2 \neq 0$ and

$$p(\langle\{a_i\}\rangle) = \left|\frac{aa_i}{a_i^2}\right|^2 = \frac{|a^2 + ab\delta_i|^2}{\left(a^2 + 2\mathrm{Re}\,(ab)\delta_i + b^2\delta_i^2\right)^2} > 1 - \varepsilon$$

for $i = 1, 2$. Finally, let $c \in \langle\{a_1\}\rangle \cap \langle\{a_2\}\rangle$. Then there exist $\alpha_1, \alpha_2 \in \mathbf{C}$ with $c = \alpha_1 a_1 = \alpha_2 a_2$, i.e., $\alpha_1(a + \delta_1 b) = \alpha_2(a + \delta_2 b)$. Since a and b are linearly independent, we conclude $\alpha_1 = \alpha_2$ and $\alpha_1\delta_1 = \alpha_2\delta_2$ which together with $\delta_1 \neq \delta_2$ implies $\alpha_1 = \alpha_2 = 0$ whence $c = 0$. This shows $\langle\{a_1\}\rangle \cap \langle\{a_2\}\rangle = \{0\}$. □

Theorem 12. *If H is a complex Hilbert space of dimension > 1 then $A(L(H)) = A_0$.*

Proof. Lemma 11 and Theorem 9. □

Remark. It should be mentioned that because of Gleason's theorem every state p on $L(H)$, where H is an at least three-dimensional Hilbert space, has the Jauch-Piron property. Hence for Hilbert spaces of dimension > 2 the result of Theorem 12 cannot be obtained by applying Theorem 10.

Finally, we will provide an asymptotic formula for $|A_0|$.

Theorem 13. $|A_0| \sim \frac{2}{\log 2}\frac{n!}{(\log 2)^n}$ *for $n \to \infty$.*

Proof. Put $a_n := |A_0|$. Because of the definition of A_0, a_n is the number of chains in $(2^N, \subseteq)$. Let b_n denote the number of chains in $(2^N \setminus \{\emptyset\}, \subseteq)$. If we put $b_0 := 1$ then we have

$$b_n = 1 + \sum_{i=1}^{n} \binom{n}{i} b_{n-i} = 1 + \sum_{i=1}^{n} \binom{n}{n-i} b_{n-i} = 1 + \sum_{i=0}^{n-1} \binom{n}{i} b_i$$

for $n \geq 1$. Since $a_n = 2b_n$ for $n \geq 0$, we have

$$a_n = 2 + \sum_{i=0}^{n-1} \binom{n}{i} a_i \text{ and hence } 2a_n = 2 + \sum_{i=0}^{n} \binom{n}{i} a_i$$

for $n \geq 0$. An easy calculation shows that $g(x) = \frac{2e^x}{2-e^x}$ is the exponentially generating function $g(x) = \sum\limits_{k=0}^{\infty} a_k \frac{x^k}{k!}$ of the sequence $a_0, a_1, a_2, \ldots$. Now

$$g(x) = -2 - 2\frac{1}{e^{x-\log 2} - 1}. \tag{4}$$

According to the Theorem of Mittag-Leffler we have

$$\frac{1}{e^x - 1} = -\frac{1}{2} + \sum_{j=-\infty}^{\infty} \frac{1}{x - 2j\pi i} \tag{5}$$

Using (4) and (5) and the formula for the sum of an infinite geometric series we obtain

$$g(x) = -1 + 2\sum_{k=0}^{\infty} x^k \sum_{j=-\infty}^{\infty} \frac{1}{(\log 2 + 2j\pi i)^{k+1}}.$$

Hence we have

$$a_n = -\delta_{n0} + 2n! \sum_{j=-\infty}^{\infty} \frac{1}{(\log 2 + 2j\pi i)^{n+1}}$$

for $n \geq 0$. Now

$$\begin{aligned} \frac{a_n}{\frac{2}{\log 2}\frac{n!}{(\log 2)^n}} &= 1 + \left(\frac{\log 2}{\log 2 + 2\pi i}\right)^{n+1} + \left(\frac{\log 2}{\log 2 - 2\pi i}\right)^{n+1} \\ &\quad + \sum_{j=2}^{\infty} \left(\frac{\log 2}{\log 2 + 2j\pi i}\right)^{n+1} + \sum_{j=2}^{\infty} \left(\frac{\log 2}{\log 2 - 2j\pi i}\right)^{n+1} \end{aligned}$$

for $n \geq 1$. Since

$$\begin{aligned} 0 &\leq \left| \sum_{j=2}^{\infty} \left(\frac{\log 2}{\log 2 + 2j\pi i}\right)^{n+1} \right| = \left| \sum_{j=2}^{\infty} \left(\frac{\log 2}{\log 2 - 2j\pi i}\right)^{n+1} \right| \\ &\leq \sum_{j=2}^{\infty} \left(\frac{1}{2j\pi}\right)^{n+1} \leq \int_1^{\infty} \left(\frac{1}{2x\pi}\right)^{n+1} dx \\ &= -\frac{1}{2n\pi}\frac{1}{(2x\pi)^n} \Big|_1^{\infty} = \frac{1}{2n\pi}\frac{1}{(2\pi)^n} \to 0 \end{aligned}$$

for $n \to \infty$, and since

$$\left|\frac{\log 2}{\log 2 + 2\pi i}\right| = \left|\frac{\log 2}{\log 2 - 2\pi i}\right| < 1$$

we have

$$\frac{a_n}{\frac{2}{\log 2}\frac{n!}{(\log 2)^n}} \to 1$$

for $n \to \infty$ and therefore

$$|A_0| \sim \frac{2}{\log 2}\frac{n!}{(\log 2)^n}$$

for $n \to \infty$. □

4. Maximal Sets of Valid Bell-type Inequalities

Theorem 14 (cf. [**3**]). *If $n = 2$ the $A(L) = A_1$ if and only if every state on L is subadditive.*

Proof. From the proof of Theorem 6 it follows that in the case $n = 2$ $A(L) = A_1$ is equivalent to the fact that $p(a) + p(b) - p(a \wedge b) \leq 1$ for all $p \in S(L)$ and all $a, b \in L$. Replacing a, b within the last inequality by a', b', respectively, yields the desired result. □

Theorem 15. *Assume L to be nearly Boolean. Then $R(L) = A(L) = A_1$.*

Proof. Let $f \in A_1$. Put

$$t(x_1, \ldots, x_n) := \bigvee_{K \subseteq N,\, f^*(K)=1} (\bigwedge_{i \in K} x_i \wedge \bigwedge_{i \in N \setminus K} x_i')$$

and let $p \in S(L)$. Since L is nearly Boolean there exists a Boolean algebra B, a state q on B and a homomorphism φ from L to B with $q \circ \varphi = p$. Let $a \in L^n$ and put $b := (\varphi(a_1), \ldots, \varphi(a_n))$ and

$$c_I := \bigwedge_{i \in I} b_i \wedge \bigwedge_{i \in N \setminus I} b_i'.$$

for all $I \subseteq N$. Then

$$\begin{aligned}
\langle f, \overline{p}_a \rangle &= \langle f, \overline{q \circ \varphi_a} \rangle = \langle f, \overline{q}_b \rangle \\
&= \sum_{I \subseteq N} f(I) q \left(\bigvee_{I \subseteq K \subseteq N} c_K \right) = \sum_{I \subseteq N} f(I) \sum_{I \subseteq K \subseteq N} q(c_K) \\
&= \sum_{K \subseteq N} q(c_K) f^*(K) = \sum_{K \subseteq N,\, f^*(K)=1} q(c_K) \\
&= q(t_B(b)) = q(\varphi(t_L(a))) = (p \circ t_L)(a).
\end{aligned}$$

Hence $f \in R(L)$. The rest of the proof follows from Theorem 3. □

Remark. The above theorem says that in Boolean algebras, for every $f \in A_1$ there exists a $g\colon L^n \to L$ such that $\langle f, \overline{p}_\bullet \rangle = p \circ g$ for all $p \in S(L)$. Since every Boolean algebra has a full set of states (consider the Stone representation and states concentrated on one element), the above g is uniquely determined by f. Observe that for $I \subseteq N$ $g(a_I) = 1$ if $f^*(I) = 1$ and $g(a_I) = 0$ if $f^*(I) = 0$. Hence, according to Lemma 5 we have

$$f(I) = \sum_{K \subseteq I} (-1)^{|I \setminus K|} g(a_K)$$

for all $I \subseteq N$. From the existence of the disjunctive normal form for term functions on Boolean algebras and from Lemma 5 it follows that in Boolean algebras the functions g corresponding to the elements of A_1 are exactly the n-ary term functions on L.

Example. Let O_6 denote the six-element ortholattice $\{0, a, a', b, b', 1\}$ where $0 < a < b < 1$. Then $R(O_6) = A(O_6) = A_1$. This can be seen as follows: Let $p \in S(L)$. Since $a \perp b'$ we have $p(a) = p(a) + p(b') - p(b') = p(a \vee b') - p(b') = p(1) - p(b') = 1 - p(b') = p(b)$. For every $\alpha \in [0,1]$ let p_α denote the mapping from L to $[0,1]$ defined by $p_\alpha(0) := 0$, $p_\alpha(a) = p_\alpha(b) := \alpha$, $p_\alpha(a') = p_\alpha(b') := 1 - \alpha$ and $p_\alpha(1) := 1$. Then, obviously, $S(O_6) = \{p_\alpha \mid \alpha \in [0,1]\}$. Now it is easy to see that every state on O_6 can be derived from a state on the four-element Boolean algebra. Hence, O_6 is nearly Boolean and therefore $R(O_6) = A(O_6) = A_1$ according to Theorem 15.

In the following we consider orthomodular logics.

Theorem 16. *Assume $n \geq 3$. Then the following are equivalent:*

(i) *L is nearly Boolean,*

(ii) *$p(t_1(a)) = p(t_2(a))$ for all $p \in S(L)$, all $m \geq 1$, all m-ary terms t_1, t_2 on L such that the law $t_1 = t_2$ holds in every Boolean algebra and all $a \in L^m$,*

(iii) *$A = A_1$,*

(iv) *A is maximal.*

Proof. The proof that (i)–(iii) are equivalent can be found e.g. in [**4**]. The equivalence of (iv) to each of (i)–(iii) now follows by Theorem 3. □

Theorem 17. *Assume $n \geq 3$. Then* (i) *and* (ii) *hold:*

(i) *If L has a separating set of states then A is maximal iff L is Boolean.*

(ii) *If L has a full set of states then A is maximal iff L is Boolean.*

Proof. Theorem 16. □

Remark. That (also in case $n \geq 3$) the maximality of A does not characterize the distributivity of L can be seen by the following observation which shows that there exist nearly Boolean logics which are not Boolean. In order to see this consider a non-trivial Boolean algebra B and a stateless logic L_0. Since every non-trivial Boolean algebra possesses at least one state (consider the Stone representation and any state concentrated on one fixed element), L_0 is not Boolean. Since L_0 is a homomorphic image of $B \times L_0$, $B \times L_0$ is also not Boolean. Let $p \in S(B \times L_0)$. Since $p(0,1) > 0$ would imply $\frac{p(0,.)}{p(0,1)} \in S(L_0)$, we have $p(0,1) = 0$ and hence $p(a,b) = p((a,0) \vee (0,b)) = p(a,0) + p(0,b) = p(a,0)$ for all $a, b \in L$ and $p(.,0) \in S(B)$. Therefore $S(B \times L_0) = \{(x,y) \mapsto q(x) \mid q \in S(B)\}$. Hence $B \times L_0$ is nearly Boolean.

Theorem 18. $|A_1| = 2^{2^n}$.

Proof. Remark after Lemma 5. □

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