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# COEXISTENCE OF SINGULAR AND REGULAR SOLUTIONS FOR THE EQUATION OF CHIPOT AND WEISSLER

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## 1. Introduction

In this paper, we are interested in the existence of positive solutions of

$$(P_{B_R}) \qquad \begin{cases} \Delta u - |\nabla u|^q + \lambda u^p = 0 & \text{on } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where $B_R$ is a ball in $\mathbb{R}^n$ of radius $R$ and

$$p > 1,\ q = \frac{2p}{p+1},\ \lambda > 0.$$

This problem was introduced in 1989 by M. Chipot and F. Weissler (cf. [**CW**]) in connection with the study of the nonlinear parabolic equation

$$\begin{aligned} u_t &= \Delta u - |\nabla u|^q + |u|^p && \text{on } B_R \times (0,T), \\ u &= 0 && \text{on } \partial B_R \times (0,T), \\ u(x,0) &= u_0 && \text{on } B_R. \end{aligned}$$

One can show that the solutions of $(P_{B_R})$ are radially symmetric (using the technique of Gidas-Ni-Nirenberg [**GNN**]) and so we consider the solution $u_a$ of

$$(P_a) \qquad \begin{cases} u'' + \dfrac{n-1}{r}u' - |u'|^q + \lambda |u|^p = 0 & \text{if } r > 0, \\ u(0) = a, \\ u'(0) = 0, \end{cases}$$

where $a > 0$.

We will denote by $z(a)$ the first zero of $u_a$ if it exists; if $u_a > 0$ on $[0, +\infty)$, we will set $z(a) = +\infty$.

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We know from [**CW**] that $z$ verifies the relation $z(a) = a^{-\frac{p-1}{2}} z(1)$; we have then only two possibilities

– either $z(a) = +\infty$ for all $a > 0$ and $(P_{B_R})$ has no solution for any $R$;

– or $z(a)$ is finite for all $a > 0$ and $z$ is a decreasing function from $[0, +\infty)$ into $[0, +\infty)$ (cf. [**CW**, Lemma 4.7]); in this case, $(P_{B_R})$ has one and only solution for any $R$.

The range of $\lambda$ is crucial for the behaviour of the map $z$.

In their paper, M. Chipot and F. Weissler show the following result:

**Theorem.** *If $q = \frac{2p}{p+1}$ and $p < \frac{n}{n-2}$ the equation*

$$(I) \qquad u''(r) + \frac{n-1}{r} u'(r) - |u'(r)|^q + \lambda |u(r)|^p = 0$$

*has a solution in the form of $u(r) = kr^{-\frac{2}{p-1}}$ if and only if $\lambda \le \lambda_{n,p}$ where*

$$\lambda_{n,p} = \frac{(2p)^p}{(p+1)^{p+1}(2p - np + n)^p} = \frac{q^p}{(p+1)\,(2p - np + n)^p}.$$

When $n = 1$ the equation (I) becomes autonomous and if $u\colon r \mapsto k\, r^{-\frac{2}{p-1}}$ is a solution of (I), the function $u_1\colon r \mapsto (r + c)^{-\frac{2}{p-1}}$, is a solution too. If $\lambda \le \lambda_{1,p}$, then it follows from the Cauchy theorem and a translation argument (see [**CW**]) that the problem $(P_{B_R})$ has no solution. This is also the case when $n \ge 1$ and $\lambda \le \lambda_{1,p}$ (see [**CW**] or [**V**, Proposition I.7]).

In a more recent paper (see [**FQ**]), M. Fila and P. Quittner show that the condition $\lambda > \lambda_{n,p}$ implies that $z(a)$ is finite for all $a > 0$; but the case $\lambda = \lambda_{n,p}$ where we could have coexistence of the singular solution $u(r) = k\, r^{-\frac{2}{p-1}}$ and solutions of $(P_a)$ with $z(a)$ finite was open. We solve this issue here. Indeed we show :

**Theorem A.** *Assume $q = \frac{2p}{p+1}$, $\lambda_{n,p} = \frac{(2p)^p}{(p+1)^{p+1}(2p-np+n)^p}$ and*

1) $n = 2$

2) $n \ge 3$ *and* $1 < p < \frac{n}{n-2}$.

*Let $u_a$ be the solution of $(P_a)$. Then there exists $\lambda'_{n,p} < \lambda_{n,p}$ such that $z(a)$ is finite for $\lambda > \lambda'_{n,p}$.*

**Remark 1.** When $\lambda = \lambda_{n,p}$, there exists only one solution of (I) of the form $u(r) = kr^{-\frac{2}{p-1}}$ (according to the proof of Proposition 5.5 in [**CW**]) and its graph cuts the one of the solution of $(P_a)$ for any $a > 0$ (see [**V**, Proposition I.6]). In the case $\lambda'_{n,p} < \lambda < \lambda_{n,p}$ the equation (I) has two distinct solutions in the form $u(r) = kr^{-\frac{2}{p-1}}$ whose graphs cut those of the solutions of $(P_a)$.

In the case $\frac{n}{n-2} \le p < \frac{n+2}{n-2}$ there always exist singular solutions of (I). We show here the following theorem:

**Theorem B.** *Assume* $1 < p < \frac{n+2}{n-2}$, $n \geq 3$ *and* $q = \frac{2p}{p+1}$. *If* $\lambda \geq \Lambda_{n,p}$ *where*

$$\Lambda_{n,p} = \frac{1}{(p+1)^{p+1}} + \frac{n\,(p-1)2^p\, q^{p+1}}{(2p+2-np+n)^{p+1}},$$

*then* $z(a)$ *is finite for any* $a > 0$.

**Remark 2.** As in the case where $p < \frac{n}{n-2}$, the graphs of regular and singular solutions are crossing. In order to prove Theorems A and B, following [**FQ**], we introduce a two dimensional autonomous system. The main properties of this system are recalled in Section 2. The Sections 3, 4 and 5 are devoted to the cases $n = 2$, $n \geq 3$ and $p < \frac{n}{n-2}$, $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, respectively.

## 2. Transformation of the Problem to an Autonomous System

Let $u$ be a solution of $(P_a)$. We consider $(X, Y)\colon t \mapsto \big(X(t), Y(t)\big)$ defined by

$$\begin{cases} X(t) = -\dfrac{ru'}{u}, \\ Y(t) = r^2 u^{p-1}, \\ r(t) = e^t. \end{cases} \tag{2}$$

We will recall some results of [**FQ**] in Propositions 1 and 2.

First we find, since $r'(t) = r(t)$

$$\begin{aligned} X'(t) &= \frac{-(ru' + r^2u'')u + u'r^2u'}{u^2} \\ &= \Big(\frac{ru'}{u}\Big)^2 - \frac{ru'}{u} - \frac{r^2u''}{u} \\ &= X^2 + X - \frac{r^2}{u}\Big((-u')^q - \lambda u^p - \frac{(n-1)}{r}u'\Big) \end{aligned}$$

and we obtain

$$X'(t) = (2-n)X + X^2 + \lambda Y - X^{\frac{2p}{p+1}}\, Y^{\frac{1}{p+1}}.$$

On the other hand,

$$\begin{aligned} Y'(t) &= 2r^2u^{p-1} + r^2(p-1)u^{p-2}u'r \\ &= r^2u^{p-1}\Big(2 + r(p-1)\frac{u'}{u}\Big) \\ &= Y\big(2 - (p-1)X\big). \end{aligned}$$

Since $u$ verifies also $u(0) = a$ and $u'(0) = 0$, we have

$$\lim_{t\to-\infty} Y(t) = \lim_{t\to-\infty} X(t) = 0, \text{ according to (2), and } \lim_{t\to-\infty}\frac{Y(t)}{X(t)} = \frac{n}{\lambda}.$$

This last equality results from $\frac{Y(t)}{X(t)} = -\frac{ru^p}{u'}$. If $t \to -\infty$, then $r \to 0$ and $\frac{u'(r)}{r} \to -\lambda\frac{u^p(0)}{n}$ since $u'' + \frac{n-1}{r}u' = |u'|^q - \lambda u^p$ and $\lim_{r\to 0}\frac{u'(r)}{r} = u''(0)$. These results are summarized in the following proposition:

**Proposition 1.** *Let $u$ be a solution of $(P_a)$. If $(X,Y)$ is defined by (2) then $(X,Y)$ is a solution of the autonomous system*

$$\begin{cases} x'(t) = (2-n)x + x^2 + \lambda y - x^{\frac{2p}{p+1}} y^{\frac{1}{p+1}}, \\ y'(t) = y\big(2-(p-1)x\big) \end{cases} \tag{3}$$

*and we have*

$$\lim_{t\to-\infty} Y(t) = \lim_{t\to-\infty} X(t) = 0, \quad \lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda}. \tag{4}$$

Let us recall also (according to a lemma in [**FQ**]) that an orbit of (3) starting when $t = t_0$ in the first quadrant $\{(x,y)\,|\,x \geq 0,\ y \geq 0\}$ stays in this quadrant when $t > t_0$. Moreover, there exists only one orbit coming from the origin $O$, its slope is $\frac{n}{\lambda}$.

The continuous dependence of solutions of $(P_a)$ on $\lambda$ implies that if $z(a)$ is finite for all $a$ when $\lambda = \lambda_{n,p}$, then there exists $\lambda'_{n,p} < \lambda_{n,p}$ such that we have the same behaviour for all $\lambda \in (\lambda'_{n,p}, \lambda_{n,p}]$.

In the computation below we set for convenience $\lambda = \lambda_{n,p}$.

Moreover, define $f$ and $g$ by

$$f(x,y) = (2-n)x + x^2 + \lambda y - x^{\frac{2p}{p+1}} y^{\frac{1}{p+1}},$$
$$g(x,y) = y\big(2-(p-1)x\big).$$

We see that $g(x,y) = 0$ if $y = 0$ or if $x = x_1$ where $x_1 := \frac{2}{p-1}$.

We are going to study the set $\Gamma$ defined by

$$\Gamma = \{(x,y)\,|\,x \geq 0,\ y \geq 0,\ f(x,y) = 0\}.$$

When $n = 2$ this set is one half of the parabola defined by $x \geq 0$ and $y = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$ (see Proposition 2). It cuts the straight line $x = x_1$ at one point only (see Figure 1).

We study also the position (with respect to $\Gamma$) of the orbit $\mathcal{O}$ of the system (3) corresponding to the map $t \mapsto \big(X(t), Y(t)\big)$. We show that

(a) $\mathcal{O}$ is located above $\Gamma$ when $0 < X(t) < x_1$ because on the corresponding part of $\Gamma$ the vector field is "vertical and oriented upwards",

(b) $\mathcal{O}$ cuts the straight line $x = x_1$ above $\Gamma$ (by linearization of the vector field around the point of intersection of $\Gamma$ with this straight line),

(c) $X(t)$ blows up in finite time (see Figure 1).

When $n \geq 3$, $\Gamma$ is tangent to the straight line $x = x_1$ (for $\lambda = \lambda_{n,p}$) and located in the half-plane defined by $x \leq x_1$ (see Figure 2). We show next that for $0 < X(t) \leq x_1$, $\mathcal{O}$ is located above $\Gamma$, and that if $X(t) > x_1$ then $X'(t) > 0$ and $Y'(t) < 0$. Then we deduce that $X(t)$ blows up in finite time.

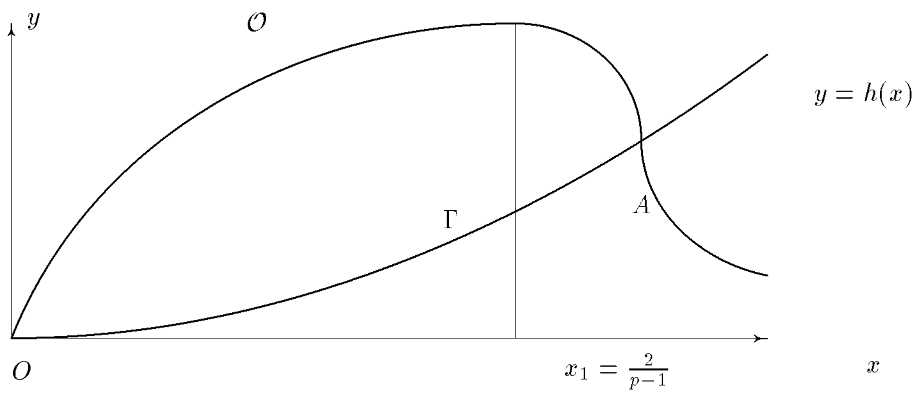


**Figure 1.**

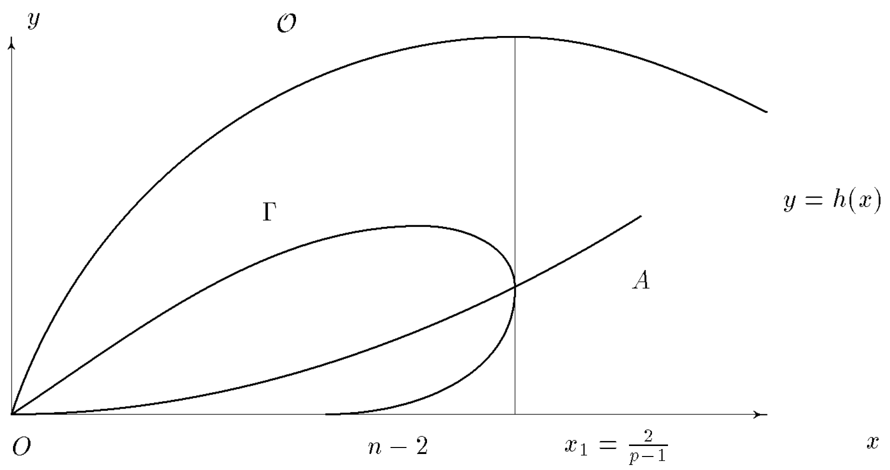


**Figure 2.**

**Proposition 2.** *Let $\lambda = \lambda_{n,p}$ and $x$ be fixed, $x > 0$, then $f(x,y)$ has a unique minimum for $y = h(x) := x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. This minimum is*

$$m(x) := f\big(x, h(x)\big) = x(n-2)\Big(\frac{p-1}{2}x - 1\Big).$$

*The vector field $\big(f(x,y), g(x,y)\big)$ has on the half straight line $x = x_1$ and $y \geq 0$ only one singular point $A = (x_1, y_1)$ with $y_1 = h(x_1)$.*

*Proof.* Put $h(x)=x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. We have

$$\frac{\partial f}{\partial y}(x,y)=\lambda-\frac{1}{p+1}x^{\frac{2p}{p+1}}y^{-\frac{p}{p+1}}$$

so that

$$\frac{\partial f}{\partial y}(x,y)<0 \ \text{ if } \ 0<y<h(x) \ \text{ and } \ \frac{\partial f}{\partial y}(x,y)>0 \ \text{ if } \ y>h(x).$$

Therefore, the map $y\mapsto f(x,y)$ has a unique minimum for $y=h(x)$. Its value is

$$f\big(x,h(x)\big)=(2-n)x+x^2\,\frac{\lambda^{\frac{1}{p}}(p+1)^{\frac{p+1}{p}}-p}{\lambda^{\frac{1}{p}}(p+1)^{1+\frac{1}{p}}}.$$

But

$$\lambda^{\frac{1}{p}}\big(p+1\big)^{\frac{p+1}{p}}=\frac{2p}{2p-np+n}$$

and

$$\frac{\lambda^{\frac{1}{p}}(p+1)^{\frac{p+1}{p}}-p}{\lambda^{\frac{1}{p}}(p+1)^{1+\frac{1}{p}}}=(p-1)\Big(\frac{n}{2}-1\Big).$$

We obtain finally

$$f\big(x,\ h(x)\big)=(2-n)x+x^2(p-1)\Big(\frac{n}{2}-1\Big)=x\ (n-2)\ \Big(\frac{p-1}{2}x-1\Big).$$

The fact that the vector field $\big(f(x,y),\ g(x,y)\big)$ has on the half straight line defined by $x=x_1$, $y\geq 0$ only one singular point $A=\big(x_1,h(x_1)\big)$ can be deduced easily from the expression for $m(x)$. This completes the proof of Proposition 2. $\square$

On the set $\mathcal{E}$ defined by $\mathcal{E}=\{(x,y)\,|\,0<x<x_1, y>0\}$, we have $g(x,y)>0$. If we consider an orbit defined by a map $t\mapsto\big(X_1(t),Y_1(t)\big)$ such that $\big(X_1(t_0),Y_1(t_0)\big)$ is in $\mathcal{E}$, we have a priori three possible behaviours since the vector field $\big(f(x,y),g(x,y)\big)$ has no singular point in $\mathcal{E}$:

1) either $Y_1(t)\to+\infty$ as $t\to\alpha$ (with $\alpha=+\infty$ or $\alpha$ real) and $X_1(t)<x_1$ for $t\geq t_0$;
2) either the orbit cuts the straight line $x=x_1$;
3) or this orbit has $A$ as the limit-point as $t\to\infty$.

First note that the case 1) cannot occur since from the formulae (3) for $X_1'(t)$ and $Y_1'(t)$ we could deduce $\limsup_{t\to\alpha}\frac{Y_1'(t)}{X_1'(t)}\leq\frac{2}{\lambda}$. Since $\lim_{t\to\alpha}Y_1(t)=+\infty$, we would get $\lim_{t\to\alpha}X_1(t)=+\infty$ which yields a contradiction with $X_1(t)<x_1$.

## 3. The Case $n = 2$

In this section we prove Theorem A for $n = 2$.

According to Proposition 2, for any $x > 0$ the map $y \mapsto f(x,\, y)$ has a unique minimum and its value is $f\big(x, h(x)\big) = 0$ when $n = 2$. The set defined by $f(x, y) = 0$, $x \geq 0$, $y \geq 0$ is then one half of the parabola defined by $y = h(x) = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}}$. Moreover, if $x > 0$, $y > 0$ and $y \neq h(x)$ then $f(x, y) > 0$.

Since $\lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda} = \frac{2}{\lambda}$, the orbit $\mathcal{O}$ defined by the map $t \mapsto \big(X(t), Y(t)\big)$ is in a neighbourhood of the origin above the parabola.

The vector field $\big(f(x, y), g(x, y)\big)$ has two singular points in the first quadrant of the plane: the origin $O = (0, 0)$ and $A = (x_1, y_1)$ with $y_1 = h(x_1)$.

If $0 < x < x_1$ and $y = h(x)$, then $f(x, y) = 0$ and $g(x, y) > 0$, so that the orbit of $t \mapsto \big(X(t), Y(t)\big)$ can cut the line $x = x_1$ at $(x_1, y)$ with $y \geq y_1$ or have $A$ as the limit point. Let us show that in fact the first possibility occurs.

For this, linearize the vector field $(x, y) \mapsto \big(f(x, y), g(x, y)\big)$ at the point $A$. We have

$$\frac{\partial f}{\partial x}(x, y) = 2x - \frac{2p}{p+1} x^{\frac{p-1}{p+1}} y^{\frac{1}{p+1}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,\, y) = \lambda - \frac{1}{p+1} x^{\frac{2p}{p+1}} y^{-\frac{p}{p+1}}.$$

Since $\big(\lambda(p+1)\big)^{\frac{p+1}{p}} = \big(\frac{p}{p+1}\big)^{p+1}$, we have $h(x) = x^2\big(\lambda(p+1)\big)^{-\frac{p+1}{p}} = x^2\big(\frac{p+1}{p}\big)^{p+1}$ and

$$\frac{\partial f}{\partial x}\big(x, h(x)\big) = 2x\Big(1 - \frac{p}{p+1} x^{-\frac{2}{p+1}} x^{\frac{2}{p+1}} \frac{p+1}{p}\Big) = 0 \quad \text{for any } x.$$

Next, since $y \mapsto f(x_1, y)$ has its minimum for $y = y_1$ then $\frac{\partial f}{\partial y}(x_1,\; y_1) = 0$. We have also $\frac{\partial g}{\partial x}(x_1, y_1) = -\frac{4}{p-1}\big(\frac{p+1}{p}\big)^{p+1}$ and $\frac{\partial g}{\partial y}(x_1, y_1) = 0$. Thus we obtain

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} (x_1, y_1) = \begin{pmatrix} 0 & 0 \\ -\frac{4}{p-1}\Big(\frac{p+1}{p}\Big)^{p+1} & 0 \end{pmatrix}. \tag{5}$$

Let us now show that the orbit cuts the straight line $x = x_1$ at $A' = (x_1, y_1')$ with $y_1' > y_1$. Consider the set $\mathcal{E}'$ of the plane defined by

$$\mathcal{E}' = \{(x, y) \,|\, 0 < x < x_1,\, h(x) \leq y \leq y_1\}$$

and let us set

$$a = x - x_1, \quad b = y - y_1.$$

On the other hand, let $C_1 = h'(x_1)$ be the slope of the tangent line to the parabola at the point $A = \big(x_1, h(x_1)\big)$. We can see (cf. Figure 1) that if $(x, y)$ is in $\mathcal{E}'$, then $\frac{b}{a} < C_1$. Next, from (5) there exists $\epsilon > 0$ such that $-\epsilon < a < 0$ and $-\epsilon < b \leq 0$ imply, as $f \geq 0$ on $\mathcal{E}$,

$$0 \leq f(x_1 + a,\; y_1 + b) < \frac{C_2}{4\, C_1(1 + C_1)}(a + b)$$

where

$$C_2 = -\frac{4}{p-1}\left(\frac{p+1}{p}\right)^{p+1} = \frac{\partial g}{\partial x}(x_1, y_1)$$

and

$$g(x_1 + a, y_1 + b) > \frac{a\,C_2}{2}. \tag{6}$$

If $(x, y) \in \mathcal{E}'$ is close to $A$ then $-\epsilon < a < 0$ and $-\epsilon < b \le 0$ so we can deduce from the fact that $aC_1 < b$ that

$$f(x_1 + a, y_1 + b) < \frac{C_2}{4C_1(1 + C_1)}(a + aC_1) = \frac{aC_2}{4C_1}$$

and, finally,

$$\frac{g(x_1 + a,\, y_1 + b)}{f(x_1 + a,\, y_1 + b)} > 2C_1$$

using (6). So we obtain that an orbit of the vector field $\big(f(x, y), g(x, y)\big)$ passing, for $t = t_0$, through a point $(x, y)$ in $\mathcal{E}'$ such that

$$-\epsilon < a < 0 \quad \text{ and } \quad -\epsilon < b \le 0 \quad \text{ where } \; a = x - x_1, \; b = y - y_1$$

(cf. Figure 2) cuts for $t_1 > t_0$ the straight line $x = x_1$ at $(x_1, y_1'')$ with $y_1'' > y_1$. Since orbits cannot intersect, the orbit $\mathcal{O}$ has to cut the line $x = \frac{2}{p-1}$ above $A$.

Now $X(t) > x_1$ implies $g\big(X(t), Y(t)\big) < 0$ and $f\big(X(t), Y(t)\big) > 0$ except if $Y(t) = h\big(X(t)\big)$; then, if $t \ge t_1'$, $Y(t) \le Y(t_1')$, on the other hand $X$ is increasing. The first equation of (3) shows that there exist $\alpha > 0$ and $x_2 > 0$ such that $x > x_2$ implies $f\big(x(t), y(t)\big) > \alpha x^2$. We deduce from this that $X$ blows up in a finite time $T$ and $z(a) = e^T$.

## 4. The Case $n \ge 3$ and $1 < p < \frac{n}{n-2}$

**Proposition 3.** *Let $\lambda = \lambda_{n,p}$. The curve $\Gamma = \{(x, y) \mid x \ge 0,\ y \ge 0,\ f(x,\ y) = 0\}$ admits a tangent line at every point. The half straight line defined by $x = c$, $y \ge 0$ cuts $\Gamma$ at one point if $c = \frac{2}{p-1}$, two points if $n - 2 \le c < \frac{2}{p-1}$, one point if $0 \le c < n - 2$.*

*Moreover, $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are the graphs of some functions $h_1$ and $h_2$,*

$$\Gamma_1 = \{(x, y) \in \Gamma \mid y \le h(x)\}, \qquad \Gamma_2 = \{(x, y) \in \Gamma \mid y \ge h(x)\}$$

*(see Figure 2).*

*Proof.* One has $f(x, 0) = (2 - n)x + x^2$ so that $f(\cdot, 0) < 0$ on $(0, n - 2)$ and $f(\cdot, 0) > 0$ on $(n - 2, +\infty)$. On the other hand, according to Proposition 2, for $x$

fixed, $x > 0$, the map $y \mapsto f(x,\ y)$ attains its unique minimum at $y = h(x)$; its value is

$$m(x) = x(n-2)\Big(\frac{p-1}{2}\,x - 1\Big).$$

This minimum is negative if $0 < x < x_1$, zero if $x = x_1$ and positive if $x > x_1$. A half line $x = C$, $y \geq 0$ has then in common with $\Gamma$

– one point if $0 < C < n-2$ or if $C = x_1$,

– two points if $n - 2 \leq C < x_1$,

– no point if $x > x_1$.

Next, if $0 < x < x_1$ and $y \neq h(x)$, then $\frac{\partial f}{\partial y}(x,y) \neq 0$. For $A = \big(x_1, h(x_1)\big)$ we have $m'(x_1) = n-2$. Since $\frac{\partial f}{\partial y}(A) = 0$ and $m' = \frac{\partial f}{\partial x} + h'\,\frac{\partial f}{\partial y}$ it follows that $\frac{\partial f}{\partial x}(A) \neq 0$. This shows that for every point $M = (x,y)$ of $\Gamma$ such that $y > 0$, either $\frac{\partial f}{\partial x}(M) \neq 0$, or $\frac{\partial f}{\partial y}(M) \neq 0$.

The above considerations show that there exist two functions $h_1 \colon [n-2, x_1] \to \mathbb{R}$ and $h_2 \colon [0, x_1] \to \mathbb{R}$ such that $f(x,y) = 0$ if and only if $y = h_1(x)$ or $y = h_2(x)$, $h_1$, $h_2$ verifying the following conditions:

1) if $n-2 < x < x_1$ then $h_1(x) < h(x)$;
2) if $0 < x < x_1$ then $h(x) < h_2(x)$;
3) $h(x_1) = h_1(x_1) = h_2(x_1)$.

Moreover, $h_2$ is differentiable at 0 since $f(x,y) = (2-n)x + \lambda y + o\big(\sqrt{x^2+y^2}\big)$, and $h_2'(0) = \frac{n-2}{\lambda}$. We can verify also that $h_1'(n-2) = 0$.

Since $\Gamma$ is differentiable at $(x_1, y_1)$, there exists $x_1' \in (0, x_1)$, such that $h_2$ is decreasing on $[x_1',\ x_1]$. $\square$

Let us consider now the orbit $\mathcal{O}$ of $t \mapsto \big(X(t), Y(t)\big)$. It is located above the graph $\Gamma_2$ of $h_2$ in a neighbourhood of $O$ since $h_2'(0) = \frac{n-2}{\lambda}$ and $\lim_{t\to-\infty} \frac{Y(t)}{X(t)} = \frac{n}{\lambda}$ according to Proposition 1. Since $g$ is continuous, for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\varepsilon < x \leq x_1'$ implies $g\big(x,\ h_2(x)\big) > \eta$. Since, on the other hand, $0 < x < x_1'$ implies $f\big(x,\ h_2(x)\big) = 0$; the orbit $\mathcal{O}$ stays above $\Gamma_2$ when $0 < X(t) < x_1'$. Finally, $x_1' < X(t) < x_1$ implies $g\big(X(t),\ Y(t)\big) \geq 0$ and $Y(t) > h_2(x_1')$ (cf. Figure 2).

$\mathcal{O}$ cuts then the straight line $x = x_1$ since, on this straight line, the only singular point of the vector field is $A = \big(x_1, h_2(x_1)\big)$ and $h_2(x_1) < h_2(x_1')$ (let us recall that according to 3) above $h_2(x_1) = h(x_1)$). We can finish as in the case $n = 2$, noting that if $X(t) > x_1$ then $f\big(X(t), Y(t)\big) > 0$ and $g\big(X(t), Y(t)\big) < 0$ which implies that $Y(t)$ is bounded and that $X(t)$ blows up in a finite time $T$, hence $z(a) = e^T$.

## 5. The Case $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$

If $p \geq \frac{n}{n-2}$ then $n - 2 \geq \frac{2}{p-1}$ and the previous method cannot be applied. Moreover, if $p \to \frac{n}{n-2}$ then $2p - np + n \to 0$ and $\lambda_{n,p} \to +\infty$. We introduce here another method which gives in both cases $\frac{n}{n-2} \le p < \frac{n+2}{n-2}$ and $1 < p < \frac{n}{n-2}$ a new value

$$\Lambda_{n,p} = \frac{1}{(p+1)^{p+1}} + \frac{2^{2p+1}(p-1)\, n\, p^{p+1}}{\big((p+1)\,(2p+2-np+n)\big)^{p+1}}$$

such that $\lambda \ge \Lambda_{n,p}$ implies that $z(a)$ is finite for any $a > 0$. When $p < \frac{n}{n-2}$, $\Lambda_{n,p} < \lambda_{n,p}$ if $p$ is near $\frac{n}{n-2}$.

The idea is to use the dissymmetry of the level lines of $f\colon (x,y) \mapsto f(x,y)$. In fact, if $0 \le y < y_0$ (where $y_0$ is given in (8) below) and $\alpha \in (0, x_1)$ then we show

$$\begin{cases} f(x_1+\alpha,\, y) > f(x_1-\alpha,\, y), \\ g(x_1+\alpha,\, y) = -g(x_1-\alpha,\, y) \end{cases} \tag{7}$$

and we show also that $\lambda \ge \Lambda_{n,p}$ implies $\mathcal{O}$ is below the line $y = y_0$.

If $\mathcal{O}_1$ is the part of $\mathcal{O}$ in the set $\{(x,y)\,|\, 0 \le x \le x_1\}$, then $\mathcal{O}$ is above $S(\mathcal{O}_1)$ in the set $\{(x,y)\,|\, x \ge x_1\}$ where $S(\mathcal{O}_1)$ is the reflection of $\mathcal{O}_1$ with respect to the straight line $x = x_1$. This shows that $X(t) > x_1$ implies $X'(t) > 0$ and we can conclude as in the proof of Theorem A (notice that $n - 2 < 2x_1$).

First, we see that $\frac{\partial f}{\partial x}(x_1, y) = 0$ if and only if

$$(2-n) + 2x - \frac{2p}{p+1}\, x^{\frac{p-1}{p+1}}\, y^{\frac{1}{p+1}} = 0$$

with $x = x_1$ i.e.

$$y = y_0 := \left(\frac{(2-n)(p-1)+4}{p-1}\right)^{p+1} \left(\frac{p+1}{2p}\right)^{p+1} \left(\frac{p-1}{2}\right)^{p-1}. \tag{8}$$

Now, we show the following lemma:

**Lemma.** *If $\beta \in (-y_0, 0)$ and $\alpha \in (0, x_1)$ then*

$$f(x_1+\alpha,\, y_0+\beta) \;>\; f(x_1-\alpha,\, y_0+\beta).$$

*Proof.* We have

$$\frac{\partial f}{\partial x}(x_1,\, y_0) = (2-n) + \frac{4}{p-1} - \frac{2p}{p+1}\left(\frac{2}{p-1}\right)^{\frac{p-1}{p+1}} y_0^{\frac{1}{p+1}} = 0 \tag{9}$$

and

$$\begin{aligned} &f(x_1+\alpha,\, y_0+\beta) \\ &\quad = (2-n)\Big(\frac{2}{p-1}+\alpha\Big) + \Big(\frac{2}{p-1}+\alpha\Big)^2 + \lambda(y_0+\beta) \\ &\qquad - \Big(\frac{2}{p-1}+\alpha\Big)^{\frac{2p}{p+1}} (y_0+\beta)^{\frac{1}{p+1}}. \end{aligned}$$

Using the Taylor-Lagrange formula for the last term we obtain

$$\begin{aligned}
&f(x_1+\alpha,\, y_0+\beta)\\
&\quad = (2-n)\frac{2}{p-1} + (2-n)\alpha + \Big(\frac{2}{p-1}\Big)^2 + \frac{4}{p-1}\alpha + \alpha^2 + \lambda(y_0+\beta)\\
&\quad - \Bigg[\Big(\frac{2}{p-1}\Big)^{\frac{2p}{p+1}} + \frac{2p}{p+1}\Big(\frac{2}{p-1}\Big)^{\frac{p-1}{p+1}}\alpha + \frac{1}{2}\frac{2p}{p+1}\frac{p-1}{p+1}\Big(\frac{2}{p-1}\Big)^{-\frac{2}{p+1}}\alpha^2\\
&\quad + \frac{1}{6}\frac{2p}{p+1}\frac{p-1}{p+1}\Big(\frac{-2}{p+1}\Big)\Big(\frac{2}{p-1}+\theta\alpha\Big)^{-\frac{p+3}{p+1}}\alpha^3\Bigg]\\
&\quad \times \Bigg[y_0{}^{\frac{1}{p+1}} + \frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\Bigg]
\end{aligned}$$

where $0 \le \theta \le 1$ and $0 \le \theta' \le 1$.

Using (9) we see that the only terms which are not symmetric in $\alpha$ are

$$-\frac{2p}{p+1}\Big(\frac{2}{p-1}\Big)^{\frac{p-1}{p+1}}\alpha\, \frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta$$

and

$$-\frac{1}{6}\frac{2p}{p+1}\frac{p-1}{p+1}\Big(\frac{-2}{p-1}\Big)\Big(\frac{2}{p-1}+\theta\alpha\Big)^{-\frac{p+3}{p+1}}\alpha^3\Bigg[y_0{}^{\frac{1}{p+1}} + \frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\Bigg].$$

The term

$$\Bigg[y_0{}^{\frac{1}{p+1}} + \frac{1}{p+1}(y_0+\theta'\beta)^{-\frac{p}{p+1}}\beta\Bigg]$$

equal to $(y_0+\beta)^{\frac{1}{p+1}}$ is positive and $(y_0+\theta'\beta)^{-\frac{p}{p+1}}$ too. Since $\beta<0$ we see that the signs of this expression and $\alpha$ are the same. This shows that $f(x_1+\alpha, y_0+\beta) > f(x_1-\alpha, y_0+\beta)$ provided $\beta \in (-y_0, 0)$ and $\alpha \in (0, x_1)$. □

It it easy to see that $g(x_1+\alpha, y_0+\beta) = -g(x_1-\alpha, y_0+\beta)$ for any $\beta$ and $\alpha$ since

$$g\Big(\frac{2}{p-1}+\alpha,\, y_0+\beta\Big) = (y_0+\beta)\Big(2-(p-1)\big(\frac{2}{p-1}+\alpha\big)\Big) = -(y_0+\beta)(p-1)\alpha.$$

Consequently, (7) is verified.

Now, we use the fact (cf. Theorem 2 of [**FQ**]) that if $(X,Y)\colon t \mapsto \big(X(t),Y(t)\big)$ corresponds to $u$ then $0 \le X(t) \le x_1$ implies

$$Y(t) \le \frac{n}{\lambda-(p+1)^{-(p+1)}}X(t).$$

In particular, if $X(t)=x_1$ then $Y(t) \le \frac{n}{\lambda-(p+1)^{-(p+1)}}\frac{2}{p-1}$. Consequently, the orbit $\mathcal{O}$ corresponding to $u$ cuts the straight line $x=x_1$ below $A_0 := (x_1, y_0)$ provided

$$\frac{n}{\lambda-(p+1)^{-(p+1)}}\frac{2}{p-1} \le y_0.$$

Since the last inequality is equivalent to the condition $\lambda \ge \Lambda_{n,p}$, we see that this is a sufficient condition to have $z(a)$ finite for any $a>0$.

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