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RESCALING OF MARKOV SHIFTS

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Abstract. Given a $\mathbb{Z}^d$ topological Markov shift Σ and a $d \times d$ integer matrix M with $\det(M) \neq 0$, we introduce the M-rescaling of Σ, denoted $\Sigma^{(M)}$. We show that some (internal) power of the $\mathbb{Z}^d$-action on $\Sigma^{(M)}$ is isomorphic to some (Cartesian, or external) power of Σ, and deduce that the two Markov shifts have the same topological entropy. Several examples from the theory of group automorphisms are discussed. Full shifts in any dimension are shown to be invariant under rescaling, and the problem of whether the reverse is true is interpreted as a higher-dimensional analogue of William's problem.

1. Introduction

Let A be a compact metric space, and let $A^{\mathbb{Z}^d}$ be the set of all functions $x\colon \mathbb{Z}^d \to A$, endowed with the product topology. For any set $F \subset \mathbb{Z}^d$, let $\rho_F\colon A^{\mathbb{Z}^d} \to A^F$ denote the restriction map, sending x to $x|_F \in A^F$. Denote by σ the natural shift action of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$,

$$\sigma_{\mathbf{n}}(x)_{\mathbf{m}} = x_{\mathbf{n}+\mathbf{m}}. \tag{1.1}$$

A closed, σ-invariant subset $\Sigma \subset A^{\mathbb{Z}^d}$ is called a (topological) Markov shift if there exists a finite set $F \subset \mathbb{Z}^d$ and a subset $P \subset A^F$ for which

$$\Sigma = \Sigma_{(F,P)} = \{x \in A^{\mathbb{Z}^d} \mid \rho_F(\sigma_{\mathbf{n}} x) \in P \text{ for all } \mathbf{n} \in \mathbb{Z}^d\}. \tag{1.2}$$

The shift action σ restricts to a shift action $\sigma^{(F,P)}$ of $\mathbb{Z}^d$ on $\Sigma_{(F,P)}$. For brevity, we shall use $\Sigma_{(F,P)}$ to denote both the set (1.2) and the $\mathbb{Z}^d$ topological dynamical system $(\Sigma_{(F,P)}, \sigma^{(F,P)})$ For a discussion of this definition and some examples, see [**S2**, Chapter 5].

Let M be a $d \times d$ integer matrix with $\det(M) \neq 0$, and let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard basis for $\mathbb{Z}^d$. For a finite set $F \subset \mathbb{Z}^d$, let $M(F) = \{\mathbf{n}M \mid \mathbf{n} \in F\}$. For $P \subset A^F$, define $M(P) \subset A^{M(F)}$ as follows. The map $x\colon M(F) \to A$ is in $M(P)$ if and only if $y\colon F \to A$ is in P, where $y(\mathbf{n}) = x(\mathbf{n}M)$.

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The M**-rescaling** of the Markov shift $\Sigma_{(F,P)}$ is then defined to be

$$\Sigma^{(M)}_{(F,P)} = \Sigma_{(M(F),M(P))}, \tag{1.3}$$

with associated $\mathbb{Z}^d$-action $\sigma^{(M(F),M(P))}$.

Notice that the rescaled shift is well-defined in the following sense: if $\Sigma_{(F,P)}$ and $\Sigma_{(G,Q)}$ are topologically conjugate Markov shifts then, for any M, so too are $\Sigma^{(M)}_{(F,P)}$ and $\Sigma^{(M)}_{(G,Q)}$.

We shall see that Σ and $\Sigma^{(M)}$ are not in general topologically conjugate, though they have the same entropy; the first theorem in Section 2 exhibits a more direct connection.

2. Rescaled Markov Shifts

Consider a topological Markov shift $(\Sigma_{(F,P)}, \sigma^{(F,P)})$ and let M be a $d\times d$ integer matrix with $\det(M) \neq 0$.

Theorem 2.1. *The $\mathbb{Z}^d$-action $\mathbf{n} \mapsto \sigma^{(M(F),M(P))}_{\mathbf{n}M}$ is topologically conjugate to the $|\det(M)|$-fold Cartesian product $\sigma^{(F,P)} \times \cdots \times \sigma^{(F,P)}$.*

Notice that if $d = 1$, then $M = [m]$ is a non-zero integer, and the action $\mathbf{n} \mapsto \sigma^{(M(F),M(P))}_{\mathbf{n}M}$ is then simply the usual m-fold power or iterate of $\sigma^{(M(F),M(P))}$.

Proof. Let $k = |\det(M)|$, and choose coset representatives $\mathbf{r}_1, \dots, \mathbf{r}_k$ for the subgroup $\mathbb{Z}^d M \subset \mathbb{Z}^d$. Define a map $\theta : \Sigma_{(M(F),M(P))} \to \left(\Sigma_{(F,P)}\right)^k$ by

$$\theta(x) = (\rho_{\mathbb{Z}^d M+\mathbf{r}_1}(x), \rho_{\mathbb{Z}^d M+\mathbf{r}_2}(x), \dots, \rho_{\mathbb{Z}^d M+\mathbf{r}_k}(x)). \tag{2.1}$$

We claim that θ is a topological conjugacy. By (1.3), θ is well-defined (each $\rho_{\mathbb{Z}^d M+\mathbf{r}_j}(x)$ is an allowed point in $\Sigma_{(F,P)}$). Moreover, every k-tuple of allowed words appears as the image of a unique point under θ in $\Sigma_{(M(F),M(P))}$. Thus θ is a homeomorphism. It is clear that θ intertwines the actions. □

Let $h(\sigma)$ denote the topological entropy of the $\mathbb{Z}^d$ shift. (For a definition, see footnote to page 56 of [**S2**] or Appendix A of [**LSW**]. When the alphabet A is finite, the entropy is given by $h(\sigma) = \limsup_{n\to\infty} \frac{1}{n^d} \log |\{\rho_{R(n)}(\Sigma)\}|$ where $R(n) = [0,n)^d \cap \mathbb{Z}^d$.)

Corollary 2.2. *Topological entropy is invariant under rescaling.*

Proof. This follows from two quite general facts.

Firstly, for any $\mathbb{Z}^d$-actions α and β on compact metric spaces X and Y by homeomorphisms, we have $h(\alpha \times \beta) = h(\alpha) + h(\beta)$. This may be proved by an easy extension of the argument in [**AKM**] from single maps to $\mathbb{Z}^d$-actions. Alternatively, notice that this is true for measure-theoretic entropy of amenable group actions by the increasing Martingale theorem (see Lemma 4.1 of [**WZ**]); the variational principle (see [**E**]) for $\mathbb{Z}^d$-actions then shows $h(\alpha \times \beta) \geq h(\alpha) + h(\beta)$. The reverse inequality is clear. It follows that

$$h(\sigma^{(F,P)} \times \cdots \times \sigma^{(F,P)}) = k\, h(\sigma^{(F,P)}). \tag{2.2}$$

Secondly, if α is any action by homeomorphisms of a compact metric space X then $h(\mathbf{n} \mapsto \alpha_{\mathbf{n}M}) = |\det(M)|\, h(\alpha)$. When $d = 1$ and $M = [m]$, $m > 0$, this is the usual power rule ([**AKM**], Theorem 2). The extension to $\mathbb{Z}^d$-actions is straightforward. It follows that

$$h(\mathbf{n} \mapsto \sigma_{\mathbf{n}M}^{(M(F),M(P))}) = k\, h(\sigma^{(M(F),M(P))}). \tag{2.3}$$

By Theorem 2.1, $h(\mathbf{n} \mapsto \sigma_{\mathbf{n}M}^{(M(F),M(P))}) = h(\sigma^{(F,P)} \times \cdots \times \sigma^{(F,P)})$; since $k \neq 0$ we deduce from (2.2) and (2.3) that $h(\sigma^{(M(F),M(P))}) = h(\sigma^{(F,P)})$. □

When the alphabet A is finite and $d = 1$, it follows that rescaling does not take one outside the finite equivalence class of the original shift (see [**P1**] and [**P2**]).

We now show how the number of periodic points is affected by rescaling. A *period* for a $\mathbb{Z}^d$-action α is a lattice of full rank $\Lambda \subset \mathbb{Z}^d$; the set of Λ-periodic points is defined by

$$F_\Lambda(\alpha) = \{x \mid \alpha_{\mathbf{n}} x = x \text{ for all } \mathbf{n} \in \Lambda\}.$$

When $d = 1$, we shall write F_n for $F_{n\mathbb{Z}}$. The symbol Λ will always be used for a lattice of full rank.

Lemma 2.3. *The number of Λ-periodic points in $\Sigma_{(F,P)}^{(M)}$ is given by*

$$\left|F_\Lambda\left(\Sigma_{(F,P)}^{(M)}\right)\right| = \left|F_{H(\Lambda)}\left(\Sigma_{(F,P)}\right)\right|^{\left|\mathbb{Z}^d/(\Lambda+\mathbb{Z}^dM)\right|},$$

where $H(\Lambda)$ is the kernel of the map $\mathbf{n} \mapsto \mathbf{n}M + (\Lambda + \mathbb{Z}^dM)$ from $\mathbb{Z}^d$ to $\mathbb{Z}^d/(\Lambda + \mathbb{Z}^dM)$.

Corollary 2.4. *If $d = 1$ and $M = [m]$, then*

$$\left|F_n(\Sigma_{(F,P)}^{(M)})\right| = \left|F_{n/(n,m)}(\Sigma_{(F,P)})\right|^{(n,m)},$$

where (n, m) denotes the highest common factor of n and m.

Proof of Lemma 2.3. This is a simple counting argument. Let x be a Λ-periodic point in $\Sigma_{(F,P)}^{(M)}$. For each coset $\mathbf{r} + \Lambda$ of Λ in $\mathbb{Z}^d$, look at the co-ordinates of x along the coset $\mathbf{r} + \mathbb{Z}^dM$. These form an element of $\Sigma_{(F,P)}$ with period $H(\Lambda)$; moreover there are $\left|\mathbb{Z}^d/(\Lambda + \mathbb{Z}^dM)\right|$ ways to extract such a point. The result follows. □

Remark 2.5. The rescaling construction may be applied to any $\mathbb{Z}^d$ action (by homeomorphisms or measure-preserving transformations) to produce an action of the same type. Let α be a $\mathbb{Z}^d$ action on a set X, and let M be a $d \times d$ integer matrix with non-zero determinant. Choose a set of coset representatives $L = \{\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_{|\det(M)|}\}$ for $\mathbb{Z}^d/\mathbb{Z}^d M$, let $c(\mathbf{n}) = \boldsymbol{\ell}_j$ if $\boldsymbol{\ell}_j + \mathbb{Z}^d M = \mathbf{n} + \mathbb{Z}^d M$ and let $s(j) = \boldsymbol{\ell}_j$. Then define a $\mathbb{Z}^d$-action $\alpha^{(M)}$ on $X^{|\det(M)|}$ by setting

$$\begin{aligned}&\alpha_{\mathbf{n}}^{(M)}(x_1, \dots, x_{|\det(M)|})\\ &\qquad = (\alpha_{(\mathbf{n}-c(\mathbf{n}))M^{-1}} x_{c(s(1)+\mathbf{n})}, \dots, \alpha_{(\mathbf{n}-c(\mathbf{n}))M^{-1}} x_{c(s(|\det(M)|)+\mathbf{n})}).\end{aligned}$$

3. Group Automorphisms

If α is an expansive action of $\mathbb{Z}^d$ by automorphisms of a compact group X (or, more generally, an action satisfying the descending chain condition on closed invariant subgroups), then α is a Markov shift in the above sense ([**KS1**]). If the group is abelian, then the system is determined by a module L over the ring $\mathcal{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$: the module L as an additive group is the dual $\widehat{X}$ of X, with multiplication by the variable u_i the automorphism of L dual to the automorphism $\alpha_{\mathbf{e}_i}$ of X. See [**KS1**] or [**LSW**] for a detailed discussion of this correspondence.

In the case of a cyclic module $L = \mathcal{R}_d/\langle f_1, \dots, f_\ell\rangle$, the correspondence takes the following form. The $\mathbb{Z}^d$-action α^L on X_L is the shift action on

$$X_L = \left\{ x \in \mathbb{T}^{\mathbb{Z}^d} \mid \sum_{\mathbf{m}} x_{\mathbf{n}+\mathbf{m}} c_{j,\mathbf{m}} = 0 \text{ mod } 1, \text{ for } j = 1, \dots, \ell, \mathbf{n} \in \mathbb{Z}^d \right\}, \tag{3.1}$$

which is a closed, shift-invariant subgroup of the compact group $\mathbb{T}^{\mathbb{Z}^d}$. Here we have written each polynomial $f_j(u_1, \dots, u_d)$ as $\sum c_{j,\mathbf{m}} \mathbf{u}^{\mathbf{m}}$, where $\mathbf{u}^{\mathbf{m}} = u_1^{m_1} \dots u_d^{m_d}$.

It is clear from Section 1 that the M-rescaling of $\alpha^{\mathcal{R}_d/\langle f_1, \dots, f_\ell\rangle}$ is the $\mathbb{Z}^d$-action corresponding to the module $\mathcal{R}_d/\langle f_1(\mathbf{u}^{\mathbf{m}_1}, \dots, \mathbf{u}^{\mathbf{m}_d}), \dots, f_\ell(\mathbf{u}^{\mathbf{m}_1}, \dots, \mathbf{u}^{\mathbf{m}_d})\rangle$ where

$$M = [\mathbf{m}_1^t \mid \cdots \mid \mathbf{m}_d^t].$$

Example 3.1. Consider $\alpha = \alpha^{\mathcal{R}_d/\langle f\rangle}$ (f non-zero). By [**LSW**], the topological entropy of α is given by

$$h(\alpha) = \log \mathrm{M}(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i s_1}, \dots, e^{2\pi i s_d})| \, ds_1 \dots ds_d$$

where $\mathrm{M}(f)$ is the Mahler measure of f. It follows from Corollary 2.2 that

$$\mathrm{M}\left(f(u_1, \dots, u_d)\right) = \mathrm{M}\left(f(\mathbf{u}^{\mathbf{m}_1}, \dots, \mathbf{u}^{\mathbf{m}_d})\right) \tag{3.2}$$

whenever $\det[\mathbf{m}_1^t \mid \cdots \mid \mathbf{m}_d^t] \neq 0$. This may of course be seen directly: the endomorphism of the d-torus given by the matrix M is Lebesgue measure-preserving.

Example 3.2. Consider $\alpha = \alpha^{\mathcal{R}_2/\langle 1+u_1+u_2\rangle}$. By [**S3**], α is isomorphic to a $\mathbb{Z}^2$ Bernoulli shift. For any $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad \neq bc$, the M-rescaling $\alpha^{(M)} = \alpha^{\mathcal{R}_2/\langle 1+u_1^a u_2^c+u_1^b u_2^d\rangle}$ is also a Bernoulli shift by Theorem 2.1 and [**OW**]. By Corollary 2.2 we deduce that α and $\alpha^{(M)}$ are measurably isomorphic for every non-singular M. Notice that by [**S1**], α and $\alpha^{(M)}$ are not topologically conjugate if $M \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 3.3. Consider $\alpha = \alpha^{\mathcal{R}_2/\langle 2,1+u_1+u_2\rangle}$. In contrast to 3.2 above, the rescalings of α are not all measurably isomorphic (for instance, if $M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\alpha^{(M)}$ is not isomorphic to α; see [**KS2**, Examples 4.3(1)]).

Example 3.4. Let $f(u) = u^n + a_{n-1}u^{n-1} + \cdots + a_1 u \pm 1$. As shown in [**KS1**], $\alpha^{\mathcal{R}_1/\langle f\rangle}$ is algebraically isomorphic to the automorphism α of the n-torus $\mathbb{T}^n$ determined by the matrix A companion to f. The $M = [m]$-rescaling of α is algebraically isomorphic to the automorphism β of the mn-torus $\mathbb{T}^{mn}$ determined by the companion matrix to the polynomial $f(u^m)$. By Corollary 2.2 and [**K**], α and β are measurably isomorphic (though they are not topologically conjugate unless $m = 1$). By Theorem 1.1, β^m is algebraically isomorphic to the m-fold Cartesian product of α. This observation is nothing more than the following matrix lemma: if A is the companion matrix to $f(u)$, and B is the companion matrix to $f(u^m)$, then B^m is conjugate in $GL(n,\mathbb{Z})$ to $A \oplus \cdots \oplus A$ (m times).

Example 3.5. Consider the group endomorphism α (the invertible case may be dealt with by an easy extension) given by the module $\mathbb{Z}[u]/\langle u-2\rangle$. This is simply the map $x \mapsto 2x$ mod 1 on the circle $\mathbb{T}$. By the above remarks, the endomorphism β of the 2-torus $\mathbb{T}^2$ given by the 2-rescaling of α (i.e. by the module $\mathbb{Z}[u]/\langle u^2-2\rangle$) is measurably isomorphic to α. Off a countable set, the map $\theta : \mathbb{T} \to \mathbb{T}\times\mathbb{T}$ sending $t = \frac{t_1}{2} + \frac{t_2}{4} + \frac{t_3}{8} + \dots$ to $\left(\frac{t_1}{2} + \frac{t_3}{4} + \frac{t_5}{8} + \dots, \frac{t_2}{2} + \frac{t_4}{4} + \frac{t_6}{8} + \dots\right)$ is an invertible measure-preserving map, and intertwines the two $\mathbb{N}$-actions. To see this, notice that the 2-rescaling of α is given by the action of $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ on $\mathbb{T}\times\mathbb{T}$, and

$$\theta^{-1}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{t_1}{2} + \frac{t_3}{4} + \frac{t_5}{8} + \dots \\ \frac{t_2}{2} + \frac{t_4}{4} + \frac{t_6}{8} + \dots \end{bmatrix} = \theta^{-1}\begin{bmatrix} \frac{t_2}{2} + \frac{t_4}{4} + \frac{t_6}{8} + \dots \\ \frac{t_3}{2} + \frac{t_5}{4} + \frac{t_7}{8} + \dots \end{bmatrix} = \frac{t_2}{2} + \frac{t_3}{4} + \dots .$$

4. Other Examples

Now consider a one-dimensional subshift of finite type $\Sigma = \Sigma_A$, where $A = (a_{ij})$ is a 0-1 valued square $k \times k$ matrix. The subshift is the shift map on

$$\Sigma_A = \{x \in \{1,2,\dots,k\}^{\mathbb{Z}} \mid a_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\}.$$

For the definition of the zeta function, see [**BL**].

Example 4.1. The rescalings of the golden mean shift give an infinite family of topologically distinct subshifts of finite type with the same entropy. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and let $\Sigma = \Sigma_A$ be the corresponding subshift of finite type. An easy calculation shows that $\Sigma^{(2)}$ is given by the matrix

$$A^{(2)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$\Sigma^{(3)}$ is given by the matrix

$$A^{(3)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and so on. All the shifts have the same entropy. By Theorem 2.1, $F_n\left(\Sigma^{(n)}\right) = F_1\left(\Sigma\right)^n = 1$; it follows that $\Sigma^{(n)}$ cannot be topologically conjugate to Σ for $n \neq 1$. Similar considerations show that $\Sigma^{(n)}$ and $\Sigma^{(m)}$ have the same zeta function (and therefore can only be topologically conjugate) if $n = m$, so this is an infinite family of topologically distinct subshifts of finite type all with topological entopy $\log\left(\frac{1+\sqrt{5}}{2}\right)$. The first few zeta functions are given by:

$$\zeta_{\Sigma}(z) = \frac{1}{1+z-z^2}, \quad \zeta_{\Sigma^{(2)}}(z) = \frac{1}{(1+z-z^2)(1+z^2)},$$

and

$$\zeta_{\Sigma^{(3)}}(z) = \frac{1}{(1+z-z^2)(1-z^3-z^6)}.$$

Example 4.2. By Corollary 2.4, the zeta function of a rescaling of any subshift of finite type is computable from the zeta function of the original subshift of finite type. This means no additional invariants of topological conjugacy can be extracted from the periodic points of the rescalings of a subshift of finite type. As an illustration, we show how to find the zeta function of $\Sigma^{(2)}$ when the zeta function of Σ is given by

$$\zeta_{\Sigma}(z) = \prod_{i=1,\ldots,s} \frac{1}{1-\lambda_i z}.$$

By Corollary 1.4,

$$\begin{aligned}\zeta_{\Sigma^{(2)}}(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{z^{2n+1}}{2n+1}\left(\lambda_1^{2n+1}+\cdots+\lambda_s^{2n+1}\right)+\sum_{m=1}^{\infty}\frac{z^{2m}}{2m}\left(\lambda_1^m+\cdots+\lambda_s^m\right)^2\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n}\left(\lambda_1^n+\cdots+\lambda_s^n\right)+\sum_{m=1}^{\infty}\frac{z^{2m}}{2m}\Big(2\sum_{i<j}\lambda_i^m\lambda_j^m\Big)\right)\end{aligned}$$

so that

$$\zeta_{\Sigma^{(2)}}(z) = \prod_{i=1,\dots,s}\frac{1}{1-\lambda_i z}\times\prod_{i<j}\frac{1}{1-\lambda_i\lambda_j z^2}. \tag{4.1}$$

Example 4.3. If σ is a $\mathbb{Z}^d$ topological Markov shift, then the k-fold Cartesian product $\sigma\times\cdots\times\sigma$ has k^{th} roots of every kind for any $k\neq 0$. (For $d=1$ a k^{th} root of σ is a subshift of finite type ϕ with the property that ϕ^k is topologically conjugate to σ; for $d>1$ we say that σ has k^{th} roots of every kind if for any integer matrix M with $\det(M)=k$, there is a $\mathbb{Z}^d$ topological Markov shift ϕ with the property that the action $\mathbf{n}\mapsto\phi_{\mathbf{n}M}$ is topologically conjugate to σ). For the given matrix M, take ϕ to be the shift $\sigma^{(M)}$ and apply Theorem 2.1.

Example 4.4. For any $N\geq 1$ there is a $\mathbb{Z}^d$ topological Markov shift with no points of period Λ with $|\mathbb{Z}^d/\Lambda|\leq N$. Enumerate the distinct subgroups with index not exceeding N as $\Lambda_1,\dots,\Lambda_k$. Define a $\mathbb{Z}^d$ Markov shift as follows:

$$\Sigma_0=\left\{\mathbf{x}\in\{1,2,3\}^{\mathbb{Z}^d}\mid a_{\mathbf{x}(\mathbf{n}),\mathbf{x}(\mathbf{n}+\mathbf{e}_i)}=1\ \forall\ \mathbf{n}\in\mathbb{Z}^d, i=1,\dots,d\right\}$$

where A is the matrix

$$A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$

and $\mathbf{e}_1,\dots,\mathbf{e}_d$ are basis vectors for $\mathbb{Z}^d$. Then Σ_0 is a shift with no fixed points (points invariant under the whole action). For each $j=1,\dots,k$, let M_j be an integer matrix with $\mathbb{Z}^dM_j=\Lambda_j$. Then the M_j-rescaled shift $\Sigma_j=\Sigma_0^{(M)}$ has no points with period Λ_j by Lemma 2.3. It follows that the shift

$$\Sigma=\Sigma_0\times\Sigma_1\times\cdots\times\Sigma_k$$

has the required property.

5. Shifts Invariant Under Rescaling

Let (Σ, σ) be a $\mathbb{Z}^d$ topological Markov shift, assumed throughout this section to have only finitely many fixed points. Then (Σ, σ) is said to be *invariant under rescaling* if for any integer matrix M with $\det(M) \neq 0$, the rescaled shift $\sigma^{(M)}$ is topologically conjugate to σ.

Lemma 5.1. *If σ is the full d-dimensional shift on s symbols, then σ is invariant under rescaling.*

Notice that Lemma 5.1 is obvious: in the notation of Section 1, the full shift on s symbols may be defined by taking $A = \{1, 2, \ldots, s\}$, $F = \{0\}$ and $P = A$.

Lemma 5.2. *If σ is a $\mathbb{Z}^d$ Markov shift that is invariant under rescaling, then, for any lattice $\Lambda \subset \mathbb{Z}^d$,*

$$|F_\Lambda(\sigma)| = s^{|\mathbb{Z}^d/\Lambda|}$$

where s is the number of points fixed by σ.

That is, if σ is invariant under rescaling, then it has the same periodic point data as a full shift.

Proof. By Lemma 2.3 and rescaling invariance,

$$|F_\Lambda(\sigma^{(M)})| = |F_\Lambda(\sigma)| = |F_{H(\Lambda)}(\sigma)|^{|\mathbb{Z}^d/(\Lambda+\mathbb{Z}^d M)|}.$$

Pick M so that $\mathbb{Z}^d M = \Lambda$. Then

$$|F_\Lambda(\sigma^{(M)})| = |F_\Lambda(\sigma)| = |F_{\mathbb{Z}^d}(\sigma)|^{|\det(M)|}.$$

The proof is completed by noting that $s = |F_{\mathbb{Z}^d}(\sigma)|$ is the number of points fixed by σ, and $|\det(M)| = |\mathbb{Z}^d/\Lambda|$. □

Corollary 5.3. *If σ is invariant under rescaling, then the topological entropy of σ is greater than or equal to* $\log s$*, where s is the number of points fixed by σ.*

Corollary 5.4. *If σ is a one-dimensional subshift of finite type that is invariant under rescaling, then it is shift equivalent to a full shift.*

Proof. By Lemma 5.2, the dynamical zeta function of σ is given by $\zeta_\sigma(z) = \frac{1}{1-sz}$. It follows that some power of σ is topologically conjugate to a full shift (see Theorem B in [**W**]). □

For definitions and results used above, see the survey paper [**P2**].

Problem. *If σ is a $\mathbb{Z}^d$ Markov shift that is invariant under rescaling, is σ topologically conjugate to a full shift?*

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