Acta Math. Univ. Comenianae
Vol. LXV, 1(1996), pp. 129–139

ON VARIANCE–COVARIANCE COMPONENTS ESTIMATION IN LINEAR MODELS WITH AR(1) DISTURBANCES

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Abstract. Estimation of the autoregressive coefficient ϱ in linear models with first-order autoregressive disturbances has been broadly studied in the literature. Based on C.R. Rao's MINQE-theory, Azaïs et al. (1993) gave a new general approach for computing locally optimum estimators of variance-covariance components in models with non-linear structure of the variance-covariance matrix. As a special case, in the linear model with AR(1) errors, we discuss a new algorithm for computing locally optimum quadratic plus constant invariant estimators of the parameters ϱ and σ^2, respectively. Moreover, simple iteration of this estimation procedure gives a maximum likelihood estimates of both, the first order parameters, and the variance-covariance components.

1. Introduction

Linear models with disturbances that follow first-order autoregressive scheme, abbreviated as AR(1), are frequently considered in econometrical applications. Great importance is given to the second-order parameters — variance-covariance components. Namely, to the autoregressive coefficient ϱ and to the variance σ^2, which are usually unknown and are to be estimated.

Estimation of variance-covariance components in models with AR(1) errors has a long history. The simplest estimators are based on very natural idea to change the unobservable disturbances by the corresponding least squares residuals in the generating scheme for AR(1) process. On the other side, under normality assumptions, full MLE's, **Maximum Likelihood Estimators**, are studied. Generally, most of those estimators reach some of optimality properties, at least they are consistent estimators of ϱ and σ^2. For more details see, e.g., Prais and Winsten (1954), Durbin (1960), Magnus (1978), and Kmenta (1986).

In a recent paper, Azaïs **et al.** (1993) gave a generalization of C.R. Rao's **Minimum Norm Quadratic Estimation Theory** of estimation variance and/or

Received March 20, 1996; revised May 13, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 62J10, 62F30.
Key words and phrases. Autoregressive disturbances, variance-covariance components, minimum norm quadratic estimation, maximum likelihood estimation, Fisher scoring algorithm, MINQE(I), LMINQE(I), AR(1), MLE, FSA.

covariance components to the models with non-linear variance-covariance structure. Their new LMINQE, **Linearized MINQE**, is defined as a MINQE, (more particularly as an invariant MINQE(I) or an unbiased and invariant MINQE(U,I)), after linearization of the variance-covariance matrix at a prior value of the parameters. Rao's MINQE-method, in contrast to MLE-method, involves only slight distributional assumptions, especially, the existence of the first four moments of the probability distribution. Moreover, MINQE's are locally optimum for a priori chosen Euclidean norm and their iteration are numerically equivalent to the well known FSA, **Fisher Scoring Algorithm** — a numerical method for finding singular points of the likelihood function, which leads to Gaussian maximum likelihood estimates. For more details on MINQE-theory and variance-covariance componets estimation see e.g. C.R. Rao (1971a), C.R. Rao (1971b), C.R. Rao and J. Kleffe (1980), C.R. Rao and J. Kleffe (1988), S.R. Searle **et al.** (1992), J. Volaufová and V. Witkovský (1992), and J. Volaufová (1993a).

In this paper, based on the above mentioned results, a new algorithm for estimation the variance-covariance components is given, i.e. ϱ and σ^2, in linear model with AR(1) disturbances. The very special character of this model allow us to find closed-form formulas for LMINQE's, locally optimum invariant estimators of ϱ and σ^2. Moreover, because of the link between iterated LMINQE's and Fisher Scoring Algorithm, we get directly an iterative method for finding Gaussian MLE's. As such, the suggested algorithm serves, in the special case of the model with constant mean, as a special alternative to the classical algorithms for computing MLE's of the parameters of the AR(1) process, see e.g. J.P. Brockwell and R.A. Davis (1987).

Finally, we note that the approach introduced by Azaïs **et al.** (1993) is quite general and can be used, e.g., for any linear model with ARMA(p, q) disturbances. However, the closed-form formulas are still a subject of further investigation.

2. Model with Linearized Variance-Covariance Structure

We consider linear regression model

$$y_t = x_t\beta + \varepsilon_t, \quad t = 1, \dots, n, \tag{1}$$

where y_t represents observation in time t, $x_t = (x_{1t}, \dots, x_{kt})$ is a vector of known constants, and $\beta = (\beta_1, \dots, \beta_k)'$ is a vector of unknown first-order parameters.

We assume that the disturbances follow a stationary AR(1) process which started at time $t = -\infty$, i.e. the errors are generated by the following scheme:

$$\begin{aligned} \varepsilon_1 &= u_1/\sqrt{1-\varrho^2}, \\ \varepsilon_t &= \varrho\varepsilon_{t-1} + u_t, \quad t = 2, \dots, n, \end{aligned} \tag{2}$$

where $|\varrho| < 1$, is an autoregressive coefficient, generally unknown parameter, and u_t, $t = 1, \dots, n$, represent uncorrelated random errors with zero mean and the variance $\sigma^2 > 0$, which is also supposed to be an unknown parameter. Generally, we do not assume normality of the probability distribution of the errors. However, we assume the existence of the third and fourth moments.

The model can be rewritten to the matrix form

$$y = X\beta + \varepsilon, \tag{3}$$

with expectation $E(y) = X\beta$ and the variance-covariance matrix $\operatorname{Var}(y) = V(\varrho, \sigma^2)$, where $y = (y_1, \dots, y_n)'$, X is $(n \times k)$-dimensional matrix with x_t beeing the t-th row, and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$. We usually denote the model as

$$(y, X\beta, V(\varrho, \sigma^2)). \tag{4}$$

It can be easily shown that under given assumptions the explicit form of the variance-covariance matrix $V(\varrho, \sigma^2)$ is given by

$$V(\varrho, \sigma^2) = \frac{\sigma^2}{1-\varrho^2} \begin{pmatrix} 1 & \varrho & \varrho^2 & \cdots & \varrho^{n-1} \\ \varrho & 1 & \varrho & \cdots & \varrho^{n-2} \\ \varrho^2 & \varrho & 1 & \cdots & \varrho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{n-1} & \varrho^{n-2} & \varrho^{n-3} & \cdots & 1 \end{pmatrix}. \tag{5}$$

Note the non-linear structure of the matrix $V(\varrho, \sigma^2)$ in its parameters, especially in the autoregressive coefficient ϱ. Under the constraints $|\varrho| < 1$ and $\sigma^2 > 0$, matrix $V(\varrho, \sigma^2)$ always remains positive definite. As a function of the parameters ϱ and σ^2, $V(\varrho, \sigma^2)$ belongs to the class $\mathcal{C}^2$, i.e. to the class of twice differentiable functions.

To get a model with linear variance-covariance structure we consider first-order Taylor expansion of $V(\varrho, \sigma^2)$ around a prior value (ϱ_0, σ_0^2). Let ϱ_0 and σ_0^2, $|\varrho_0| < 1$ and $\sigma_0^2 > 0$, denote prior values for the parameters ϱ and σ^2. Then the linearized variance-covariance matrix $V(\varrho, \sigma^2)$ around (ϱ_0, σ_0^2) is approximately given by

$$V(\varrho, \sigma^2) \approx V_0 + (\varrho - \varrho_0)V_1 + (\sigma^2 - \sigma_0^2)V_2, \tag{6}$$

where $V_0 = V(\varrho_0, \sigma_0^2)$, and

$$V_1 = \left.\frac{\partial V(\varrho, \sigma^2)}{\partial \varrho}\right|_{\varrho_0, \sigma_0^2} \quad \text{and} \quad V_2 = \left.\frac{\partial V(\varrho, \sigma^2)}{\partial \sigma^2}\right|_{\varrho_0, \sigma_0^2}. \tag{7}$$

Relation $V_0 = \sigma_0^2 V_2$ implies

$$V(\varrho, \sigma^2) \approx (\varrho - \varrho_0)V_1 + \sigma^2 V_2. \tag{8}$$

If we denote $W(\varrho, \sigma^2) = (\varrho - \varrho_0)V_1 + \sigma^2 V_2$, the linear approximation of the variance-covariance matrix, then the linear model

$$(y, X\beta, W(\varrho, \sigma^2))$$

is the linear model with variance and covariance components, as usually considered in MINQE-theory, i.e. $W(\varrho, \sigma^2)$ is a linear combination of known symmetrical matrices, V_1 and V_2, and unknown variance-covariance components, $(\varrho - \varrho_0)$ and σ^2. It should be emphasized, however, that we can not generally ensure the positive definiteness of the matrix $W(\varrho, \sigma^2)$. It is positive definite only in sufficiently close neighborhood of a priori chosen point of the parameter space (ϱ_0, σ_0^2).

3. Locally Optimum Estimators of ϱ and σ^2

Following Azaïs **et al.** (1993), MINQE's, $\widehat{(\varrho - \varrho_0)}$ and $\hat{\sigma}^2$, of the variance-covariance components $(\varrho - \varrho_0)$ and σ^2, respectively, computed for the prior values 0 and σ_0 in the linearized model $(y, X\beta, W(\varrho, \sigma^2))$, leads directly to LMINQE's, $\tilde{\varrho}$ and $\tilde{\sigma}^2$, of the parameters ϱ and σ^2 in the original model $(y, X\beta, V(\varrho, \sigma^2))$:

$$\tilde{\varrho} = \varrho_0 + \widehat{(\varrho - \varrho_0)} \quad \text{and} \quad \tilde{\sigma}^2 = \hat{\sigma}^2. \tag{10}$$

More particularly, MINQE(U,I) in the linearized model leads to LMINQE(U,I) in original model, and similarly, MINQE(I) leads to LMINQE(I). In the present paper we concentrate our attention to invariant (with respect to the group of translations $y \mapsto y + X\alpha$, for arbitrary α) estimation only, i.e. to LMINQE(I)'s.

The following theorem gives the explicit form of the LMINQE(I)'s in the model $(y, X\beta, V(\varrho, \sigma^2))$.

Theorem 1. *Consider the linear model* (4) *with autoregressive disturbances. Let* $|\varrho_0| < 1$ *and* $\sigma_0^2 > 0$ *denote the prior values for the autoregressive coefficient* ϱ *and the variance* σ^2*. Further, let* $e = (e_1, \dots, e_n)'$ *denotes the vector of* (ϱ_0, σ_0^2)*-locally best least squares residuals:* $e = y - X(X'V_0^{-1}X)^- X'V_0^{-1}y$*, where* $V_0 = V(\varrho_0, \sigma_0^2)$.

Then the LMINQE(I)'s, $\tilde{\varrho}$ *and* $\tilde{\sigma}^2$*, of the parameters* ϱ *and* σ^2*, respectively, computed for the prior value* (ϱ_0, σ_0)*, are given by*

$$\tilde{\varrho} = \varrho_0 + \frac{\delta}{\sigma_0^2(n-1)} \Big\{ -\varrho_0 \sum_{t=1}^{n} e_t^2 + \kappa \sum_{t=2}^{n} e_t e_{t-1} - \varrho_0(\kappa - \varrho_0^2) \sum_{t=3}^{n} e_{t-1}^2 \Big\} \tag{11}$$

$$\tilde{\sigma}^2 = \frac{1}{(n-1)} \Big\{ (1-\delta) \sum_{t=1}^{n} e_t^2 - 2\varrho_0 \sum_{t=2}^{n} e_t e_{t-1} + (1+\delta)\, \varrho_0^2 \sum_{t=3}^{n} e_{t-1}^2 \Big\} \tag{12}$$

where $\kappa = n - (n-2)\varrho_0^2$*, and* $\delta = (1 - \varrho_0^2)/\kappa$.

Proof. According to (10), we compute the MINQE(I)'s for the parameters $(\varrho - \varrho_0)$ and σ^2, respectively, at prior value $(0, \sigma_0^2)$ in the model $(y, X\beta, W(\varrho, \sigma^2))$.

By the MINQE-theory, the solution is a linear combination of two quadratics, given by

$$\begin{pmatrix} \widehat{\varrho - \varrho_0} \\ \sigma^2 \end{pmatrix} = K_{(I)}^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{13}$$

where $K_{(I)}$ denotes the (2×2) criterion matrix for invariant quadratic estimation, with its elements given by

$$\{K_{(I)}\}_{ij} = \operatorname{tr}(V_0^{-1} V_i V_0^{-1} V_j), \quad i, j = 1, 2, \tag{14}$$

where $V_0 = V(\varrho_0, \sigma_0^2)$. ("tr" stands for the **trace** operator which sums the diagonal elements of a matrix). The quadratics q_1 and q_2 are defined as

$$\begin{aligned} q_1 &= e' V_0^{-1} V_1 V_0^{-1} e, \\ q_2 &= e' V_0^{-1} V_2 V_0^{-1} e, \end{aligned} \tag{15}$$

where $e = y - X(X'V_0^{-1}X)^{-}X'V_0^{-1}y$ is the vector of (ϱ_0, σ_0^2)-locally best wighted least squares residuals.

Considering the special form of the inversion of the matrix V_0, see e.g. Kmenta (1986), the form of the matrices V_1 and V_2, defined by (7), and after some algebra, we get

(16)

$$K_{(I)}^{-1} = \frac{1}{(n-1)(n-(n-2)\varrho_0^2)} \begin{pmatrix} \dfrac{n(1-\varrho_0^2)^2}{2} & -\varrho_0(1-\varrho_0^2)\sigma_0^2 \\ -\varrho_0(1-\varrho_0^2)\sigma_0^2 & \left(n-1-(n-3)\varrho_0^2\right)\sigma_0^4 \end{pmatrix},$$

and

$$\begin{aligned} q_1 &= \frac{2}{\sigma_0^2} \left\{ \sum_{t=2}^{n} e_t e_{t-1} - \varrho_0 \sum_{t=3}^{n} e_{t-1}^2 \right\}, \\ q_2 &= \frac{1}{\sigma_0^4} \left\{ \sum_{t=1}^{n} e_t^2 - 2\varrho_0 \sum_{t=2}^{n} e_t e_{t-1} + \varrho_0^2 \sum_{t=3}^{n} e_{t-1}^2 \right\}. \end{aligned} \tag{17}$$

The explicit forms of the LMINQE(I)'s, given by (11) and (12), are a direct consequence of the previous calculations and the equation (13). □

Remark 1. Generally, LMINQE's are quadratic plus constant estimators, i.e. they are of the type $c + y'Ay$, where c is a constant and A is a given symmetrical matrix. The optimality properties of $\tilde{\varrho}$ and $\tilde{\sigma}$ are direct consequence of the results given in Azaïs **et al.** (1993):

> "LMINQE(I)'s of ϱ and σ^2 are (ϱ_0, σ_0^2)-locally optimum in QCE(I) — the class of quadratic plus constant and invariant estimators".

Similar result holds true for LMINQE(U,I), see Proposition 5 in Azaïs **et al.** (1993).

Remark 2. Following the lines of the proof, we can easily find that the expectation of the LMINQE(I)-vector $(\tilde{\varrho}, \tilde{\sigma}^2)'$, locally at the point $(\varrho_0, \sigma_0^2)'$, is equal to

$$E\left((\tilde{\varrho}, \tilde{\sigma}^2)'\right) = (\varrho_0, 0)' + K_{(I)}^{-1} K_{(UI)} (0, \sigma_0^2)' \tag{18}$$
$$= (\varrho_0, 0)' + K_{(I)}^{-1} \left(\operatorname{tr}(V_1 (M V_0 M)^+), \frac{n - \operatorname{rank}(X)}{\sigma_0^2}\right)',$$

where $K_{(I)}^{-1}$ is defined by (16), $K_{(UI)}$ denotes the criterion matrix for unbiased and invariant quadratic estimation, with its elements given by

$$\{K_{(UI)}\}_{ij} = \operatorname{tr}((M V_0 M)^+ V_i (M V_0 M)^+ V_j), \quad i, j = 1, 2, \tag{19}$$

and where $(M V_0 M)^+ = V_0^{-1} - V_0^{-1} X (X' V_0^{-1} X)^- X' V_0^{-1}$.

Under normality assumptions, i.e. if the AR(1) process is Gaussian, the variance-covariance matrix of the vector $(\tilde{\varrho}, \tilde{\sigma}^2)'$, locally at the point $(\varrho_0, \sigma_0^2)'$, is equal to

$$\operatorname{Var}\left((\tilde{\varrho}, \tilde{\sigma}^2)'\right) = 2 K_{(I)}^{-1} K_{(UI)} K_{(I)}^{-1}. \tag{20}$$

For more details see, e.g. J. Volaufová and V. Witkovský (1992).

Remark 3. The statistical properties of the estimator of the linear function $p'\beta$ of the first-order parameters β in linear model with variance and covariance components, $\widehat{p'\beta} = p'(X'\tilde{V}^{-1}X)^- X'\tilde{V}^{-1}y$, the so-called **plug-in** or **two-stage** estimator, based on the estimate of the variance-covarince matrix $\tilde{V} = V(\tilde{\varrho}, \tilde{\sigma}^2)$ are generally unknown. However, one approach to determinig the upper bound for the difference in variances of the plug-in estimator and the BLUE, under the assumption of symetry of the distribution of ε and the existence of finite moments up to the tenth order, was proposed by J. Volaufová, (1993b).

The LMINQE(I)'s of the autoregressive coefficient ϱ and the variance σ^2 are sensitive to the choice of the prior values ϱ_0 and σ_0^2, respectively. An inappropriate choice of the prior parameters, i.e. inconsistent with the observed data, may leads to the estimate $\tilde{\varrho}$ out of the parameter space $|\varrho| < 1$.

For practical applications, if there is no strong evidence about the prior values, we suggest the so-called **two-stage** LMINQE(I)'s, which are based on two iterations of the following **two-step** method:

1. **First step:** Compute $\tilde{\sigma}^2$, according to (12), for the prior choice ϱ_0. (Note, that the estimator $\tilde{\sigma}^2$ does not depend on a prior value σ_0^2, it depends only on ϱ_0).

2. **Second step:** Compute $\tilde{\varrho}$, according to (11), at the prior value $(\varrho_0, \tilde{\sigma}^2)$.

Note, that this two-step method computed for the prior value $\varrho_0 = 0$, leads to the estimators

$$\tilde{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n} e_t^2 \quad \text{and} \quad \tilde{\varrho} = \frac{n}{n-1}\,\frac{\sum_{t=2}^{n} e_t e_{t-1}}{\sum_{t=1}^{n} e_t^2}, \tag{21}$$

with e_t, $t = 1, \dots, n$, being the ordinary least squares residuals given by $e = y - X(X'X)^{-}X'y$.

The second iteration of the two-step method, computed for the prior value $\varrho_0 = \tilde{\varrho}$, $\tilde{\varrho}$ given by (21), gives the **two-stage** LMINQE(I)'s of σ^2 and ϱ, respectively.

4. Maximum Likelihood Estimates

The two-step method could be naturally expanded to the **iterative algorithm** by a repeated application of those two steps until convergence is reached.

Provided that $\tilde{\varrho}_N$ and $\tilde{\sigma}_N^2$ denote the estimates after the Nth stage of the iteration procedure, the $(N+1)$st stage of the algorithm is given by:

1. **First step:** Compute the new vector of residuals

$$e = y - X(X'V_{N+1}^{-1}X)^{-}X'V_{N+1}^{-1}y, \tag{22}$$

where $V_{N+1} = V(\tilde{\varrho}_N, \tilde{\sigma}_N^2)$. Further, compute $\tilde{\sigma}_{N+1}^2$ following the formula (12), and replacing ϱ_0 by $\tilde{\varrho}_N$.

2. **Second step:** Compute $\tilde{\varrho}_{N+1}$, following the formula (11), as a function of $\tilde{\varrho}_N$ and $\tilde{\sigma}_{N+1}^2$.

Note, that the eventual limit points of the above iterative algorithm are equivalent to the eventual limit points of **iterated** LMINQE(I)'s, denoted as **ILMINQE(I)'s**, of the parameters ϱ and σ^2, respectively. The limit points we denote by $\mathring{\varrho}$ and $\mathring{\sigma}^2$, respectively.

Azaïs **et al.** (1993) have proved the equivalence between the **Fisher Scoring Algorithm** for finding singular points of the Gaussian likelihood, and ILMINQE(I)'s. Following those results, the point $(\mathring{\beta}, \mathring{\varrho}, \mathring{\sigma}^2)$, with ILMINQE(I)'s $\mathring{\varrho}$ and $\mathring{\sigma}^2$, and $\mathring{\beta}$ such that

$$X\mathring{\beta} = X\left(X'\left(V(\mathring{\varrho}, \mathring{\sigma}^2)\right)^{-1}X\right)^{-}X'\left(V(\mathring{\varrho}, \mathring{\sigma}^2)\right)^{-1}y, \tag{23}$$

is the singular point for the Gaussian log-likelihood function

$$\mathcal{L}(\beta, \varrho, \sigma^2) = \text{const} - \frac{1}{2}\log|V(\varrho, \sigma^2)| - \frac{1}{2}(y - X\beta)'\left(V(\varrho, \sigma^2)\right)^{-1}(y - X\beta), \tag{24}$$

i.e. $\mathcal{L}(\mathring{\beta}, \mathring{\varrho}, \mathring{\sigma}^2) = \max \mathcal{L}(\beta, \varrho, \sigma^2)$. Hence, the following theorem holds true:

Theorem 2. *Eventual limit points of the above iteration procedure, $\mathring{\varrho}$ and $\mathring{\sigma}^2$, are the maximum likelihood estimates, MLE's, of the autoregressive coefficient ϱ and the variance σ^2 in the linear model with autoregressive disturbances $(y, X\beta, V(\varrho, \sigma^2))$.*

5. Discussion

To illustrate the properties of the proposed estimators we consider the simple form model of the quantity theory of money, originally disscused by Friedman and Meiselman (1963), see Table 1:

$$C_t = \alpha + \beta M_t + \varepsilon_t, \tag{25}$$

with C — consumer expenditure and M — stock of money, both measured in billions of current dollars. It is assumed that the disturbances follow a first-order autoregressive scheme.

Year	Quarter	Consumer expenditure	Money stock	Year	Quarter	Consumer expenditure	Money stock
1952	I	214.6	159.3	1954	III	238.7	173.9
	II	217.7	161.2		IV	243.2	176.1
	III	219.6	162.8	1955	I	249.4	178.0
	IV	227.2	164.6		II	254.3	179.1
1953	I	230.9	165.9		III	260.9	180.2
	II	233.3	167.9		IV	263.3	181.2
	III	234.1	168.3	1956	I	265.6	181.6
	IV	232.3	169.7		II	268.2	182.5
1954	I	233.7	170.5		III	270.4	183.3
	II	236.5	171.6		IV	275.6	184.3

Table 1. Consumer expenditure and stock of money, 1952(I) — 1956(IV), both measured in billions of current dollars. Source: M. Friedman and D. Meiselman: "The relative stability of monetary velocity and the investment multiplier in the United States, 1897–1958", In: Commision on Money and Credit, Stabilization Policies (Englewood Cliffs, NJ: Prentice-Hall, 1963), p. 266.

We will consider three types of LMINQE(I)'s of the autoregressive parameter ϱ in the model (25), which differ in the choice of the prior values of the parameters ϱ_0 and σ_0^2, respectively:

1. **Estimator I:** LMINQE(I) of ϱ computed for all prior values $(\varrho_0, \tilde{\sigma^2})$, where $\tilde{\sigma^2}$ is given by (12), and $|\varrho_0| < 1$.

This estimator we get as a single iteration of the two-step method for computing LMINQE(I) of the parameter ϱ. The estimator seems to be highly sensitive to the choice of the prior value of ϱ.

2. **Estimator II:** LMINQE(I) of ϱ computed for all prior values $(\varrho_0, \sigma^2_{LSE})$, where σ^2_{LSE} is the estimate based on the ordinary least squares residuals, given by (21), and with $|\varrho_0| < 1$.

   This estimator seems to be quite robust to all possible choices of the prior value of ϱ_0. The reason is that σ^2_{LSE} is the upper bound for the all possible estimates of σ^2.

3. **Estimator III:** LMINQE(I) of ϱ computed for all prior values $(\varrho_0, (1-\varrho_0^2)\,\sigma^2_{LSE})$, where σ^2_{LSE} is the estimate based on the ordinary least squares residuals, given by (21), and $|\varrho_0| < 1$.

   The estimator is a modification to the previous one. Here we put the variance of the disturbance, under given prior values ϱ_0 and σ_0^2, $\mathrm{Var}\,(\varepsilon_t) = \sigma_0^2/(1-\varrho_0^2)$, to be constant and equal to σ^2_{LSE}.

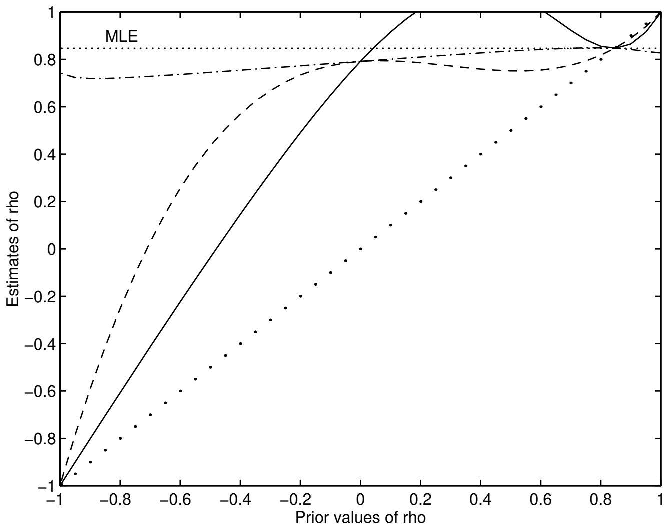

Figure 1. Estimates of the autoregressive coefficient ϱ for different prior values of ϱ_0. Estimator I — solid line; Esimator II — dashed line; Estimator III — dashed dotted line; MLE — dotted line.

The values of the considered estimators can be easily seen from the Figure 1. Note, that the Estimator I, i.e. LMINQE(I) computed by two-step method, is highly sensitive to the choice of the prior value ϱ_0 of the parameter ϱ. Each of those estimators, if iterated, leads to the maximum likelihood estimate.

Stage	ϱ	σ^2	α	β	Log-likelihood
1	0.7915	14.2593	-154.7929	2.3008	-48.7363
2	0.8588	4.5273	-157.7919	2.3255	-44.0237
3	0.8483	4.4785	-156.2356	2.3192	-44.0010
4	0.8470	4.4771	-156.6078	2.3209	-44.0003
5	0.8470	4.4771	-156.6474	2.3210	-44.0003
6	0.8470	4.4771	-156.6496	2.3211	-44.0003

Table 2. Iteration steps of the algorithm for computing the MLE's of the parameters ϱ, σ^2, α and β in the model (25).

Finally, in the Table 2 we illustrate the speed of convergence of the proposed algorithm for computing MLE's of the parameters of the model (25). For each stage the table gives the estimates of ϱ, σ^2, α, and β, together with the value of log-likelihood function.

References

1. Azaïs J. M., Bardin A. and Dhorne T., *MINQE, maximum likelihood estimation and Fisher scoring algorithm for non linear variance models*, Statistics **24(3)** (1993), 205–213.

2. Brockwell J. P. and Davis R. A., *Time Series: Theory and Methods*, Springer-Verlag, New York Berlin Heidelberg, 1987.

3. Friedman M. and Meiselman D., *The relative stability of monetary velocity and the investment multiplier in the United States, 1897–1958*, In: Commision on Money and Credit, Stabilization Policies, Englewood Cliffs, NJ: Prentice-Hall, 1963..

4. Durbin. J., *Estimation of parameters in time-series regression models*, Journal of the Royal Statistical Society Series B **22** (1960), 139–153.

5. Kmenta. J., *Elements of Econometrics*, Macmillan Publishing Company, New York, 1986.

6. Magnus J. R., *Maximum likelihood estimation of the GLS model with unknown parameters in the disturbance covariance matrix*, Journal of Econometrics **7** (1978), 305–306.

7. Prais S. J. and Winsten C. B., *Trend Estimators and Serial Correlation*, Cowles Commision Discussion Paper No. 383 (1954), Chicago.

8. Rao C. R., *Estimation of variance and covariance components — MINQUE theory*, Journal of Multivariate Analysis **1** (1971a), 257–275.

9. ______, *Minimum variance quadratic unbiased estimation of variance components*, Journal of Multivariate Analysis **1** (1971b), 445–456.

10. Rao C. R. and Kleffe J., *Estimation of variance components*, Handbook of Statistics I, Analysis of Variance, Chapter 1 (P.R. Krishnaiah, ed.), North-Holland, Amsterdam New York Oxford Tokyo, 1980, pp. 1–40.

11. ______, *Estimation of Variance Components and Applications*, volume 3 of Statistics and probability, North-Holland, Amsterdam New York Oxford Tokyo, 1988.

12. Searle S. R., Casella G. and McCulloch Ch. E., *Variance Components*, Wiley series in probability and mathematical statistics, John Wiley & Sons Inc., New York Chichester Brisbane Toronto Singapore, 1992.

13. Volaufová J., *A brief survey on the linear methods in variance-covariance components model*, Model-Oriented Data Analysis $\mathcal{P}$185–196, St. Petersburg, Russia, May 1993 or Physica-Verlag Heidelberg, 1993a. (W. G. Müller, H. P. Wynn, and A. A. Zhigljavsky, eds.).

14. ______, *On variance of the two-stage estimator in variance-covariance components model*, *Applications of Mathematics*, vol. 38(1), 1993b, pp. 1–9.
15. J. Volaufová and V. Witkovský., *Estimation of variance components in mixed linear model.*, Applications of Mathematics **37**(2) (1992), 139–148.

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