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RNP AND KMP ARE INCOMPARABLE PROPERTIES IN NONCOMPLETE SPACES

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Abstract. We exhibit an example in a noncomplete space of a closed, bounded and convex subset verifying KMP and failing RNP and, another such example verifying RNP and failing KMP.

We begin this note by recalling some definitions: (See [**2**] and [**3**]).

Let X be a normed linear space and let C be a closed, bounded and convex subset of X.

C is said to be dentable if for each $\varepsilon > 0$ there is $x \in C$ such that $x \notin \overline{\text{co}}(C \setminus B(x,\varepsilon))$, where $\overline{\text{co}}$ denotes the closed convex hull and $B(x,\varepsilon)$ is the closed ball with centrum x and radius ε.

C is said to have the Radon-Nikodym property (RNP) if every nonempty subset of C is dentable.

C is said to have the Krein-Milman property if every closed and convex subset, F, of C verifies $F = \overline{\text{co}}(\text{Ext } F)$, where Ext F denotes the set of extreme points of F.

It is known that C has KMP if every closed and convex subset of C has some extreme point. (Even in noncomplete spaces.)

The above definition of RNP working in noncomplete spaces and, today, the most authors define RNP in Banach spaces as here.

For a Banach space X it is known that RNP implies KMP and the converse is an well known open problem.

We prove that KMP does not imply RNP in noncomplete spaces. For this we consider a closed, bounded and convex subset, STS, which appears in [**1**], of $c_0(\Gamma)$.

In [**1**] it is shown that $\overline{STS_0} = STS$ in $c_0(\Gamma)$.

Our goal is to prove that STS_0 is a closed, bounded and convex subset of $c_{00}(\Gamma)$ verifying KMP and failing RNP.

Now we descript briefly the set STS_0 of $c_{00}(\Gamma)$.

Γ denotes the set of finite sequences of natural numbers and 0 denotes the empty sequence in Γ.

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For α, $\beta \in \Gamma$ we define $\alpha \leq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha_i = \beta_i$ for $1 \leq i \leq |\alpha|$, where $|\alpha|$ is the lenght of α. Of course $|0| = 0$ and $0 \leq \alpha \quad \forall\, \alpha \in \Gamma$.

$$c_{00}(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\alpha \in \Gamma : x(\alpha) \neq 0\} \text{ is finite}\}$$

For each $\alpha \in \Gamma$ we define $b_\alpha \in c_{00}(\Gamma)$ by $b_\alpha(\gamma) = 1$ if $\gamma \leq \alpha$ and $b_\alpha(\gamma) = 1$ in other case.

And $STS_0 = co\{b_\alpha : \alpha \in \Gamma\} \subset c_{00}(\Gamma)$.

So, STS_0 is a nonempty closed, bounded and convex subset of $c_{00}(\Gamma)$.

Theorem. *STS_0 has KMP and fails RNP.*

Proof. It is easy to see that

$$b_\beta \in \overline{\text{co}}(A \setminus B(b_\beta, 1)) \quad \forall \beta \in \Gamma,$$

where $A = \{b_\alpha : \alpha \in \Gamma\}$, because

$$\lim_{n \to +\infty} \frac{b_{(\alpha,1)} + \ldots + b_{(\alpha,n)}}{n} = b_\alpha \quad \forall\, \alpha \in \Gamma.$$

Then A is not dentable and so STS_0 fails RNP.

Now let C be a nonempty closed and convex subset of STS_0. We will see that $\text{Ext}\,(C) \neq \emptyset$.

Let $z \in C$, and $K = \{x \in C : \text{supp}(x) \subseteq \text{supp}(z)\}$, where for each $x \in C$, $\text{supp}(x) = \{\alpha \in \Gamma : x(\alpha) \neq 0\}$.

Now K is a nonempty, convex and compact face of C. The Krein-Milman theorem says us that $\text{Ext}\,(K) \neq \emptyset$ and so, $\text{Ext}\,(C) \neq \emptyset$ because K is a face of C.□

Remark. As in [**1**] it is easy to see that STS_0 fails PCP (the point of continuity property) because $\{b_{(\alpha,i)}\}$ converges weakly to b_α when $i \to +\infty$, $\forall \alpha \in \Gamma$ and $\|b_{(\alpha,i)} - b_\alpha\| = 1 \quad \forall \alpha \in \Gamma$. (This is not inmediate because our environment space is not complete.)

Now, we give an example of a closed, bounded and convex set in a noncomplete space verifying RNP and failing KMP.

For this, we consider c_0 the Banach space of real null sequences with the maximum norm and, c_{00} the nonclosed subspace of c_0 of real sequences with a finite numbers of terms nonzero. So, c_{00} is a noncomplete normed linear space. We define:

$$F_0 = \left\{x \in c_{00} : |x_n| \leq \frac{1}{n}\ \forall\, n \in \mathbb{N}\right\}$$

Then F_0 is a closed, bounded and convex subset of c_{00}.

It is clear that F_0 has not extreme points because if $x \in F_0$ and $k \in \mathbb{N}$ such that $x(n) = 0\ \forall\, n \geq k$, then $y = x + \frac{1}{k}e_k$ and $z = x - \frac{1}{k}e_k$ are elements of F_0 such that $x = \frac{y+z}{2}$. (e_k is the sequence with value 1 in k and value 0 in $n \neq k$.)

Therefore, F_0 fails KMP.

Let us see, now, that F_0 has RNP. If C is a subset of F_0, then $\overline{C}$ is a weakly compact of c_0, since the closure of F_0 in c_0, F is it. So C is dentable. (See [**2**, Th. 2.3.6].)

Then F_0 has RNP and fails KMP.

References

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