Acta Math. Univ. Comenianae
Vol. LXV, 2(1996), pp. 161–169

ADJOINTABILITY OF OPERATORS ON HILBERT C^*–MODULES

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Abstract. Can a functional $f \in H_A^* = \mathrm{Hom}_A(H_A; A)$ on the non-self-dual Hilbert module H_A over a C^*-algebra A be represented as an operator of some inner product by an element of the module H_A, this inner product being equivalent to the given one? We discuss this question and prove that for some classes of C^*-algebras the closure with respect to the given norm of unification of such functionals for all equivalent inner products coincides with the dual module H_A^*. We discuss the notion of compactness of operators in relation to representability of functionals. We also show how an operator on H_A in some situations (e.g. if it is Fredholm) can be made adjointable by change of the inner product to an equivalent one.

Introduction

Let A be a C^*-algebra with unity. We consider Hilbert A-modules over A [**8**], i.e. (right) A-modules M together with an A-valued inner product $\langle\cdot,\cdot\rangle\colon M \times M \longrightarrow A$ satisfying the following conditions:

(i) $\langle x, x\rangle \geq 0$ for every $x \in M$ and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii) $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in M$,
(iii) $\langle\cdot,\cdot\rangle$ is A-linear in the second argument,
(iv) M is complete with respect to the norm $\|x\|^2 = \|\langle x, x\rangle\|_A$.

By $M^* = \mathrm{Hom}_A(M; A)$ we denote A-module dual to M consisting of continuous A-valued functionals. Let H_A be the right Hilbert A-module of sequences $a = (a_k)$, $a_k \in A$, $k \in \mathbf{N}$ such that the series $\sum a_k^* a_k$ converges in A in norm with the standard basis $\{e_k\}$ and let $L_n(A) \subset H_A$ be the submodule generated by the elements $e_1, \dots, e_n$ of the basis. An inner A-valued product on the module H_A can be given by $\langle x, y\rangle = \sum x_k^* y_k$ for $x, y \in A$. It is known [**9**] that in the case when A is a W^*-algebra an inner product can be naturally extended to the dual module H_A^*.

By an operator S from a Hilbert C^*-module M to another module N we mean a bounded A-homomorphism from M to N which possesses an adjoint operator S^* from N to M (of course there always exists an adjoint operator S^* from N (or N^*) to M^*).

Received June 24, 1996; revised November 7, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 47C15.

One of the essential distinctions between Hilbert C^*-modules and Hilbert spaces is the existence of non-self-dual modules (i.e. such that $M^* \neq M$). In other words there is no Riesz representation theorem for Hilbert C^*-modules and not all operators have an adjoint. In some papers [**3**], [**11**] it is shown that it is useful sometimes to consider on M other inner products defining norms equivalent to the given one. We show how in some situations one can make an operator have an adjoint by changing an inner product to another one equivalent to the given one. Partial result in this direction was announced in [**6**].

1. Representability of Functionals on Hilbert C^*-modules

In this section we study the question of representability of functionals on a Hilbert C^*-module M as inner products by elements from M. Define F to be the set of functionals of the form

$$x \longmapsto \langle y, x\rangle_\beta \quad (x, y \in M, \beta \in \mathbf{B})$$

where $\mathbf{B}$ is the set of all inner products $\langle \cdot, \cdot\rangle_\beta$, defining norms equivalent to the given one. We call a functional $f \in M^*$ representable if $f \in F$. Let A^{**} be an enveloping W^*-algebra for A. By H_A we denote the standard Hilbert A-module $l_2(A)$ with the standard basis $\{e_i\}$. The extension of the given inner product from $H_{A^{**}}$ to $H^*_{A^{**}}$ [**9**] we also denote by $\langle \cdot, \cdot\rangle$. Obviously we have $H^*_A \subset H^*_{A^{**}}$.

Proposition 1.1. *If $f \in H^*_A$ is representable then there exists an element $z \in H_A$ such that the operator inequality*

$$\alpha\langle z, z\rangle \leq \beta\langle f, f\rangle \leq \langle f, z\rangle \leq \gamma\langle f, f\rangle \leq \delta\langle z, z\rangle \tag{1}$$

holds for some positive numbers α, β, γ, δ.

Proof. It was proved in [**3**] using results of L. Brown [**1**] that due to the fact that the module H_A is countably generated, any inner product equivalent to the given one is of the form $\langle x, y\rangle_\beta = \langle Sx, Sy\rangle$, where $S \in \mathrm{End}_A(H_A)$ is an invertible bounded operator. S need not have an adjoint operator in the module H_A but it has an adjoint operator acting from H_A to H^*_A. If f is representable then $f = S^*Sz$ for some S and some $z \in H_A$, the operator $\langle f, z\rangle \in A$ is positive and we have

$$\langle f, z\rangle = \langle S^*Sz, z\rangle = \langle Sz, Sz\rangle = \langle z, z\rangle_\beta.$$

Due to invertibility of S we can find [**3**] positive numbers a and b such that

$$a\langle z, z\rangle \leq \langle z, z\rangle_\beta \leq b\langle z, z\rangle$$

and consequently

$$a\langle z, z\rangle \leq \langle f, z\rangle \leq b\langle z, z\rangle. \tag{2}$$

We now estimate $\langle f, f\rangle = \langle S^*Sz, S^*Sz\rangle$. Since $a^2 \le S^*S \le b^2$, so we have

$$a^2\langle z, z\rangle \le \langle f, f\rangle \le b^2\langle z, z\rangle. \tag{3}$$

Gluing together (2) and (3) we obtain (1). □

We call a functional $F \in M^*$ non-singular if there exists an element $z \in M$ such that spectrum of the operator $\langle f, z\rangle \in A$ is separated from zero. In this case without any loss of generality we can consider the operator $\langle f, z\rangle$ as positive with $\langle f, z\rangle \ge c > 0$ for some number c. The following example shows that there exist singular functionals with the property $\langle f, f\rangle = 1$.

Example 1.2. Let $A = L^\infty([0; 1])$. Define $f \in H_A^*$ as a sequence of functions $f = (f_k(t))$, where

$$f_k(t) = \begin{cases} 1, & t \in \left[\frac{1}{2^k}; \frac{1}{2^{k-1}}\right], \\ 0, & \text{for other } t. \end{cases}$$

It is obvious that $\langle f, f\rangle = 1$. We show that the spectrum of $\langle f, z\rangle$ is not separated from zero for all $z = (z_k) \in H_A$. Since the series $\sum_{k=1}^\infty z_k^* z_k$ is convergent in norm in A, for any $\varepsilon > 0$ we can find a number n such that $\|\sum_{k=n+1}^\infty z_k^* z_k\| < \varepsilon$. Then if $t < 1/2^n$ we have $|f_k(t)z_k(t)| < \varepsilon$, and consequently $|\langle f, z\rangle(t)| < \varepsilon$. Hence f is singular. As $\langle f, z\rangle$ is not invertible for all $z \in H_A$ and $\langle f, f\rangle = 1$, the inequality (1) is violated, hence f is not representable.

Proposition 1.3. *Let $f \in M^*$ be non-singular. Then it is representable.*

Proof. The Cauchy-Schwarz inequality gives us

$$0 < c \le \langle f, z\rangle \le \|f\| \langle z, z\rangle^{1/2},$$

therefore the module $\mathrm{Span}_A z$ is isomorphic to A. One can check that M can be decomposed into the (not orthogonal) direct sum:

$$M = \mathrm{Span}_A z \oplus \mathrm{Ker}\, f.$$

If $x \in M$, then put

$$a = \langle f, z\rangle^{-1} \cdot \langle f, x\rangle; \quad y = x - za.$$

Then $x = za + y$, and $y \in \mathrm{Ker}\, f$. Uniqueness of such decomposition is obvious. The rest of the proof follows from a simple corollary of the following result of E. V. Troitsky [**13**]:

Proposition 1.4. *Let $M = M_1 \oplus M_2$ be a topological decomposition into a direct sum (not necessarily orthogonal) of closed modules. Then there exists a new* *inner product on M equivalent to the given one with respect to which the given decomposition is orthogonal.*

Proof. Define an operator $J\colon M \longrightarrow M$ by

$$Jx = \begin{cases} x, & \text{if } x \in M_1, \\ -x, & \text{if } x \in M_2. \end{cases}$$

As it is shown in [**13**], the inner product

$$\langle x, y\rangle_\beta = \langle x, y\rangle + \langle Jx, Jy\rangle$$

is equivalent to the given one because the operator J is bounded due to the fact that M_1 and M_2 are closed. Orthogonality of these modules with respect to this inner product is obvious. □

If we take $M_1 = \mathrm{Span}_A z$ and $M_2 = \operatorname{Ker} f$ and take a new inner product as in the previous proposition then z would be orthogonal to $\operatorname{Ker} f$. Put $z' = z \cdot \langle z, z\rangle_\beta^{-1} \cdot \langle f, z\rangle$. Then we have $\langle f, x\rangle = \langle z', x\rangle_\beta$. Hence f is representable. □

Theorem 1.5. *Let A be a C^*-algebra of stable topological rank one. Then the set of representable functionals on H_A is dense in H_A^* with respect to the given norm.*

Proof. The proof is reduced to the verification of density of non-singular functionals in H_A^*. As A has stable topological rank one, so the set of invertible elements is dense in A. If $f = (f_i) \in H_A^*$, then we can find in A an invertible element g_1 such that $\|g_1 - f_1\| < \varepsilon$. Putting $g = (g_1, f_2, f_3, \dots) \in H_A^*$ and taking $z = e_1 = (1, 0, 0, \dots) \in H_A^*$, we obtain the invertibility of $\langle g, z\rangle$ and $\|g - f\| < \varepsilon$.□

Notice that the class of C^*-algebras of stable topological rank one evidently includes commutative *-algebras of functions on spaces of dimension one. It was shown in [**10**] that this class includes also C^*-algebras of irrational rotation. It includes also finite W^*-algebras.

The situation with respect to representability of functionals in the general case is more complicated. Namely for purely infinite algebras there exist open sets of non-representable functionals. To show that consider the following

Example 1.6. Let A be the C^*-algebra of all bounded operators on an infinite-dimensional Hilbert space. As it is shown in [**3**], there exists an isomorphism of modules $S\colon A \longrightarrow H_A^*$. Put $f = S(a)$ with $a \in A$. Then the condition $\langle f, x\rangle = 0$ can be written as $\langle S(a), S(b)\rangle = 0$ with $b = S^{-1}(x) \in A$, hence $\langle a, b\rangle = a^* b = 0$. If a is invertible then we have $\operatorname{Ker} f = 0$ — a situation which is impossible for Hilbert spaces. But if f is representable, $\langle f, \cdot\rangle = \langle z, \cdot\rangle_\beta$ with $z \in H_A$, then $\operatorname{Ker} f$ cannot be zero. Therefore the functional $f = S(1_A) \in H_A^*$ is not representable and it possesses a neighborhood consisting of non-representable functionals.

2. Compactness and Adjointability of Operators on Hilbert C^*-modules

It was noticed recently [**13**] that in some cases we can make an operator on a Hilbert C^*-module adjointable by change of the inner product to an equivalent one. Namely E. V. Troitsky showed that if an operator $T \in \mathrm{End}_A(H_A)$ lies in the image of a representation of a compact group G, $T = T_g$ for $g \in G$, then averaging on G the given inner product we can obtain a new inner product equivalent to the given one such that T is unitary with respect to this inner product. In this section we show some other situations where operators on Hilbert C^*-modules can be made adjointable.

Consider the closed ideal $BK(H_A)$ of Banach-compact operators generated by operators of the form $y\langle f, \cdot\rangle$ with $y \in H_A$, $f \in H_A^*$ in the Banach algebra $\mathrm{End}_A(H_A)$. By $K^\beta(H_A)$ we denote the C^*-algebra of compact operators [**5**], [**7**] in the Hilbert C^*-module $(H_A, \langle\cdot,\cdot\rangle_\beta)$.

Theorem 2.1. *Let A be a C^*-algebra with dense set of invertible elements. Then we have*

$$BK(H_A) = \overline{\bigcup_{\beta\in\mathbf{B}} K^\beta(H_A)},$$

where bar denotes the closure with respect to the given norm.

Proof. It is sufficient to approximate operators of the form

$$x \longmapsto \sum_{i=1}^{n} y_i \langle f^{(i)}, x\rangle.$$

Without loss of generality we can suppose that the i-th coordinates $f^{(i)}$ of f are invertible operators. Suppose that we have found an inner product $\langle\cdot,\cdot\rangle_\beta$ with respect to which the functionals $f^{(i)}$ $(i \le k)$ are of the form $\langle \sum_{j=1}^{i} e_j a_j, \cdot\rangle_{\beta_k}$ with $a_j \in A$. Define an inner product $\langle\cdot,\cdot\rangle_{\beta_{k+1}}$ approximating the operator $\sum_{i=1}^{k+1} y_i\langle f^{(i)}, \cdot\rangle$ as follows. Let $\langle\cdot,\cdot\rangle_{\beta_{k+1}}$ coincide with $\langle\cdot,\cdot\rangle_{\beta_k}$ on the module $L_k(A)$ generated by the first k elements of the standard basis $e_1, \dots, e_k$, and on its orthogonal complement $L_k(A)^\perp$ we define $\langle\cdot,\cdot\rangle_{\beta_{k+1}}$ so that it is equivalent to $\langle\cdot,\cdot\rangle_{\beta_k}$ and

$$f^{(k+1)}|_{L_k(A)^\perp} = \langle e_{k+1}a_{k+1}, \cdot\rangle_{\beta_{k+1}}$$

with some $a_{k+1} \in A$. Then we have

$$f^{(k+1)} = \left(\sum_{i=1}^{k} e_i f_i^{(k+1)} + e_{k+1}a_{k+1}, \cdot \right)_{\beta_{k+1}}.$$

Notice that the representations of the functionals $f^{(i)}$, $i \leq k$ have not changed because $L_k(A)$ and $L_k(A)^\perp$ are orthogonal with respect to both inner products $\langle\cdot,\cdot\rangle_{\beta_k}$ and $\langle\cdot,\cdot\rangle_{\beta_{k+1}}$. $\square$

Remark. Although the set $\bigcup_{\beta\in\mathbf{B}} K^\beta(H_A)$ has no natural structure of an algebra, its norm closure surprisingly is a Banach algebra. For comparison we would like to state also the following result of M. Frank [**3**]:

Proposition 2.2. *One has*

$$\bigcap_{\beta\in\mathbf{B}} K^\beta(H_A) = 0.$$

Now we give a geometrical description of compact operators. Let $S \subset H_A$ be a bounded subset. We call it A-precompact if for every $\varepsilon > 0$ there exists a free finitely generated A-module $N \cong A^n$; $N \subset H_A$ such that $\operatorname{dist}(S, N) < \varepsilon$.

Proposition 2.3. *Let T be an operator on H_A (resp. an operator having an adjoint). Then the following conditions are equivalent:*

(i) $T \in BK(H_A)$ *(resp. T is compact);*
(ii) *the image $T(B_1(H_A))$ of the unit ball $B_1(H_A)$ is A-precompact.*

Proof. If the first statement is valid then it is sufficient to prove that one can find an approximating module $N \cong A^n$ for a finite set of elements of H_A. This can be done by the method of [**2**]. So suppose now that (ii) is valid. Then for any $\varepsilon > 0$ we can find elements $b_1, \dots, b_k \in H_A$ with $\langle b_i, b_j\rangle = \delta_{ij}$ which generate a module $N \subset H_A$ and $\operatorname{dist}(T(B_1(H_A)), N) < \varepsilon$. Denote by P_N a projection onto N and consider the operator $P_N T$. It can be decomposed in the form

$$P_N T x = b_1\langle f_1, x\rangle + \cdots + b_n\langle f_n, x\rangle \tag{4}$$

with $f_i \in H_A^*$. As for any $x \in B_1(H_A)$ we can find an element $b \in N$ with $\|Tx - b\| < \varepsilon$, so

$$\begin{aligned}\|Tx - P_N Tx\| &= \|Tx - b + b - P_N Tx\| \\ &= \|Tx - b\| + \|P_N(b - Tx)\| \ \leq\ \varepsilon + \|P_N\|\,\varepsilon = 2\varepsilon,\end{aligned}$$

hence $\|T - P_N T\| \leq 2\varepsilon$ and T belongs to the norm closure of operators of the form (4). If T is adjointable then $P_N T$ is also adjointable, hence $f_i \in H_A$ and T is compact. $\square$

Remember that an operator is called Fredholm if it is invertible modulo the ideal of compact operators. As in the case of Hilbert modules one has two definitions of compactness (with and without adjointness), so there are also two definitions of Fredholmness. By a Banach-Fredholm operator we understand an operator which is invertible modulo the ideal of the Banach-compact operators. Notice that this definition does not depend on a Hilbert structure. Now we show that Banach-Fredholm operators in Hilbert modules over arbitrary unital C^*-algebras can be made adjontable by a change of inner product. Unfortunately the case of C^*-algebras without unit is much more difficult and we intend to consider it somewhere else. The interest in the space of Banach-Fredholm operators lies in the fact that due to contractibility of the general linear group of all bounded operators of the module H_A [**12**] this space can be considered as a classifying space for the topological K-theory with coefficients in the C^*-algebra A.

Theorem 2.4. *Let M, N be Hilbert C^*-modules isomorphic to H_A, $T\colon M \longrightarrow N$ be a Banach-Fredholm operator having no adjoint. Then there exist new inner products on these modules equivalent to the given ones so that T is adjointable with respect to these inner products.*

Proof. Consider a Banach-compact operator $K \in BK(M, N)$,

$$K = \sum_{i=1}^{\infty} y_i \langle f_i, \cdot\rangle \tag{5}$$

with $y_i \in N$, $f_i \in M^*$. □

Lemma 2.5. *Let $S\colon M \longrightarrow N$ be an invertible operator, $S \in \mathrm{End}_A(M; N)$. Then the operator $S + K$ is diagonal for some decompositions $M = M_1 \oplus M_2$; $N = N_1 \oplus N_2$, where M_1, N_1 are finitely generated projective modules, direct sums are not necessarily orthogonal and $(S + K)|_{M_2}$ is an isomorphism.*

Proof. Without loss of generality we can consider the sum (5) to be finite and the elements y_i to be such that $\langle y_i, y_i\rangle = \delta_{ij}$. Denote by $N_1 \cong A^n$ the A-module generated by these y_i, $i = 1, \dots, n$. Put

$$M_1 = S^{-1}(N_1); \quad M_2 = S^{-1}(N_1^{\perp})$$

and define an operator

$$R\colon N_1^{\perp} \longrightarrow N_1 \ \text{ by } \ Ry = K(S^{-1}y), \ \ y \in N_1^{\perp}.$$

Then the module $(S + K)(M_2)$ is of the form

$$N_2 = (S + K)(M_2) = \{y + Ry,\ y \in N_1^{\perp}\}.$$

This module is obviously closed and $N = N_1 \oplus N_2$ (not orthogonal direct sum). Indeed, if we denote by P_1 and P_2 the orthoprojections on N_1 and on $N_1^{\perp}$ respectively, then an element $z \in N$ can be decomposed:

$$z = (P_1 z - RP_2 z; P_2 z + RP_2 z).$$

As we have

$$(S+K)(M_1) \subset N_1, \quad (S+K)(M_2) = N_2,$$

so the operator $S+K$ is diagonal with respect to the chosen decompositions of M and N. $\square$

If $T\colon M \longrightarrow N$ is Banach-Fredholm then there exists an operator $Q\colon N \longrightarrow M$ such that $TQ-1$ and $QT-1$ are Banach-compact. Then by standard methods [**7**], [**4**] one can find a decomposition

$$T\colon M_1 \oplus M_2 \longrightarrow N_1 \oplus N_2$$

where M_1 and N_1 are projective and finitely generated, $T|_{M_2}\colon M_2 \longrightarrow N_2$ is an isomorphism, but direct sums need not be orthogonal. By the Proposition 1.4 we can make them orthogonal by changing inner products on M and N to equivalent ones. Further, as $T|_{M_2}$ is an isomorphism, we can correct the inner product on N_2 so that this isomorphism preserves Hilbert module structure. Then $T|_{M_2}$ is an identity, hence adjointable. But $T|_{M_2}$ is an operator acting in the auto-dual modules, hence it is also adjointable, so T is adjointable. $\square$

Finally we show how the averaging theorem of [**13**] can be generalized from compact to amenable groups in the case of W^*-algebras in order to find an appropriate inner product. We state the following theorem for the group $\mathbf{Z}$ but the proof is valid for all amenable groups.

Theorem 2.6. *Let A be a W^*-algebra and let $T\colon M \longrightarrow M$ be an operator such that all its powers are uniformly bounded, $\|T^n\| \le C$, $n \in \mathbf{Z}$. Then there exists an inner product $\langle \cdot, \cdot \rangle_\beta$ equivalent to the given one so that T is unitary with respect to this inner product.*

Proof. Let A_* be a predual Banach space for A. For any $\phi \in A_*$ define a function $f_{x,y}$ on $\mathbf{Z}$ by

$$f_{x,y}(n) = \phi(\langle T^n x, T^n y \rangle)$$

for $x, y \in M$. By supposition this function is bounded. Put

$$\phi_{x,y} = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f_{x,y}(k).$$

Fixing x and y we obtain a linear bounded map

$$a_{x,y}\colon A_* \longrightarrow \mathbf{C}; \quad \phi \longmapsto \phi_{x,y}.$$

This map is an element of $(A_*)^* = A$. Define a new inner product on M by $\langle x, y \rangle_\beta = a_{x,y} \in A$. We must check that $\langle \cdot, \cdot \rangle_\beta$ is an inner product. Its sesquilinearity is obvious. If $\phi \in A_*$ is a state then $f_{x,x}(n) \ge 0$, hence $\phi(\langle x, x \rangle_\beta) = \phi_{x,x} \ge 0$. Suppose that $\langle x, x \rangle_\beta = 0$ for some $x \in M$. Then we have $\phi_{x,x} = 0$. But as

$$\langle x, x \rangle = \langle T^{-k}(T^k x), T^{-k}(T^k x) \rangle \le C^2 \langle T^k x, T^k x \rangle,$$

so $\frac{1}{C^2}f_{x,x}(0) \le f_{x,x}(n)$ and

$$\frac{1}{2n+1}\sum_{k=-n}^{n} f_{x,x}(k) \ge \frac{1}{C^2}f_{x,x}(0),$$

hence $\phi_{x,x} \ge \frac{1}{C^2}f_{x,x}(0)$ and by supposition we must have $f_{x,x}(0) = 0$, i.e. $\phi(\langle x,x\rangle) = 0$ for an arbitrary state ϕ. But then $\langle x,x\rangle = 0$, hence $x = 0$. So $\langle\cdot,\cdot\rangle_\beta$ is an inner product. The property $\langle Tx,Ty\rangle_\beta = \langle x,y\rangle_\beta$ is obvious, so T is unitary. Equivalence of $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\beta$ follows directly from the estimate

$$\frac{1}{C^2}\langle x,x\rangle \le \langle T^k x, T^k x\rangle \le C^2\langle x,x\rangle$$

being valid for all k. □

Acknowledgement. The author is grateful to M. Frank and to E. V. Troitsky for submitting copies of their preprints and fruitful discussions and to A. S. Mishchenko for helpful discussions. This paper was partially supported by Russian Foundation for Fundamental Research (grant N 96-01-00182-a) and DAAD.

References

1. Brown L. G., *Close hereditary C^*-subalgebras and the structure of quasimultipliers*, MSRI preprint # 11211-85, Purdue University, 1985.

2. Dupré M. J. and Fillmore P. A., *Triviality theorems for Hilbert modules*, Topics in modern operator theory, 5-th Internat. conf. on operator theory. Timisoara and Herculane (Romania), 1980, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1981, pp. 71–79.

3. Frank M., *Geometrical aspects of Hilbert C^*-modules*, Københavns Universitet preprint series 22/1993.

4. Irmatov A., *On a new topology in the space of Fredholm operators*, Ann. Global Anal. and Geom. **7** (1989), 93–106.

5. Kasparov G. G., *Hilbert modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133–150.

6. Manuilov V. M., *Representability of functionals on Hilbert C^*-modules*, Func. Anal. i Pril., (to appear).

7. Mishchenko A. S. and Fomenko A. T., *The index of elliptic operators over C^*-algebras*, Izv. Akad. Nauk SSSR. Ser. Mat. **43** (1979), 831–859. (in Russian)

8. Paschke W. L., *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

9. ———, *The double B-dual of an inner product module over a C^*-algebra*, Canad. J. Math. **26** (1974), 1272–1280.

10. Putnam I. F., *The invertible elements are dense in the irrational rotation C^*-algebras*, J. Reine Angew. Math. **410** (1990), 160–166.

11. Takesaki M., *Theory of operator algebras, 1*, Springer Verlag, New York-Heidelberg-Berlin.

12. Troitsky E. V., *Contractibility of the full general linear group of the C^*-Hilbert module $l_2(A)$*, Func. Anal. i Pril. **20**, N. 4 (1986), 58–64. (in Russian)

13. ———, *Some aspects of geometry of operators in Hilbert modules*, Ruhr-Universität Bochum, Preprint N. 173, 1994.

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