Acta Math. Univ. Comenianae
Vol. LXVI, 1(1997), pp. 117–134

REMARKS ON GERMS IN INFINITE DIMENSIONS

A. KRIEGL

Abstract. Smooth, real analytic and holomorphic mappings defined on non-open subsets of infinite dimensional vector spaces are treated.

0. Introduction

In this paper we will generalize the concept of differentiable maps $f\colon E \supseteq X \to F$ defined on open subsets to such on more general subsets of infinite dimensional vector spaces. We will refer to the theories for open domains as they have been developed in [**K82**], [**K83**] and [**F-K**] for smooth (i.e. C^∞) maps, in [**K-N**] for holomorphic maps and in [**K-M**] for real analytic maps.

But before we start the general discussion, let us recall the finite dimensional situation for smooth maps. Let first $E = F = \mathbb{R}$ and X be a non-trivial closed interval. Then a map $f\colon X \to \mathbb{R}$ is usually called smooth, if it is infinite often differentiable on the interior of X and the one-sided derivatives of all orders exist. The later condition is equivalent to the condition, that all derivatives extend continuously from the interior of X to X. Furthermore, by Whitney's extension theorem (see [**W34**]) these maps can also be described as being the restrictions to X of smooth maps on (some open neighborhood of X in) $\mathbb{R}$. In case where $X \subseteq \mathbb{R}$ is more general, these conditions fall apart.

Now what happens if one changes to $X \subseteq \mathbb{R}^n$. For closed convex sets with non-empty interior the corresponding conditions to the one dimensional situation still agree.

In case of holomorphic and real analytic maps the germ on such a subset is already defined by the values on the subset. Hence we are actually speaking about germs in this situation.

In infinite dimensions we will consider maps on just those convex subsets. So we do not claim greatest achievable generality, but rather restrict to a situation which is quite manageable. We will show that even in infinite dimensions the conditions above often coincide, and that real analytic and holomorphic maps on such sets are often germs of that class. Furthermore we have exponential laws for all three

Received February 1, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 26E15; Secondary 46G05.

classes, more precisely, the maps on a product correspond uniquely to maps from the first factor into the corresponding function space on the second.

1. Smooth Maps on Non-Open Domains

In this section we will discuss smooth maps $f\colon E \supseteq X \to F$, where E and F are convenient vector spaces, see [**F-K**], and X are certain not necessarily open subsets of E.

We will use the setting of [**F-K**]. There a map $f\colon E \supseteq X \to F$ from an arbitrary subset $X \subseteq E$ of a convenient vector space E to a convenient vector space F is smooth iff for all smooth curves $c\colon \mathbb{R} \to X \subseteq E$ the composite $f \circ c\colon \mathbb{R} \to F$ is a smooth curve. And it was shown that a curve $c\colon \mathbb{R} \to F$ is smooth iff for all $\ell \in E'$ the composite $\ell \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth. Furthermore it was shown, that in case where X is c^∞-open, i.e. the inverse image $c^{-1}(X) \subseteq \mathbb{R}$ is open for all smooth curves $c\colon \mathbb{R} \to F$, there exist smooth derivatives $f^{(n)}\colon X \to L^n(E;F)$ which satisfy the chain rule. Finally, cartesian closedness holds. More precisely there is a (unique) convenient vector space structure on $C^\infty(X_2, F)$ such that a map $f\colon X_1 \times X_2 \to F$ is smooth if and only if the corresponding map $\check{f}\colon X_1 \to C^\infty(X_2, F)$ is smooth.

1.1. Lemma. (Convex sets with non-void interior)
Let $K \subseteq E$ be a convex set with non-void c^∞-interior K^o. Then the segment $(x, y] := \{x + t(y-x) : 0 < t \le 1\}$ is contained in K^o for every $x \in K$ and $y \in K^o$. The interior K^o is convex and open even in the locally convex topology. And K is closed if and only if it is c^∞-closed.

Proof. Let $y_0 := x + t_0(y - x)$ be an arbitrary point on the segment $(x, y]$, i.e. $0 < t_0 \le 1$. Then $x + t_0(K^o - x)$ is an c^∞-open neighborhood of y_0, since homotheties are c^∞-continuous. It is contained in K, since K is convex.

In particular, the c^∞-interior K^o is convex, hence it is not only c^∞-open but open in the locally convex topology [**F-K**, 6.2.2].

Without loss of generality we now assume that $0 \in K^o$. We claim that the closure of K is the set $\{x : tx \in K^o \text{ for } 0 < t < 1\}$. This implies the statement on closedness. Let $U := K^o$ and consider the Minkowski-functional $q_U(x) := \inf\{t > 0 : x \in tU\}$. Since U is convex, the function q_U is convex, see [**J81**, 6.3.2]. Using that U is c^∞-open it can easily be shown that $U = \{x : q_U(x) < 1\}$. From [**F-K**, 6.4.2] we conclude that q_U is c^∞-continuous, and thus by [**F-K**, 6.4.3] even continuous for the locally convex topology. Hence the set $\{x : tx \in K^o \text{ for } 0 < t < 1\} = \{x : q_U(x) \le 1\} = \{x : q_K(x) \le 1\}$ is the closure of K in the locally convex topology by [**J81**, 6.4.2]. □

1.2. Theorem. (Derivative of smooth maps)
Let $K \subseteq E$ be a convex subset with non-void interior K^o, and let $f\colon K \to \mathbb{R}$ be a smooth map. Then $f|_{K^o}\colon K^o \to F$ is smooth, and its derivative $(f|_{K^o})'$ extends (uniquely) to a smooth map $K \to L(E, F)$.

Proof. Only the extension property is to be shown. Let us first try to find a candidate for $f'(x)(v)$ for $x \in K$ and $v \in E$ with $x + v \in K^o$. By convexity the smooth curve $c_{x,v} : t \mapsto x + t^2 v$ has for $0 < |t| < 1$ values in K^o and $c_{x,v}(0) = x \in K$, hence $f \circ c_{x,v}$ is smooth. In the special case where $x \in K^o$ we have by the chain rule that $(f \circ c_{x,v})'(t) = f'(x)(c_{x,v}(t))(c'_{x,v}(t))$, hence $(f \circ c_{x,v})''(t) = f''(c_{x,v}(t))(c'_{x,v}(t), c'_{x,v}(t)) + f'(c_{x,v}(t))(c''_{x,v}(t))$, and for $t = 0$ in particular $(f \circ c_{x,v})''(0) = 2\, f'(x)(v)$. Thus we define

$$2\, f'(x)(v) := (f \circ c_{x,v})''(0) \text{ for } x \in K \text{ and } v \in K^o - x.$$

Note that for $0 < \varepsilon < 1$ we have $f'(x)(\varepsilon\, v) = \varepsilon\, f'(x)(v)$, since $c_{x,\varepsilon\, v}(t) = c_{x,v}(\sqrt{\varepsilon}\, t)$.

Let us show next that $f'(\_)(v) : \{x \in K : x + v \in K^o\} \to \mathbb{R}$ is smooth. So let $s \mapsto x(s)$ be a smooth curve in K, and let $v \in K^o - x(0)$. Then $x(s) + v \in K^o$ for all sufficiently small s. And thus the map $(s,t) \mapsto c_{x(s),v}(t)$ is smooth from some neighborhood of $(0,0)$ into K. Hence $(s,t) \mapsto f(c_{x(s),v}(t))$ is smooth and also its second derivative $s \mapsto (f \circ c_{x(s),v})''(0) = 2\, f'(x(s))(v)$.

In particular, let $x_0 \in K$ and $v_0 \in K^o - x_0$ and $x(s) := x_0 + s^2 v_0$. Then

$$2f'(x_0)(v) := (f \circ c_{x_0,v})''(0) = \lim_{s\to 0}(f \circ c_{x(s),v})''(0) = \lim_{s\to 0} 2\, f'(x(s))(v),$$

with $x(s) \in K^o$ for $0 < |s| < 1$. Obviously this shows that the given definition of $f'(x_0)(v)$ is the only possible smooth extension of $f'(\_)(v)$ to $\{x_0\} \cup K^o$.

Now let $v \in E$ be arbitrary. Choose a $v_0 \in K^o - x_0$. Since the set $K^o - x_0 - v_0$ is a c^∞-open neighborhood of 0, hence absorbing, there exists some $\varepsilon > 0$ such that $v_0 + \varepsilon v \in K^o - x_0$. Thus

$$f'(x)(v) = \tfrac{1}{\varepsilon} f'(x)(\varepsilon v) = \tfrac{1}{\varepsilon}\big(f'(x)(v_0 + \varepsilon v) - f'(x)(v_0)\big)$$

for all $x \in K^o$. By what we have shown above the right side extends smoothly to $\{x_0\} \cup K^o$, hence the same is true for the left side. I.e. we define $f'(x_0)(v) := \lim_{s\to 0} f'(x(s))(v)$ for some smooth curve $x\colon (-1,1) \to K$ with $x(s) \in K^o$ for $0 < |s| < 1$. Then $f'(x)$ is linear as pointwise limit of $f'(x(s)) \in L(E,\mathbb{R})$ and is bounded by the Banach-Steinhaus theorem (applied to E_B). This shows at the same time, that the definition does not depend on the smooth curve x, since for $v \in x_0 + K^o$ it is the unique extension.

In order to show that $f'\colon K \to L(E,F)$ is smooth it is by [**F-K**, 3.6.5] enough to show that

$$s \mapsto f'(x(s))(v), \quad \mathbb{R} \xrightarrow{x} K \xrightarrow{f'} L(E,F) \xrightarrow{ev_v} F$$

is smooth for all $v \in E$ and all smooth curves $x\colon \mathbb{R} \to K$. For $v \in x_0 + K^o$ this was shown above. For general $v \in E$, this follows since $f'(x(s))(v)$ is a linear combination of $f'(x(s))(v_0)$ for two $v_0 \in x_0 + K^o$ not depending on s locally. $\square$

By (1.2) the following lemma applies in particular to smooth maps.

1.3. Lemma. (Chain rule)
Let $K \subseteq E$ be a convex subset with non-void interior K^o, let $f : K \to \mathbb{R}$ be smooth on K^o and let $f' : K \to L(E, F)$ be an extension of $(f|_{K^o})'$, which is continuous for the c^∞-topology of K, and let $c : \mathbb{R} \to K \subseteq E$ be a smooth curve. Then $(f \circ c)'(t) = f'(c(t))(c'(t))$.

Proof.
Claim. Let $g : K \to L(E, F)$ be continuous along smooth curves in K, then $\hat{g} : K \times E \to F$ is also continuous along smooth curves in $K \times E$.
In order to show this let $t \mapsto (x(t), v(t))$ be a smooth curve in $K \times E$. Then $g \circ x : \mathbb{R} \to L(E, F)$ is by assumption continuous (for the bornological topology on $L(E, F)$) and $v^* : L(E, F) \to C^\infty(\mathbb{R}, F)$ is bounded and linear [**F-K**, 4.4.8 and 4.4.1]. Hence the composite $v^* \circ g \circ x : \mathbb{R} \to C^\infty(\mathbb{R}, F) \to C(\mathbb{R}, F)$ is continuous. Thus $(v^* \circ g \circ x)\hat{}: \mathbb{R}^2 \to F$ is continuous, and in particular when restricted to the diagonal in $\mathbb{R}^2$. But this restriction is just $g \circ (x, v)$.

Now choose a $y \in K^o$. And let $c_s(t) := c(t) + s^2(y - c(t))$. Then $c_s(t) \in K^o$ for $0 < |s| \leq 1$ and $c_0 = c$. Furthermore $(s, t) \mapsto c_s(t)$ is smooth and $c_s'(t) = (1 - s^2)c'(t)$. And for $s \neq 0$

$$\frac{f(c_s(t)) - f(c_s(0))}{t} = \int_0^1 (f \circ c_s)'(t\tau)\, d\tau = (1 - s^2) \int_0^1 f'(c_s(t\tau))(c'(t\tau))\, d\tau\ .$$

Now consider the specific case where $c(t) := x + tv$ with $x,\ x + v \in K$. Since f is continuous along $(t, s) \mapsto c_s(t)$, the left side of the above equation converges to $\frac{f(c(t)) - f(c(0))}{t}$ for $s \to 0$. And since $f'(\cdot)(v)$ is continuous along $(t, \tau, s) \mapsto c_s(t\tau)$ we have that $f'(c_s(t\tau))(v)$ converges to $f'(c(t\tau))(v)$ uniformly with respect to $0 \leq \tau \leq 1$ for $s \to 0$. Thus the right side of the above equation converges to $\int_0^1 f'(c(t\tau))(v)\, d\tau$. Hence we have

$$\frac{f(c(t)) - f(c(0))}{t} = \int_0^1 f'(c(t\tau))(v)\, d\tau \to \int_0^1 f'(c(0))(v)\, d\tau = f'(c(0))(c'(0))$$

for $t \to 0$.

Now let $c : \mathbb{R} \to K$ be an arbitrary smooth curve. Then $(s, t) \mapsto c(0) + s(c(t) - c(0))$ is smooth and has values in K for $0 \leq s \leq 1$. By the above consideration we have for $x = c(0)$ and $v = (c(t) - c(0))/t$ that $\frac{f(c(t)) - f(c(0))}{t} = \int_0^1 f'(c(0) + \tau(c(t) - c(0)))(\frac{c(t) - c(0)}{t})$ which converges to $f'(c(0))(c'(0))$ for $t \to 0$, since f' is continuous along smooth curves in K and thus $f'(c(0) + \tau(c(t) - c(0))) \to f'(c(0))$ uniformly on the bounded set $\{\frac{c(t) - c(0)}{t} : t \text{ near } 0\}$. Thus $f \circ c$ is differentiable with derivative $(f \circ c)'(t) = f'(c(t))(c'(t))$. □

Since f' can be considered as a map $df : E \times E \supseteq K \times E \to F$ it is important to study sets $A \times B \subseteq E \times F$. Clearly $A \times B$ is convex provided $A \subseteq E$ and $B \subseteq F$

are. Remains to consider the openness condition. In the locally convex topology $(A \times B)^o = A^o \times B^o$, which would be enough to know in our situation. However we are also interested in the corresponding statement for the c^∞-topology. This topology on $E \times F$ is in general not the product topology $c^\infty E \times c^\infty F$. Thus we cannot conclude that $A \times B$ has non-void interior with respect to the c^∞-topology on $E \times F$, even if $A \subseteq E$ and $B \subseteq F$ have it. However in case where $B = F$ everything is fine.

1.4. Lemma. (Interior of a product)
Let $X \subseteq E$. Then the interior $(X \times F)^o$ of $X \times F$ with respect to the c^∞-topology on $E \times F$ is just $X^o \times F$, where X^o denotes the interior of X with respect to the c^∞-topology on E.

Proof. Let W be the saturated hull of $(X \times F)^o$ with respect to the projection $\mathrm{pr}_1 : E \times F \to E$, i.e. the c^∞-open set $(X \times F)^o + \{0\} \times F \subseteq X \times F$. Its projection to E is c^∞-open, since it agrees with the intersection with $E \times \{0\}$. Hence it is contained in X^o, and $(X \times F)^o \subseteq X^o \times F$. The converse inclusion is obvious since pr_1 is continuous. □

1.5. Theorem. (Smooth maps on convex sets)
Let $K \subseteq E$ be a convex subset with non-void interior K^o, and let $f : K \to F$ be a map. Then f is smooth if and only if f is smooth on K^o and all derivatives $(f|_{K^o})^{(n)}$ extend continuously to K with respect to the c^∞-topology of K.

Proof.
($\Rightarrow$) It follows by induction using (1.2) that $f^{(n)}$ has a smooth extension $K \to L^n(E; F)$.
($\Leftarrow$) By (1.3) we conclude that for every $c : \mathbb{R} \to K$ the composite $f \circ c : \mathbb{R} \to F$ is differentiable with derivative $(f \circ c)'(t) = f'(c(t))(c'(t)) =: df(c(t), c'(t))$.
The map df is smooth on the interior $K^o \times E$, linear in the second variable, and its derivatives $(df)^{(p)}(x, w)(y_1, w_1; \dots, y_p, w_p)$ are universal linear combinations of

$$f^{(p+1)}(x)(y_1, \dots, y_p; w) \text{ and of } f^{(k+1)}(x)(y_{i_1}, \dots, y_{i_k}; w_{i_0}) \text{ for } k \le p.$$

These summands have unique extensions to $K \times E$. The first one is continuous along smooth curves in $K \times E$, because for such a curve $(t \mapsto (x(t), w(t))$ the extension $f^{(k+1)} : K \to L(E^k, L(E, F))$ is continuous along the smooth curve x, and $w^* : L(E, F) \to C^\infty(\mathbb{R}, F)$ is continuous and linear, so the map $t \mapsto (s \mapsto f^{(k+1)}(x(t))(y_{i_1}, \dots, y_{i_k}; w(s)))$ is continuous from $\mathbb{R} \to C^\infty(\mathbb{R}, F)$ and thus as map from $\mathbb{R}^2 \to F$ it is continuous, and in particular if restricted to the diagonal. And the other summands only depend on x, hence have a continuous extension by assumption.
So we can apply (1.3) inductively using (1.4), to conclude that $f \circ c : \mathbb{R} \to F$ is smooth. □

In view of the preceding theorem (1.5) it is important to know the c^∞-topology $c^\infty X$ of X, i.e. the final topology generated by all the smooth curves $c\colon \mathbb{R} \to X \subseteq E$. So the first question is whether this is the trace topology $c^\infty E|_X$ of the c^∞-topology of E.

1.6. Lemma. (The c^∞-topology is the trace topology)
In the following cases of subsets $X \subseteq E$ the trace topology $c^\infty E|X$ equals the topology $c^\infty X$:

(1) *X is $c^\infty E$-open.*
(2) *X is convex and locally c^∞-closed.*
(3) *The topology $c^\infty E$ is sequential and $X \subseteq E$ is convex and has non-void interior.*

(3) applies in particular to the case where E is metrizable, see [**F-K**, 6.1.4]. A topology is called sequential iff the closure of any subset equals its adherence, i.e. the set of all accumulation points of sequences in it. By [**F-K**, 2.3.10] the adherence of a set X with respect to the c^∞-topology, is formed by the limits of all Mackey-converging sequences in X.

Proof. Remark that the inclusion $X \to E$ is by definition smooth in the sense of [**F-K**], hence the identity $c^\infty X \to c^\infty E|_X$ is always continuous.

(1) Let $U \subseteq X$ be $c^\infty X$-open and let $c\colon \mathbb{R} \to E$ be a smooth curve with $c(0) \in U$. Since X is $c^\infty E$-open, $c(t) \in X$ for all small t. By composing with a smooth map $h\colon \mathbb{R} \to \mathbb{R}$ which satisfies $h(t) = t$ for all small t, we obtain a smooth curve $c \circ h\colon \mathbb{R} \to X$, which coincides with c locally around 0. Since U is $c^\infty X$-open we conclude that $c(t) = (c \circ h)(t) \in U$ for small t. Thus U is $c^\infty E$-open.

(2) Let $A \subseteq X$ be $c^\infty X$-closed. And let $\bar{A}$ be the $c^\infty E$-closure of A. We have to show that $\bar{A} \cap X \subseteq A$. So let $x \in \bar{A} \cap X$. Since X is locally $c^\infty E$-closed, there exists a $c^\infty E$-neighborhood U of $x \in X$ with $U \cap X$ c^∞-closed in U. For every $c^\infty E$-neighborhood U of x we have that x is in the closure of $A \cap U$ in U with respect to the $c^\infty E$-topology (otherwise some open neighborhood of x in U does not meet $A \cap U$, hence also not A). Let $a_n \in A \cap U$ be Mackey converging to $a \in U$. Then $a_n \in X \cap U$ which is closed in U thus $a \in X$. Since X is convex the infinite polygon through the a_n lies in X and can be smoothly parameterized by the special curve lemma [**F-K**, 2.3.4]. Using that A is $c^\infty X$-closed, we conclude that $a \in A$. Thus $A \cap U$ is $c^\infty U$-closed and $x \in A$.

(3) Let $A \subseteq X$ be $c^\infty X$-closed. And let $\bar{A}$ denote the closure of A in $c^\infty E$. We have to show that $\bar{A} \cap X \subseteq A$. So let $x \in \bar{A} \cap X$. Since $c^\infty E$ is sequential there is a Mackey converging sequence $A \ni a_n \to x$. By the special curve lemma [**F-K**, 2.3.4] the infinite polygon through the a_n can be smoothly parameterized. Since X is convex this curve gives a smooth curve $c\colon \mathbb{R} \to X$ and thus $c(0) = x \in A$, since A is $c^\infty X$-closed. $\square$

1.7. Example. (The c^∞-topology is not trace topology)
Let $A \subseteq E$ be such that the c^∞-adherence $Adh(A)$ of A is not the whole c^∞-closure $\bar{A}$ of A. So let $a \in \bar{A} \setminus Adh(A)$. Then consider the convex subset $K \subseteq E \times \mathbb{R}$ defined by $K := \{(x,t) \in E \times \mathbb{R} : t \geq 0$ and $(t = 0 \Rightarrow x \in A \cup \{a\})\}$ which has non-empty interior $E \times \mathbb{R}^+$. However the topology $c^\infty K$ is not the trace topology of $c^\infty(E \times \mathbb{R})$ which equals $c^\infty(E) \times \mathbb{R}$ by [**F-K**, 3.3.4].

Remark that this situation occurs quite often, see [**F-K**, 6.1.6] and [**F-K**, 6.3.3] where A is even a linear subspace.

Proof. Consider $A = A \times \{0\} \subseteq K$. This set is closed in $c^\infty K$, since $E \cap K$ is closed in $c^\infty K$ and the only point in $(K \cap E) \setminus A$ is a, which cannot be reached by a Mackey converging sequence in A, since $a \notin Adh(A)$.

It is however not the trace of a closed subset in $c^\infty(E) \times \mathbb{R}$, since such a set has to contain A and hence $\bar{A} \ni a$. $\square$

1.8. Theorem. (Smooth maps on subsets with collar)
Let $M \subseteq E$ have a smooth collar, i.e. the boundary ∂M of M is a smooth submanifold of E and there exists a neighborhood U of ∂M and a diffeomorphism $\psi \colon \partial M \times \mathbb{R} \to U$ which is the identity on ∂M and such that $\psi(M \times \{t \in \mathbb{R} : t \geq 0\}) = M \cap U$. Then every smooth map $f \colon M \to F$ extends to a smooth map $\tilde{f} \colon M \cup U \to F$.

Proof. Due to [**S64**] (see [**F-K**, 7.1.4] for a reformulation in this setting) there is a continuous linear right inverse S to the restriction map $C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(I, \mathbb{R})$, where $I := \{t \in \mathbb{R} : t \geq 0\}$. Now let $x \in U$ and $(p_x, t_x) := \psi^{-1}(x)$. Then $f(\psi(p_x, \cdot)) \colon I \to F$ is smooth, since $\psi(p_x, t) \in M$ for $t \geq 0$. Thus we have a smooth map $S(f(\psi(p_x, \cdot))) \colon \mathbb{R} \to F$ and we define $\tilde{f}(x) := S(f(\psi(p_x, \cdot)))(t_x)$. Then $\tilde{f}(x) = f(x)$ for all $x \in M \cap U$, since for such an x we have $t_x \geq 0$. Now we extend the definition by $\tilde{f}(x) = f(x)$ for $x \in M^o$. Remains to show that $\tilde{f}$ is smooth (on U). So let $s \mapsto x(s)$ be a smooth curve in U. Then $s \mapsto (p_s, t_s) := \psi^{-1}(x(s))$ is smooth. Hence $s \mapsto (t \mapsto f(\psi(p_s, t))$ is a smooth curve $\mathbb{R} \to C^\infty(I, F)$. Since S is continuous and linear the composite $s \mapsto (t \mapsto S(f\psi(p_s, \cdot))(t))$ is a smooth curve $\mathbb{R} \to C^\infty(\mathbb{R}, F)$ and thus the associated map $\mathbb{R}^2 \to F$ is smooth, and also the composite $\tilde{f}(x_s)$ of it with $s \mapsto (s, t_s)$. $\square$

In particular the previous theorem applies to the following convex sets:

1.9. Proposition. (Convex sets with smooth boundary have a collar)
Let $K \subseteq E$ be a closed convex subset with non-empty interior and smooth boundary ∂K. Then K has a smooth collar as defined in (1.8).

Proof. Without loss of generality let $0 \in K^o$.

In order to show that the set $U := \{x \in E : tx \notin K$ for some $t > 0\}$ is c^∞-open let $s \mapsto x(s)$ be a smooth curve $\mathbb{R} \to E$ and assume that $t_0 x(0) \notin K$ for some $t_0 > 0$. Since K is closed we have that $t_0 x(s) \notin K$ for all small $|s|$.

For $x \in U$ let $r(x) := \sup\{t \geq 0 : tx \in K^o\} > 0$, i.e. $r = \frac{1}{q_{K^o}}$ as defined in the proof of (1.1) and $r(x)x$ is the unique intersection point of $\partial K \cap (0, +\infty)x$. We claim that $r\colon U \to \mathbb{R}^+$ is smooth. So let $s \mapsto x(s)$ be a smooth curve in U and $x_0 := r(x(0))x(0) \in \partial K$. Choose a local diffeomorphism $\psi\colon (E, x_0) \to (E, 0)$ which maps ∂K locally to some closed hyperplane $F \subseteq E$. Any such hyperplane is the kernel of a continuous linear functional $\ell\colon E \to \mathbb{R}$, hence $E \cong F \times \mathbb{R}$.

We claim that $v := \psi'(x_0)(x_0) \notin F$. If this were not the case, then we consider the smooth curve $c\colon \mathbb{R} \to \partial K$ defined by $c(t) = \psi^{-1}(-tv)$. Since $\psi'(x_0)$ is injective its derivative is $c'(0) = -x_0$ and $c(0) = x_0$. Since $0 \in K^o$, we have that $x_0 + \frac{c(t)-c(0)}{t} \in K^o$ for all small $|t|$. By convexity $c(t) = x_0 + t\frac{c(t)-c(0)}{t} \in K^o$ for small $t > 0$, a contradiction.

So we may assume that $\ell(\psi'(x)(x)) \neq 0$ for all x in a neighborhood of x_0.

For s close enough to 0 we have that $r(x(s))$ is given by the implicit equation $\ell(\psi(r(x(s))x(s))) = 0$. So let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be the locally defined smooth map $g(t,s) := \ell(\psi(tx(s)))$. For $t \neq 0$ its first partial derivative is $\partial_1 g(t,s) = \ell(\psi'(tx(s))(x(s))) \neq 0$. So by the classical implicit function theorem the solution $s \mapsto r(x(s))$ is smooth.

Now let $\Psi\colon U \times \mathbb{R} \to U$ be the smooth map defined by $(x,t) \mapsto e^{-t}r(x)x$. Restricted to $\partial K \times \mathbb{R} \to U$ is injective, since $tx = t'x'$ with x, $x' \in \partial K$ and t, $t' > 0$ implies $x = x'$ and hence $t = t'$. Furthermore it is surjective, since the inverse mapping is given by $x \mapsto (r(x)x, \ln(r(x)))$. Use that $r(\lambda x) = \frac{1}{\lambda}r(x)$. Since this inverse is also smooth, we have the required diffeomorphism Ψ. In fact $\Psi(x,t) \in K$ iff $e^{-t}r(x) \leq r(x)$, i.e. $t \leq 0$. □

2. Real Analytic Maps on Non-Open Domains

In this section we will consider real analytic mappings defined on the same type of convex subsets as in the previous section. Here we will use the cartesian closed setting of [**K-M**] for real analytic maps defined on open subsets.

2.1. Theorem. (Power series in Fréchet spaces)
Let E be a Fréchet space and (F, F') be a dual pair. Assume that a Baire vector space topology on E' exists for which the point evaluations are continuous. Let f_k be k-linear symmetric bounded functionals from E to F, for each $k \in \mathbb{N}$. Assume that for every $\ell \in F'$ and every x in some open subset $W \subseteq E$ the power series $\sum_{k=0}^{\infty} \ell(f_k(x^k))t^k$ has positive radius of convergence. Then there exists a 0-neighborhood U in E, such that $\{f_k(x_1, \dots, x_k) : k \in \mathbb{N}, x_j \in U\}$ is bounded and thus the power series $x \mapsto \sum_{k=0}^{\infty} f_k(x^k)$ converges Mackey on some 0-neighborhood in E.

Proof. Choose a fixed but arbitrary $\ell \in F'$. Then $\ell \circ f_k$ satisfy the assumptions of [**K-M**, 2.2.1] for an absorbing subset in a closed cone C with non-empty interior. Since this cone is also complete metrizable we can proceed with the proof as in

[**K-M**, 2.2] to obtain a set $A_{K,r} \subseteq C$ whose interior in C is non-void. But this interior has to contain a non-void open set of E and as in the proof of [**K-M**, 2.2] there exists some $\rho_\ell > 0$ such that for the ball U_{ρ_ℓ} in E with radius ρ_ℓ and center 0 the set $\{\ell(f_k(x_1, \dots, x_k)) : k \in \mathbb{N}, x_j \in U_{\rho_\ell}\}$ is bounded.

Now let similarly to [**K-M**, 1.5]

$$A_{K,r,\rho} := \bigcap_{k \in \mathbb{N}} \bigcap_{x_1,\dots x_n \in U_\rho} \{\ell \in F' : |\ell(f_k(x_1, \dots, x_k))| \le K r^k\}$$

for $K, r, \rho > 0$. These sets $A_{K,r,\rho}$ are closed in the Baire topology, since evaluation at $f_k(x_1, \dots, x_k)$ is assumed to be continuous.

By the first part of the proof the union of these sets is F'. So by the Baire property, there exist $K, r, \rho > 0$ such that the interior U of $A_{K,r,\rho}$ is non-empty. As in the proof of [**K-M**, 1.5] we choose an $\ell_0 \in U$. Then for every $\ell \in F'$ there exists some $\varepsilon > 0$ such that $\ell_\varepsilon := \varepsilon \ell \in U - \ell_0$. So $|\ell(y)| \le \frac{1}{\varepsilon}(|\ell_\varepsilon(y) + \ell_0(y)| + |\ell_0(y)|) \le \frac{2}{\varepsilon} K r^n$ for every $y = f_k(x_1, \dots, x_k)$ with $x_i \in U_\rho$. Thus $\{f_k(x_1, \dots, x_k) : k \in \mathbb{N}, x_i \in U_{\frac{\rho}{r}}\}$ is bounded.

On every smaller ball we have therefore that the power series with terms f_k converges Mackey. $\square$

Remark that if the vector spaces are real and the assumption above hold, then the conclusion is even true for the complexified terms by [**K-M**, 2.2].

2.2. Theorem. (Real analytic maps $I \to \mathbb{R}$ are germs)
Let $f\colon I := \{t \in \mathbb{R} : t \ge 0\} \to \mathbb{R}$ be a map. Suppose $t \mapsto f(t^2)$ is real analytic $\mathbb{R} \to \mathbb{R}$. Then f extends to a real analytic map $\tilde{f}\colon \tilde{I} \to \mathbb{R}$, where $\tilde{I}$ is an open neighborhood of I in $\mathbb{R}$.

Proof. We show first that f is smooth. Consider $g(t) := f(t^2)$. Since $g\colon \mathbb{R} \to \mathbb{R}$ is assumed to be real analytic it is smooth and clearly even. We claim that there exists a smooth map $h\colon \mathbb{R} \to \mathbb{R}$ with $g(t) = h(t^2)$ (This is due to [**W43**]). In fact by $h(t^2) := g(t)$ a continuous map $h : \{t :\in \mathbb{R} : t \ge 0\} \to \mathbb{R}$ is uniquely determined. Obviously $h|_{\{t \in \mathbb{R} : t > 0\}}$ is smooth. Differentiating for $t \ne 0$ the defining equation gives $h'(t^2) = \frac{g'(t)}{2t} =: g_1(t)$. Since g is smooth and even, g' is smooth and odd, so $g'(0) = 0$. Thus

$$t \mapsto g_1(t) = \frac{g'(t) - g'(0)}{2t} = \frac{1}{2} \int_0^1 g''(ts)\, ds$$

is smooth. Hence we may define h' on $\{t \in \mathbb{R} : t \ge 0\}$ by the equation $h'(t^2) = g_1(t)$ with even smooth g_1. By induction we obtain continuous extensions of $h^{(n)} : \{t \in \mathbb{R} : t > 0\} \to \mathbb{R}$ to $\{t \in \mathbb{R} : t \ge 0\}$, and hence h is smooth on $\{t \in \mathbb{R} : t \ge 0\}$ and so can be extended to a smooth map $h\colon \mathbb{R} \to \mathbb{R}$.

From this we get $f(t^2) = g(t) = h(t^2)$ for all t. Thus $h: \mathbb{R} \to \mathbb{R}$ is a smooth extension of f.

Composing with the exponential map $\exp : \mathbb{R} \to \mathbb{R}^+$ shows that f is real analytic on $\{t : t > 0\}$, and has derivatives $f^{(n)}$ which extend by (1.5) continuously to maps $I \to \mathbb{R}$. It is enough to show that $a_n := \frac{1}{n!} f^{(n)}(0)$ are the coefficients of a power series p with positive radius of convergence and for $t \in I$ this map p coincides with f.

Claim. We show that a smooth map $f: I \to \mathbb{R}$, which has a real analytic composite with $t \mapsto t^2$, is the germ of a real analytic mapping.
Consider the real analytic curve $c: \mathbb{R} \to I$ defined by $c(t) = t^2$. Thus $f \circ c$ is real analytic. By the chain rule the derivative $(f \circ c)^{(p)}(t)$ is for $t \neq 0$ a universal linear combination of terms $f^{(k)}(c(t))c^{(p_1)}(t) \cdots c^{(p_k)}(t)$, where $1 \leq k \leq p$ and $p_1 + \ldots + p_k = p$. Taking the limit for $t \to 0$ and using that $c^{(n)}(0) = 0$ for all $n \neq 2$ and $c''(0) = 2$ shows that there is a universal constant c_p satisfying $(f \circ c)^{(2p)}(0) = c_p \cdot f^{(p)}(0)$. Take as $f(x) = x^p$ to conclude that $(2p)! = c_p \cdot p!$. Now we use [**K-M**, 1.3.3] to show that the power series $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)t^k$ converges locally. So choose a sequence (r_k) with $r_k t^k \to 0$ for all $t > 0$. Define a sequence $(\bar{r}_k)$ by $\bar{r}_{2n} = \bar{r}_{2n+1} := r_n$ and let $\bar{t} > 0$. Then $\bar{r}_k \bar{t}^k = r_n t^n$ for $2n = k$ and $\bar{r}_k \bar{t}^k = r_n t^n \bar{t}$ for $2n + 1 = k$, where $t := \bar{t}^2 > 0$, hence $(\bar{r}_k)$ satisfies the same assumptions as (r_k) and thus by [**K-M**, 1.4($1 \Rightarrow 3$)] the sequence $\frac{1}{k!}(f \circ c)^{(k)}(0)\bar{r}_k$ is bounded. In particular this is true for the subsequence

$$\tfrac{1}{(2p)!}(f \circ c)^{(2p)}(0)\bar{r}_{2p} = \tfrac{c_p}{(2p)!} f^{(p)}(0) r_p = \tfrac{1}{p!} f^{(p)}(0) r_p.$$

Thus by [**K-M**, 1.4($1 \Leftarrow 3$)] the power series with coefficients $\frac{1}{p!} f^{(p)}(0)$ converges locally to a real analytic function $\tilde{f}$.

Remains to show that $p = f$ on J. But since $p \circ c$ and $f \circ c$ are both real analytic near 0, and have the same Taylor series at 0, they have to coincide locally, i.e. $p(t^2) = f(t^2)$ for small t. □

Remark however that the more straight forward attempt of a proof of the first step, namely to show that $f \circ c$ is smooth for all $c: \mathbb{R} \to \{t \in \mathbb{R} : t \geq 0\}$ by showing that for such c there is a smooth map $h: \mathbb{R} \to \mathbb{R}$, satisfying $c(t) = h(t)^2$, is doomed to fail as the following example shows.

2.3. Example. (A smooth function without smooth square root)
Let $c: \mathbb{R} \to \{t \in \mathbb{R} : t \geq 0\}$ be defined by the general curve lemma [**F-K**, 4.2.5] *using pieces of parabolas $c_n : t \mapsto \frac{2n}{2^n} t^2 + \frac{1}{4^n}$. Then there is no smooth square root of c.*

Proof. The curve c constructed in [**F-K**, 4.2.5] has the property that there exists a converging sequence t_n such that $c(t + t_n) = c_n(t)$ for small t. Assume there were a smooth map $h\colon \mathbb{R} \to \mathbb{R}$ satisfying $c(t) = h(t)^2$ for all t. At points where $c(t) \neq 0$ we have in turn:

$$\begin{aligned} c'(t) &= 2h(t)h'(t) \\ c''(t) &= 2h(t)h''(t) + 2h'(t)^2 \\ 2c(t)c''(t) &= 4h(t)^3h''(t) + c'(t)^2. \end{aligned}$$

Choosing t_n for t in the last equation gives $h''(t_n) = 2n$, which is unbounded in n. Thus h cannot be C^2. □

2.4. Definition. (Real analytic maps $I \to F$)
Let $I \subseteq \mathbb{R}$ be a non-trivial interval. Then a map $f : I \to F$ is called real analytic iff the composites $\ell \circ f \circ c\colon \mathbb{R} \to \mathbb{R}$ are real analytic for all real analytic $c\colon \mathbb{R} \to I \subseteq \mathbb{R}$ and all $\ell \in F'$. If I is an open interval then this definition coincides with [**K-M**, 1.2, 2.6].

2.5. Lemma. (Bornological description of real analyticity)
Let $I \subseteq \mathbb{R}$ be a compact interval. A curve $c\colon I \to E$ is real analytic if and only if c is smooth and the set $\{\frac{1}{k!}\, c^{(k)}(a)\, r_k : a \in I, k \in \mathbb{N}\}$ is bounded for all sequences (r_k) with $r_k\, t^k \to 0$ for all $t > 0$.

Proof. We use [**K-M**, 1.5]. Since both sides can be tested with $\ell \in E'$ we may assume that $E = \mathbb{R}$.

($\Rightarrow$) By (2.2) we may assume that $c\colon \tilde{I} \to \mathbb{R}$ is real analytic for some open neighborhood $\tilde{I}$ of I. Thus the required boundedness condition follows from [**K-M**, 1.5].

($\Leftarrow$) By (2.2) we only have to show that $f : t \mapsto c(t^2)$ is real analytic. For this we use again [**K-M**, 1.5]. So let $K \subseteq \mathbb{R}$ be compact. Then the Taylor series of f is obtained by that of c composed with t^2. Thus the composite f satisfies the required boundedness condition, and hence is real analytic. □

This characterization of real analyticity can not be weakened by assuming the boundedness conditions only for single pointed K as the map $c(t) := e^{-\frac{1}{t^2}}$ for $t \neq 0$ and $c(0) = 0$ shows. It is real analytic on $\mathbb{R} \setminus \{0\}$ thus the condition is satisfied at all points there, and at 0 the power series has all coefficients equal to 0, hence the condition is satisfied there as well.

2.6. Corollary. (Real analytic maps into inductive limits)
Let $T_\alpha\colon E \to E_\alpha$ be a family of bounded linear maps that generates the bornology on E. Then a map $c\colon I \to F$ is real analytic if and only if all the composites $T_\alpha \circ c\colon I \to F_\alpha$ are real analytic.

Proof. This follows either directly from (2.5) or from (2.2) by using the corresponding statement for maps $\mathbb{R} \to E$, see [**K-M**, 1.11]. □

2.7. Definition. (Real analytic maps $K \to F$)
For an arbitrary subset $K \subseteq E$ let us call a map $f\colon E \supseteq K \to F$ real analytic iff $\lambda \circ f \circ c\colon I \to \mathbb{R}$ is a real analytic (resp. smooth) for all $\lambda \in F'$ and all real analytic (resp. smooth) maps $c\colon I \to K$, where $I \subset \mathbb{R}$ is some compact non-trivial interval. Remark however that it is enough to use all real analytic (resp. smooth) curves $c\colon \mathbb{R} \to K$ by (2.2).

With $C^\omega(K,F)$ we denote the vector space of all real analytic maps $K \to F$. And we topologize this space with the initial structure induced by the cone $c^*\colon C^\omega(K,F) \to C^\omega(\mathbb{R},F)$ (for all real analytic $c\colon \mathbb{R} \to K$) together with the cone $c^*\colon C^\omega(K,F) \to C^\infty(\mathbb{R},F)$ (for all smooth $c\colon \mathbb{R} \to K$). The space $C^\omega(\mathbb{R},F)$ should carry the structure of [**K-M, 5.4**] and the space $C^\infty(\mathbb{R},F)$ that of [**F-K**].

For an open $K \subseteq E$ the definition for $C^\omega(K,F)$ given here coincides with that of [**K-M**, 2.6 and 5.4].

2.8. Proposition. ($C^\omega(K,F)$ is convenient)
Let $K \subseteq E$ and F be arbitrary. Then the space $C^\omega(K,F)$ is a convenient vector space and satisfies the $\mathcal{S}$-uniform boundedness principle (see [**K-M**, 4.1]*), where $\mathcal{S} := \{ev_x : x \in K\}$.*

Proof. Since both spaces $C^\omega(\mathbb{R},\mathbb{R})$ and $C^\infty(\mathbb{R},\mathbb{R})$ are c^∞-complete and satisfy the uniform boundedness principle for the set of point evaluations the same is true for $C^\omega(K,F)$, by the usual arguments, cf. [**K-M**, 5.5 and 5.6]. $\square$

2.9. Theorem. (Real analytic maps $K \to F$ are often germs)
Let $K \subseteq E$ be a convex subset with non-empty interior of a Fréchet space and let (F,F') be a complete dual pair for which a Baire topology on F' exists, as required in (2.1)*. Let $f\colon K \to F$ be a real analytic map. Then there exists an open neighborhood $U \subseteq E_\mathbb{C}$ of K and a holomorphic map $\tilde f\colon U \to F_\mathbb{C}$ such that $\tilde f|_K = f$.*

Proof. By (1.5) the map $f\colon K \to F$ is smooth, i.e. the derivatives $f^{(k)}$ exist on the interior K^o and extend continuously (with respect to the c^∞-topology of K) to the whole of K. So let $x \in K$ be arbitrary and consider the power series with coefficients $f_k = \frac{1}{k!}f^{(k)}(x)$. This power series has the required properties of (2.1), since for every $\ell \in F'$ and $v \in K^o - x$ the series $\sum_k \ell(f_k(v^k))t^k$ has positive radius of convergence. In fact $\ell(f(x+tv))$ is by assumption a real analytic germ $I \to \mathbb{R}$, by (1.8) hence locally around any point in I it is represented by its converging Taylor series at that point. Since $(x, v-x] \subseteq K^o$ and f is smooth on this set, $(\frac{d}{dt})^k(\ell(f(x+tv)) = \ell(f^{(k)}(x+tv)(v^k)$ for $t > 0$. Now take the limit for $t \to 0$ to conclude that the Taylor coefficients of $t \mapsto \ell(f(x+tv))$ at $t = 0$ are exactly $k!\ell(f_k)$. Thus by (2.1) the power series converges locally and hence represents a holomorphic map in a neighborhood of x. Let $y \in K^o$ be an arbitrary point in this neighborhood. Then $t \mapsto \ell(f(x+t(y-x)))$ is real analytic $I \to \mathbb{R}$ and hence the series converges at $y - x$ towards $f(y)$. So the restriction of the power series to the interior of K coincides with f.

We have to show that the extensions f_x of $f : K \cap \tilde{U}_x \to F_{\mathbb{C}}$ to star shaped neighborhoods $\tilde{U}_x$ of x in $E_{\mathbb{C}}$ fit together to give an extension $\tilde{f} : \tilde{U} \to F_{\mathbb{C}}$. So let $\tilde{U}_x$ be such a domain for the extension and let $U_x := \tilde{U}_x \cap E$.

For this we claim that we may assume that U_x has the following additional property: $y \in U_x \Rightarrow [0,1]y \subseteq K^o \cup U_x$. In fact let $U_0 := \{y \in U_x : [0,1]y \subseteq K^o \cup U_x\}$. Then U_0 is open, since $f : (t,s) \mapsto ty(s)$ being smooth, and $f(t,0) \in K^o \cup U_x$ for $t \in [0,1]$, implies that a $\delta > 0$ exists such that $f(t,s) \in K^o \cup U_x$ for all $|s| < \delta$ and $-\delta < t < 1 + \delta$. The set U_0 is star shaped, since $y \in U_0$ and $s \in [0,1]$ implies that $t(x + s(y - x)) \in [x, t'y]$ for some $t' \in [0,1]$, hence lies in $K^o \cup U_x$. The set U_0 contains x, since $[0,1]x = \{x\} \cup [0,1)x \subseteq \{x\} \cup K^o$. Finally U_0 has the required property, since $z \in [0,1]y$ for $y \in U_0$ implies that $[0,1]z \subseteq [0,1]y \subseteq K^o \cup U_x$, i.e. $z \in U_0$.

Furthermore, we may assume that for $x + iy \in \tilde{U}_x$ and $t \in [0,1]$ also $x + ity \in \tilde{U}_x$ (replace $\tilde{U}_x$ by $\{x + iy : x + ity \in \tilde{U}_x$ for all $t \in [0,1]\}$).

Now let $\tilde{U}_1$ and $\tilde{U}_2$ be two such domains around x_1 and x_2, with corresponding extensions f_1 and f_2. Let $x + iy \in \tilde{U}_1 \cap \tilde{U}_2$. Then $x \in U_1 \cap U_2$ and $[0,1]x \subseteq K^o \cup U_i$ for $i = 1, 2$. If $x \in K^o$ we are done, so let $x \notin K^o$. Let $t_0 := \inf\{t > 0 : tx \notin K^o\}$. Then $t_0 x \in U_i$ for $i = 1, 2$ and by taking t_0 a little smaller we may assume that $x_0 := t_0 x \in K^o \cap U_1 \cap U_2$. Thus $f_i = f$ on $[x_0, x_i]$ and the f_i are real analytic on $[x_0, x]$ for $i = 1, 2$. Hence $f_1 = f_2$ on $[x_0, x]$ and thus $f_1 = f_2$ on $[x, x + iy]$ by the 1-dimensional uniqueness theorem. □

That the result corresponding to (1.8) is not true for manifolds with real analytic boundary shows the following

2.10. Example. (No real analytic extension exists)
Let $I := \{t \in \mathbb{R} : t \geq 0\}$, $E := C^\omega(I, \mathbb{R})$, and let $ev : E \times \mathbb{R} \supseteq E \times I \to \mathbb{R}$ be the real analytic map $(f, t) \mapsto f(t)$. Then there is no real analytic extension of ev to a neighborhood of $E \times I$.

Proof. Suppose there is some open set $U \subseteq E \times \mathbb{R}$ containing $\{(0,t) : t \geq 0\}$ and a C^ω-extension $\varphi : U \to \mathbb{R}$. Then there exists a c^∞-open neighborhood V of 0 and some $\delta > 0$ such that U contains $V \times (-\delta, \delta)$. Since V is absorbing in E, we have for every $f \in E$ that there exists some $\varepsilon > 0$ such that $\varepsilon f \in V$ and hence $\frac{1}{\varepsilon}\varphi(\varepsilon f, \cdot) : (-\delta, \delta) \to \mathbb{R}$ is a real analytic extension of f. This cannot be true, since there are $f \in E$ having a singularity inside $(-\delta, \delta)$. □

The following theorem generalizes [**K-M**, 5.11].

2.11. Theorem. (Mixing of C^∞ and C^ω)
Let (E, E') be a complete dual pair, let $X \subseteq E$, let $f : \mathbb{R} \times X \to \mathbb{R}$ be a mapping that extends for every B locally around every point in $\mathbb{R} \times (X \cap E_B)$ to a holomorphic

map $\mathbb{C} \times (E_B)_{\mathbb{C}} \to \mathbb{C}$, *and let* $c \in C^\infty(\mathbb{R}, X)$. *Then* $c^* \circ \check{f} \colon \mathbb{R} \to C^\omega(X, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ *is real analytic.*

Proof. Let $I \subseteq \mathbb{R}$ be open and relatively compact, let $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Now choose an open and relatively compact $J \subseteq \mathbb{R}$ containing the closure $\bar{I}$ of I. There is a bounded subset $B \subseteq E$ such that $c|_J \colon J \to E_B$ is a $\mathcal{L}ip^k$-curve in the Banach space E_B generated by B. This is [**K82**, Folgerung on p. 114]. Let X_B denote the subset $X \cap E_B$ of the Banach space E_B. By assumption on f there is a holomorphic extension $f \colon V \times W \to \mathbb{C}$ of f to an open set $V \times W \subseteq \mathbb{C} \times (E_B)_{\mathbb{C}}$ containing the compact set $\{t\} \times c(\bar{I})$. By cartesian closedness of the category of holomorphic mappings $\check{f} \colon V \to \mathcal{H}(W, \mathbb{C})$ is holomorphic. Now recall that the bornological structure of $\mathcal{H}(W, \mathbb{C})$ is induced by that of $C^\infty(W, \mathbb{C}) := C^\infty(W, \mathbb{R}^2)$. And $c^* \colon C^\infty(W, \mathbb{C}) \to \mathcal{L}ip^k(I, \mathbb{C})$ is a bounded $\mathbb{C}$-linear map, by [**F-K**]. Thus $c^* \circ \check{f} \colon V \to \mathcal{L}ip^k(I, \mathbb{C})$ is holomorphic, and hence its restriction to $\mathbb{R} \cap V$, which has values in $\mathcal{L}ip^k(I, \mathbb{R})$, is (even topologically) real analytic by [**K-M**, 1.7]. Since $t \in \mathbb{R}$ was arbitrary we conclude that $c^* \circ \check{f} \colon \mathbb{R} \to \mathcal{L}ip^k(I, \mathbb{R})$ is real analytic. But the bornology of $C^\infty(\mathbb{R}, \mathbb{R})$ is generated by the inclusions into $\mathcal{L}ip^k(I, \mathbb{R})$, [**F-K**, 4.2.7], and hence $c^* \circ \check{f} \colon \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is real analytic. □

This can now be used to show cartesian closedness with the same proof as in [**K-M**, 5.12] for certain non-open subsets of convenient vector spaces. In particular the previous theorem applies to real analytic mappings $f \colon \mathbb{R} \times X \to \mathbb{R}$, where $X \subseteq E$ is convex with non-void interior. Since for such a set the intersection X_B with E_B has the same property and since E_B is a Banach space, the real analytic mapping is the germ of a holomorphic mapping.

2.12. Theorem. (Exponential law for real analytic germs)
Let K and L be two convex subsets with non-empty interior in convenient vector spaces. A map $f \colon K \to C^\omega(L, F)$ is real analytic if and only if the associated mapping $\hat{f} \colon K \times L \to F$ is real analytic.

Proof. $(\Rightarrow)$ Let $c = (c_1, c_2) \colon \mathbb{R} \to K \times L$ be C^α (for $\alpha \in \{\infty, \omega\}$) and let $\ell \in F'$. We have to show that $\ell \circ \hat{f} \circ c \colon \mathbb{R} \to \mathbb{R}$ is C^α. By cartesian closedness of C^α it is enough to show that the map $\ell \circ \hat{f} \circ (c_1 \times c_2) \colon \mathbb{R}^2 \to \mathbb{R}$ is C^α. This map however is associated to $\ell_* \circ (c_2)_* \circ f \circ c_1 \colon \mathbb{R} \to K \to C^\omega(L, F) \to C^\alpha(\mathbb{R}, \mathbb{R})$, hence is C^α by assumption on f and the structure of $C^\omega(L, F)$.

$(\Leftarrow)$ Let conversely $f \colon K \times L \to F$ be real analytic. Then obviously $f(x, \cdot) \colon L \to F$ is real analytic, hence $\check{f} \colon K \to C^\omega(L, F)$ makes sense. Now take an arbitrary C^α-map $c_1 \colon \mathbb{R} \to K$. We have to show that $\check{f} \circ c_1 \colon \mathbb{R} \to C^\omega(L, F)$ is C^α. Since the structure of $C^\omega(L, F)$ is generated by $C^\beta(c_1, \ell)$ for C^β-curves $c_2 \colon \mathbb{R} \to L$ (for $\beta \in \{\infty, \omega\}$) and $\ell \in F'$, it is by [**K-M**, 1.5] enough to show that $C^\beta(c_2, \ell) \circ \check{f} \circ c_1 \colon \mathbb{R} \to C^\beta(\mathbb{R}, \mathbb{R})$ is C^α. For $\alpha = \beta$ it is by cartesian closedness of C^α maps [**K-M**, 5.1] enough to show that the associate map $\mathbb{R}^2 \to \mathbb{R}$ is C^α. Since this map is just $\ell \circ f \circ (c_1 \times c_2)$, this is clear. In fact take for $\gamma \leq \alpha, \gamma \in \{\infty, \omega\}$ an arbitrary

C^γ-curve $d = (d_1, d_2)\colon \mathbb{R} \to \mathbb{R}^2$. Then $(c_1 \times c_2) \circ (d_1, d_2) = (c_1 \circ d_1, c_2 \circ d_2)$ is C^γ, and so the composite with $\ell \circ f$ has the same property.

Remains to show the mixing case, where c_1 is real analytic and c_2 is smooth or conversely.

First the case c_1 real analytic, c_2 smooth. Then $\ell \circ f \circ (c_1 \times id)\colon \mathbb{R} \times L \to \mathbb{R}$ is real analytic, hence extends to some holomorphic map by (2.9), and by (2.11) the map

$$C^\infty(c_2, \ell) \circ \check{f} \circ c_1 = c_2^* \circ (\ell \circ f \circ (c_1 \times id))^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

is real analytic.

Now the case c_1 smooth and c_2 real analytic. Then $\ell \circ f \circ (id \times c_2)\colon K \times \mathbb{R} \to \mathbb{R}$ is real analytic, so by the same reasoning as just before applied to $\tilde{f}$ defined by $\tilde{f}(x, y) := f(y, x)$, the map

$$C^\infty(c_1, \ell) \circ (\tilde{f})^\vee \circ c_2 = c_1^* \circ (\ell \circ \tilde{f} \circ (id \times c_2))^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

is real analytic. By [**K-M**, 5.10] the associated mapping

$$(c_1^* \circ (\ell \circ \tilde{f} \circ (id \times c_2))^\vee)^\sim = C^\omega(c_2, \ell) \circ \check{f} \circ c_1 : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$$

is smooth. □

The following example shows that Theorem (2.12) does not extend to arbitary domains.

2.13. Example. (The exponential law for general domains is false) *Let $X \subseteq \mathbb{R}^2$ be the graph of the map $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(t) := e^{-t^{-2}}$ for $t \neq 0$ and $h(0) = 0$. Let, furthermore, $f\colon \mathbb{R} \times X \to \mathbb{R}$ be the mapping defined by $f(t, s, r) := \frac{r}{t^2+s^2}$ for $(t, s) \neq (0, 0)$ and $f(0, 0, r) := 0$. Then $f\colon \mathbb{R} \times X \to \mathbb{R}$ is real analytic, however the associated mapping $\check{f}\colon \mathbb{R} \to C^\omega(X, \mathbb{R})$ is not.*

Proof. Obviously f is real analytic on $\mathbb{R}^3 \setminus \{(0,0)\} \times \mathbb{R}$. If $u \mapsto (t(u), s(u), r(u))$ is real analytic $\mathbb{R} \to \mathbb{R} \times X$, then $r(u) = h(s(u))$. Suppose s is not constant and $t(0) = s(0) = 0$, then we have that $r(u) = h(u^n s_0(u))$ cannot be real analytic, since it is not constant but the Taylor series at 0 is identical 0, contradiction. Thus $s = 0$, $r = h \circ s = 0$ and therefore $u \mapsto f(t(u), s(u), r(u)) = 0$ is real analytic.

Remains to show that $u \mapsto f(t(u), s(u), r(u))$ is smooth for all smooth curves $(t, s, r)\colon \mathbb{R} \to \mathbb{R} \times X$. Since $f(t(u), s(u), r(u)) = \frac{h(s(u))}{t(u)^2+s(u)^2}$ it is enough to show that $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(t, s) = \frac{h(s)}{t^2+s^2}$ is smooth. This is obviously the case, since each of its partial derivatives is of the form $h(s)$ multiplied by some rational function of t and s, hence extend continuously to $\{(0,0)\}$.

Now we show that $\check{f}\colon \mathbb{R} \to C^\omega(X, \mathbb{R})$ is not real analytic. Take the smooth curve $c\colon u \mapsto (u, h(u))$ into X and consider $c^* \circ \check{f}\colon \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$, which is given by $t \mapsto (s \mapsto f(t, c(s)) = \frac{h(s)}{t^2+s^2})$. Suppose it is real analytic into $C([-1, +1], \mathbb{R})$.

Then it has to be locally representable by a converging power series $\sum a_n t^n \in C([-1,+1],\mathbb{R})$. So there has to exist a $\delta > 0$ such that $\sum a_n(s) z^n = \frac{h(s)}{s^2}\sum_{k=0}^{\infty}(-1)^k(\frac{z}{s})^{2k}$ converges for all $|z| < \delta$ and $|s| < 1$. This is impossible, since at $z = si$ there is a pole. □

3. Holomorphic Maps on Non-Open Domains

In this section we will consider holomorphic maps defined on two types of convex subsets. First the case where the set is contained in some real part of the vector space and has non-empty interior there. Here we use the cartesian closed setting of [**K-N**] for holomorphic mappings.

Recall that for a subset $X \subseteq \mathbb{R} \subseteq \mathbb{C}$ the space of germs of holomorphic maps $X \to \mathbb{C}$ is the complexification of that of germs of real analytic maps $X \to \mathbb{R}$, [**K-M**, 3.11]. Thus we give the following

3.1. Definition. (Holomorphic maps $K \to F$)
Let $K \subseteq E$ be a convex set with non-empty interior in a real convenient vector space. And let F be a complex convenient vector space. We call a map $f\colon E_{\mathbb{C}} \supseteq K \to F$ holomorphic iff $f\colon E \supseteq K \to F$ is real analytic.

3.2. Lemma. (Holomorphic maps can be tested by functionals)
Let $K \subseteq E$ be a convex set with non-empty interior in a real convenient vector space. And let F be a complex convenient vector space. Then a map $f\colon K \to F$ is holomorphic if and only if the composites $\ell \circ f\colon K \to \mathbb{C}$ are holomorphic for all $\ell \in L_{\mathbb{C}}(E,\mathbb{C})$, where $L_{\mathbb{C}}(E,\mathbb{C})$ denotes the space of $\mathbb{C}$-linear maps.

Proof. ($\Rightarrow$) Let $\ell \in L_{\mathbb{C}}(F,\mathbb{C})$. Then the real and imaginary part $\Re\ell, \Im\ell \in L_{\mathbb{R}}(F,\mathbb{R})$ and since by assumption $f\colon K \to F$ is real analytic so are the composites $\Re\ell \circ f$ and $\Im\ell \circ f$, hence $\ell \circ f\colon K \to \mathbb{R}^2$ is real analytic, i.e. $\ell \circ f\colon K \to \mathbb{C}$ is holomorphic.

($\Leftarrow$) We have to show that $\ell \circ f\colon K \to \mathbb{R}$ is real analytic for every $\ell \in L_{\mathbb{R}}(F,\mathbb{R})$. So let $\tilde{\ell}\colon F \to \mathbb{C}$ be defined by $\tilde{\ell}(x) = i\ell(x) + \ell(ix)$. Then $\tilde{\ell} \in L_{\mathbb{C}}(F,\mathbb{C})$, since $i\tilde{\ell}(x) = -\ell(x) + i\ell(ix) = \tilde{\ell}(ix)$. Remark that $\ell = \Im \circ \tilde{\ell}$. By assumption $\tilde{\ell} \circ f\colon K \to \mathbb{C}$ is holomorphic, hence its imaginary part $\ell \circ f\colon K \to \mathbb{R}$ is real analytic. □

3.3. Theorem. (Holomorphic maps $K \to F$ are often germs)
Let $K \subseteq E$ be a convex subset with non-empty interior in a real Fréchet space E and let F be a complex convenient vector space such that F' carries a Baire topology as required in (2.1). Then a map $f\colon E_{\mathbb{C}} \supseteq K \to F$ is holomorphic if and only if it extends to a holomorphic map $\tilde{f}\colon \tilde{K} \to F$ for some neighborhood $\tilde{K}$ of K in $E_{\mathbb{C}}$.

Proof. Using (2.9) we conclude that f extends to a holomorphic map $\tilde{f}\colon \tilde{K} \to F_{\mathbb{C}}$ for some neighborhood $\tilde{K}$ of K in $E_{\mathbb{C}}$. The map $\mathrm{pr}\colon F_{\mathbb{C}} \to F$, given by $\mathrm{pr}(x,y) = x + iy \in F$ for $(x,y) \in F^2 = F \otimes_{\mathbb{R}} \mathbb{C}$, is $\mathbb{C}$-linear and restricted to

$F \times \{0\} = F$ it is the identity. Thus $\mathrm{pr} \circ \tilde{f} \colon \tilde{K} \to F_{\mathbb{C}} \to F$ is a holomorphic extension of f.

Conversely let $\tilde{f} \colon \tilde{K} \to F$ be a holomorphic extension to a neighborhood $\tilde{K}$ of K. So it is enough to show that the holomorphic map $\tilde{f}$ is real analytic. By [**K-N**] it is smooth. So it remains to show that it is real analytic. For this it is enough to consider a topological real analytic curve in $\tilde{K}$ by [**K-M**, 2.8]. Such a curve is extendable to a holomorphic curve $\tilde{c}$ by [**K-M**, 1.7], hence the composite $\tilde{f} \circ \tilde{c}$ is holomorphic and its restriction $\tilde{f} \circ c$ to $\mathbb{R}$ is real analytic. □

3.4. Definition. (Holomorphic maps on complex vector spaces)
Let $K \subseteq E$ be a convex subset with non-empty interior in a complex convenient vector space. And map $f \colon E \supseteq K \to F$ is called holomorphic iff it is real analytic and the derivative $f'(x)$ is $\mathbb{C}$-linear for all $x \in K^o$.

3.5. Theorem. (Holomorphic maps are germs)
Let $K \subseteq E$ be a convex subset with non-empty interior in a complex convenient vector space. Then a map $f \colon E \supseteq K \to F$ into a complex convenient vector space F is holomorphic if and only if it extends to a holomorphic map defined on some neighborhood of K in E.

Proof. Since $f \colon K \to F$ is real analytic, it extends by (2.9) to a real analytic map $\tilde{f} \colon E \supseteq U \to F$, where we may assume that U is connected with K by straight line segments. We claim that $\tilde{f}$ is in fact holomorphic. For this it is enough to show that $f'(x)$ is $\mathbb{C}$-linear for all $x \in U$. So consider the real analytic mapping $g \colon U \to F$ given by $g(x) := if'(x)(v) - f'(x)(iv)$. Since it is zero on K^o it has to be zero everywhere by the uniqueness theorem. □

3.6. Remark. (There is no definition for holomorphy analogous to (2.7))
In order for a map $K \to F$ to be holomorphic it is not enough to assume that all composites $f \circ c$ for holomorphic $c \colon \mathbb{D} \to K$ are holomorphic, where $\mathbb{D}$ is the open unit disk. Take as K the closed unit disk, then $c(\mathbb{D}) \cap \partial K = \phi$. In fact let $z_0 \in \mathbb{D}$ then $c(z) = (z - z_0)^n (c_n + (z - z_0) \sum_{k>n} c_k (z - z_0)^{k-n-1})$ for z close to z_0, which covers a neighborhood of $c(z_0)$. So the boundary values of such a map would be completely arbitrary.

3.7. Lemma. (Holomorphy is a bornological concept)
Let $T_\alpha \colon E \to E_\alpha$ be a family of bounded linear maps that generates the bornology on E. Then a map $c \colon K \to F$ is holomorphic if and only if all the composites $T_\alpha \circ c \colon I \to F_\alpha$ are holomorphic.

Proof. It follows from (2.6) that f is real analytic. And the $\mathbb{C}$-linearity of $f'(x)$ can certainly be tested by point separating linear functionals. □

3.8. Theorem. (Exponential law for holomorphic maps)
Let K and L be convex subsets with non-empty interior in complex convenient

vector spaces. Then a map $f\colon K \times L \to F$ is holomorphic if and only if the associated map $\check{f}\colon K \to H(L,F)$ is holomorphic.

Proof. This follows immediately from the real analytic result (2.12), since the $\mathbb{C}$-linearity of the involved derivatives translates to each other. In fact we have $f'(x_1,x_2)(v_1,v_2) = ev_{x_2}((\check{f})'(x_1)(v_1)) + (\check{f}(x_1))'(x_2)(v_2)$ for $x_1 \in K$ and $x_2 \in L$. $\square$

References

[F-K] Frölicher A. and Kriegl A., *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

[J81] Jarchow H., *Locally convex spaces*, Teubner, Stuttgart, 1981.

[K82] Kriegl A., *Die richtigen Räume für Analysis im Unendlich – Dimensionalen*, Monatshefte für Math. **94** (1982), 109–124.

[K83] ———, *Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen*, Monatshefte für Math. **95** (1983), 287–309.

[K-M] Kriegl A. and Michor P. W., *The convenient setting for real analytic mappings*, Acta Mathematica **165** (1990), 105–159.

[K-N] Kriegl A. and Nel L. D., *A convenient setting for holomorphy*, Cahiers Top. Géo. Diff. **26** (1985), 273–309.

[S64] Seeley R. T., *Extensions of C^∞-functions defined in a half space*, Proc. AMS **15** (1964), 625–626.

[W34] Whitney H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. AMS **36** (1934), 63–89.

[W43] ———, *Differentiable Even Functions*, Duke Math. J. **10** (1943), 159–166.

A. Kriegl, Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria, *e-mail:* Andreas.Kriegl@univie.ac.at