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SUPERREFLEXIVITY AND J–CONVEXITY OF BANACH SPACES

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Abstract. A Banach space X is superreflexive if each Banach space Y that is finitely representable in X is reflexive. Superreflexivity is known to be equivalent to J-convexity and to the non-existence of uniformly bounded factorizations of the summation operators S_n through X.

We give a quantitative formulation of this equivalence.

This can in particular be used to find a factorization of S_n through X, given a factorization of S_N through $[L_2, X]$, where N is 'large' compared to n.

1. Introduction

Much of the significance of the concept of superreflexivity of a Banach space X is due to its many equivalent characterizations, see e.g. Beauzamy [**1**, Part 4].

Some of these characterizations allow a quantification, that makes also sense in non superreflexive spaces. Here are two examples.

Definition. Given n and $0 < \varepsilon < 1$, we say that a Banach space X is **$J(n,\varepsilon)$-convex**, if for all elements $z_1, \dots, z_n \in U_X$ we have

$$\inf_{1 \le k \le n} \Big\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \Big\| < n(1-\varepsilon).$$

We let $\boldsymbol{J}_n(X)$ denote the infimum of all ε, such that X is not $J(n,\varepsilon)$-convex.

Definition. Given n and $\sigma \ge 1$, we say that a Banach space X **factors the summation operator** S_n with norm σ, if there exists a factorization $S_n = B_n A_n$ with $A_n : l_1^n \to X$ and $B_n : X \to l_\infty^n$ such that $\|A_n\| \, \|B_n\| = \sigma$.

We let $\boldsymbol{S}_n(X)$ denote the infimum of all σ, such that X factors S_n with norm σ.

Here, the summation operator $S_n : l_1^n \to l_\infty^n$ is given by

$$(\xi_k) \mapsto \left(\sum_{h=1}^{k} \xi_h \right)$$

and U_X denotes the unit ball of the Banach space X.

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It is known that a Banach space is superreflexive if and only if it is $J(n,\varepsilon)$-convex for some n and $\varepsilon > 0$, or equivalently, if it does not factor the summation operators with uniformly bounded norm; see James [**5**, Th. 5, Lem. B], and Schäffer/Sundaresan [**9**, Th. 2.2].

Using the terminology introduced above, this can be reformulated as follows:

Theorem 1. *For a Banach space X the following properties are equivalent:*

(i) *X is not superreflexive.*
(ii) *For all $n \in \mathbb{N}$ we have $\boldsymbol{J}_n(X) = 0$.*
(iii) *There is a constant $\sigma \geq 1$ such that for all $n \in \mathbb{N}$ we have $\boldsymbol{S}_n(X) \leq \sigma$.*
(iv) *For all $n \in \mathbb{N}$ we have $\boldsymbol{S}_n(X) = 1$.*

There are two conceptually different methods to prove that X is superreflexive if and only if $[L_2, X]$ is. The one is to use Enflo's renorming result [**2**, Cor. 3], which is not suited to be localized, the other is the use of J-convexity, see Pisier [**7**, Prop. 1.2]. It turns out that for fixed n

$$\boldsymbol{J}_n([L_2, X]) \leq \boldsymbol{J}_n(X) \leq 2n^2 \boldsymbol{J}_n([L_2, X]); \tag{1}$$

see Section 2 for a proof. Similar results hold also in the case of B-convexity; see [**8**, p. 30].

Theorem 2. *If for some n and all $\varepsilon > 0$, $[L_2, X]$ contains $(1+\varepsilon)$ isomorphic copies of l_1^n, then X contains $(1+\varepsilon)$ isomorphic copies of l_1^n.*

Theorem 3. *If for some n and all $\varepsilon > 0$, $[L_2, X]$ contains $(1+\varepsilon)$ isomorphic copies of l_∞^n, then X contains $(1+\varepsilon)$ isomorphic copies of l_∞^n.*

On the other hand, no result of this kind for the factorization of S_n is known, i.e. if for some n and all $\varepsilon > 0$, $[L_2, X]$ factors S_n with norm $(1+\varepsilon)$, does it follow that X factors S_n with norm $(1+\varepsilon)$?

Assuming $\boldsymbol{S}_n([L_2, X]) \leq \sigma$ for some constant σ and **all** $n \geq 1$, one can use Theorem 1 to obtain that $\boldsymbol{J}_n([L_2, X]) = 0$ for **all** $n \geq 1$ and consequently $\boldsymbol{S}_n(X) = 1$.

The intent of our paper is to keep n fixed in this reasoning. Unfortunately, we don't get a result as smooth as Theorems 2 and 3. Instead, we have to consider two different values n and N. If $\boldsymbol{S}_N([L_2, X]) = \sigma$ for some 'large' N, then $\boldsymbol{S}_n(X) \leq (1+\varepsilon)$ for some 'small' n. To make this more precise, let us introduce the iterated exponential (or TOWER) function $P_g(m)$. We let

$$P_0(m) := m \quad \text{and} \quad P_{g+1}(m) := 2^{P_g(m)}.$$

We will prove the following two theorems.

Theorem 4. *For fixed $n \in \mathbb{N}$ and $\sigma > 1$ there is $\varepsilon > 0$ such that $\boldsymbol{J}_n(X) \leq \varepsilon$ implies $\boldsymbol{S}_n(X) < \sigma$. In particular $\boldsymbol{J}_n(X) = 0$ implies $\boldsymbol{S}_n(X) = 1$.*

Theorem 5. *For fixed $n \in \mathbb{N}$, $\varepsilon > 0$ and $\sigma \geq 1$ there is a number $N(\varepsilon, n, \sigma)$, such that $\boldsymbol{S}_N(X) \leq \sigma$ implies $\boldsymbol{J}_n(X) < \varepsilon$. The number N can be estimated by*

$$N \leq P_m(cn),$$

where m and c depend on σ and ε only.

Using (1), we obtain the following consequence.

Corollary 6. *For fixed $n \in \mathbb{N}$, $\sigma_1 > 1$, and $\sigma_2 \geq 1$ there is a number $N(\sigma_1, n, \sigma_2)$ such that $\boldsymbol{S}_N([L_2, X]) \leq \sigma_2$ implies $\boldsymbol{S}_n(X) \leq \sigma_1$.*

Proof. Determine ε as in Theorem 4 such that $\boldsymbol{J}_n(X) \leq \varepsilon$ implies $\boldsymbol{S}_n(X) < \sigma_1$. Choose $N = N(\frac{\varepsilon}{2n^2}, n, \sigma_2)$ as in Theorem 5 such that $\boldsymbol{S}_N([L_2, X]) \leq \sigma_2$ implies $\boldsymbol{J}_n([L_2, X]) < \frac{\varepsilon}{2n^2}$. By (1) we obtain $\boldsymbol{J}_n(X) < \varepsilon$ and hence $\boldsymbol{S}_n(X) < \sigma_1$. $\square$

The estimate in Theorem 5 seems rather crude, and we have no idea, whether or not it is optimal.

2. Proofs

First of all, we list some elementary properties of the sequences $\boldsymbol{S}_n(X)$ and $\boldsymbol{J}_n(X)$.

Fact.

(i) *The sequence $(\boldsymbol{S}_n(X))$ is non-decreasing.*
(ii) $1 \leq \boldsymbol{S}_n(X) \leq (1 + \log n)$ *for all infinite dimensional Banach spaces X.*
(iii) *The sequence $(n\boldsymbol{J}_n(X))$ is non-decreasing.*
(iv) *For all $n, m \in \mathbb{N}$ we have $\boldsymbol{J}_n(X) \leq \boldsymbol{J}_{nm}(X) \leq \boldsymbol{J}_n(X) + 1/n$.*
(v) *If $\boldsymbol{J}_n(X) \to 0$ then for all $n \in \mathbb{N}$ we have $\boldsymbol{J}_n(X) = 0$.*
(vi) $\boldsymbol{J}_n(\mathbb{R}) \geq 1 - 1/n$ *for all $n \in \mathbb{N}$.*
(vii) *If q and ε are related by $\varepsilon \geq (1-\varepsilon)^{q-1}$ then $\boldsymbol{J}_n(l_q) \leq 4\varepsilon$ for all $n \in \mathbb{N}$.*

Proof. The monotonicity properties (i) and (iii) are trivial.

The bound for $\boldsymbol{S}_n(X)$ in (ii) follows from the fact that the summation operator S_n factors through l_2^n with norm $(1 + \log n)$ and from Dvoretzky's Theorem.

To see (iv) assume that X is $J(n, \varepsilon)$-convex. Given $z_1, \ldots, z_{nm} \in U_X$, let

$$x_h := \frac{1}{m} \sum_{k=1}^{m} z_{(h-1)m+k} \quad \text{for } h = 1, \ldots, n.$$

Then

$$\inf_{1 \leq k \leq nm} \Big\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{nm} z_h \Big\| \leq m \inf_{1 \leq k \leq n} \Big\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{n} x_h \Big\| < mn(1-\varepsilon),$$

which proves that X is $J(nm, \varepsilon)$-convex, and consequently $\boldsymbol{J}_n(X) \leq \boldsymbol{J}_{nm}(X)$.

Assume now that X is $J(nm,\varepsilon)$-convex. Given $z_1,\dots,z_n \in U_X$, let

$$\begin{aligned} x_1 =\dots= \ & x_m := z_1 \\ \vdots \qquad & \qquad \vdots \\ x_{(n-1)m+1} =\dots= \ & x_{nm} := z_n . \end{aligned}$$

If

$$\inf_{1\le k\le nm}\Big\|\sum_{h=1}^{k} x_h - \sum_{h=k+1}^{nm} x_h\Big\| \quad \text{is attained for } k_0,$$

there is $l \in \{0,\dots,n\}$ such that $m/2+(l-1)m < k_0 \le m/2+lm$, hence

$$\Big\|\sum_{h=1}^{k_0} x_h - \sum_{h=k_0+1}^{nm} x_h\Big\| \ge \Big\|\sum_{h=1}^{lm} x_h - \sum_{h=lm+1}^{nm} x_h\Big\| - 2\sum_{h\in I}\|x_h\|,$$

where $I=\{k_0+1,\dots,lm\}$ or $I=\{lm+1,\dots,k_0\}$ according to whether $k_0\le lm$ or $k_0>lm$. It follows that

$$nm(1-\varepsilon) > m \inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| - m,$$

and hence $\boldsymbol{J}_n(X)\ge \varepsilon - 1/n$. This proves (iv).

(v) is a consequence of (iv).

For (vi) and (vii) see Section 3. □

For the convenience of the reader, let us repeat the argument for the proof of (1) from [**1**]. The left-hand part of (1) is obvious, since X can be isometrically embedded into $[L_2,X]$. To see the right-hand inequality, assume that for all $z_1,\dots,z_n\in U_X$

$$\inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| < n(1-\varepsilon).$$

Obviously, if $\|z_1\|=\dots=\|z_n\|$ it follows by homogeneity that

$$\inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| < (1-\varepsilon)\sum_{k=1}^{n}\|z_k\|. \tag{2}$$

If $z_1,\dots,z_n$ are arbitrary, let $m:=\min_{1\le k\le n}\|z_k\|$, $\lambda_k := m/\|z_k\|$, and $\tilde z_k := (1-\lambda_k)z_k$. It turns out that $\|z_k-\tilde z_k\| = m$ and therefore by (2)

$$\begin{aligned} \inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| &< \sum_{k=1}^{n}\|\tilde z_k\| + (1-\varepsilon)\sum_{k=1}^{n}\|z_k-\tilde z_k\| \\ &\le \sum_{k=1}^{n}\big((1-\lambda_k)+(1-\varepsilon)\lambda_k\big)\|z_k\| \le \Big(\sum_{k=1}^{n}(1-\varepsilon\lambda_k)^2\Big)^{1/2}\Big(\sum_{k=1}^{n}\|z_k\|^2\Big)^{1/2}. \end{aligned}$$

Now, at least one of the λ_k's equals one, while the others are greater than or equal to zero. This yields

$$\inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| < \big((1-\varepsilon)^2 + n - 1\big)^{1/2}\Big(\sum_{k=1}^{n}\|z_k\|^2\Big)^{1/2}$$
$$\le (n - 2\varepsilon + \varepsilon^2)^{1/2}\Big(\sum_{k=1}^{n}\|z_k\|^2\Big)^{1/2}.$$

On the other hand, we trivially get that for all $1 \le k \le n$

$$\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| \le n^{1/2}\Big(\sum_{k=1}^{n}\|z_k\|^2\Big)^{1/2}.$$

Therefore

$$\sum_{k=1}^{n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\|^2 < \big((n-2\varepsilon+\varepsilon^2) + (n-1)n\big)\sum_{k=1}^{n}\|z_k\|^2$$

for all $z_1, \ldots, z_n \in X$. If in particular $f_1, \ldots, f_n \in U_{[L_2,X]}$, then

$$\sum_{k=1}^{n}\Big\|\sum_{h=1}^{k} f_h(t) - \sum_{h=k+1}^{n} f_h(t)\Big\|^2 < (n^2 - 2\varepsilon + \varepsilon^2)\sum_{k=1}^{n}\|f_k(t)\|^2.$$

Integration with respect to t yields

$$\sum_{k=1}^{n}\Big\|\sum_{h=1}^{k} f_h - \sum_{h=k+1}^{n} f_h\Big|L_2\Big\|^2 < (n^2 - 2\varepsilon + \varepsilon^2)\sum_{k=1}^{n}\|f_k|L_2\|^2 \le n(n^2 - 2\varepsilon + \varepsilon^2).$$

This implies that

$$\inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} f_h - \sum_{h=k+1}^{n} f_h\Big|L_2\Big\| < (n^2 - 2\varepsilon + \varepsilon^2)^{1/2} \le n(1-\delta)$$

for $\delta = \varepsilon/2n^2$. Therefore $\boldsymbol{J}_n([L_2, X]) \ge \boldsymbol{J}_n(X)/2n^2$. □

Let us now prove Theorem 4.

Proof of Theorem 4. Choose $\varepsilon < \frac{1}{2(n+2)!}$ such that $1 + 2(n+2)!\varepsilon < \sigma$. If $\boldsymbol{J}_n(X) \le \varepsilon$, we find $z_1, \ldots, z_n \in U_X$ be such that

$$\inf_{1\le k\le n}\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| \ge n(1-\varepsilon).$$

By the Hahn-Banach theorem, we find $y_k \in U_{X^*}$ such that

$$n(1-\varepsilon) \le \sum_{h=1}^{k} \langle z_h, y_k\rangle - \sum_{h=k+1}^{n} \langle z_h, y_k\rangle .$$

Obviously $|\langle z_h, y_k\rangle| \le 1$. If for some $h \le k$ we even have

$$\langle z_h, y_k\rangle < 1 - n\varepsilon,$$

then

$$n(1-\varepsilon) \le \sum_{l=1}^{k} \langle z_l, y_k\rangle - \sum_{l=k+1}^{n} \langle z_l, y_k\rangle < (n-1) + (1-n\varepsilon) = n(1-\varepsilon),$$

which is a contradiction. Hence

$$1 - n\varepsilon \le \langle z_h, y_k\rangle \le 1 \quad \text{for all } h \le k. \tag{3}$$

Similarly

$$1 - n\varepsilon \le -\langle z_h, y_k\rangle \le 1 \quad \text{for all } h > k. \tag{4}$$

Let $x_h := (z_1 + z_h)/2$. Then it follows from (3) and (4) that there are $x_1, \dots, x_n \in U_X$ and $y_1, \dots, y_n \in U_{X'}$ so that

$$\langle x_h, y_k\rangle \in \begin{cases} (1-n\varepsilon, 1] & \text{if } h \le k, \\ (-n\varepsilon, +n\varepsilon) & \text{if } h > k. \end{cases}$$

The assertion now follows from the following distortion lemma. $\square$

Lemma 7. *Suppose that* $\varepsilon < \frac{1}{2(n+1)!}$ *and that there are* $x_1, \dots, x_n \in U_X$ *and* $y_1, \dots, y_n \in U_{X^*}$ *such that*

$$\langle x_h, y_k\rangle \in \begin{cases} (1-\varepsilon, 1] & \textit{if } h \le k, \\ (-\varepsilon, +\varepsilon) & \textit{if } h > k. \end{cases}$$

Then $\boldsymbol{S}_n(X) \le 1 + 2(n+1)!\varepsilon$.

Proof. Fix $h \in \{1, \dots, n\}$. Let $\alpha_{lk} := \langle x_l, y_k\rangle$. Consider the system of linear equations

$$\sum_{l=1}^{n} \alpha_{lk}\xi_l = \begin{cases} 1 - \alpha_{hk} & \text{if } h \le k, \\ -\alpha_{hk} & \text{if } h > k, \end{cases} \qquad k = 1, \dots, n$$

in the n variables $\xi_1, \ldots, \xi_n$. Its solution is given by

$$\xi_m^{(h)} = \frac{\det(\beta_{lk}^{(m)})}{\det(\alpha_{lk})},$$

where $(\beta_{lk}^{(m)})$ is the matrix (α_{lk}) but with its m-th column replaced by the right-hand side of our system of equations. It follows that

$$|\det(\beta_{lk}^{(m)})| = \Big|\sum_{\pi} \operatorname{sgn}(\pi) \prod_{k=1}^{n} \beta_{k\pi(k)}^{(m)}\Big| \le n!|\beta_{m\pi(m)}^{(m)}| \le n!\varepsilon.$$

Since for all permutations π that are not the identity, there exists at least one k such that $\pi(k) > k$, we have $|\alpha_{\pi(k)k}| < \varepsilon$ and hence

$$\begin{aligned} |\det(\alpha_{lk})| = \Big|\sum_{\pi} \operatorname{sgn}(\pi) \prod_{k=1}^{n} \alpha_{\pi(k)k}\Big| &\ge \Big|\prod_{k=1}^{n} \alpha_{kk}\Big| - \sum_{\pi\neq id} \varepsilon \\ &\ge (1-\varepsilon)^n - n!\varepsilon \ge 1 - n\varepsilon - n!\varepsilon \ge 1 - (n+1)!\varepsilon. \end{aligned}$$

Hence if $\varepsilon < \frac{1}{2(n+1)!}$ the solutions $\xi_m^{(h)}$ satisfy

$$|\xi_m^{(h)}| \le 2n!\varepsilon.$$

Defining $A_n : l_1^n \to X$ by

$$A_n e_h := \sum_{m=1}^{n} x_m \xi_m^{(h)} + x_h,$$

we get that $\|A_n\| \le 1 + \sup_h \sum_{m=1}^{n} |\xi_m^{(h)}| \le 1 + 2(n+1)!\varepsilon$. Defining $B_n : X \to l_\infty^n$ by

$$B_n x := (\langle x, y_k\rangle)_{k=1}^{n},$$

we get that $\|B_n\| \le 1$ and $S_n = B_n A_n$. This completes the proof, since $\boldsymbol{S}_n(X) \le \|A_n\|\,\|B_n\| \le 1 + 2(n+1)!\varepsilon$. $\square$

Interlude on Ramsey theory

Our proof of Theorem 5 makes massive use of the general form of Ramsey's Theorem. Therefore, for the convenience of the reader, let us recall, what it says; see [**3**] and [**6**].

For a set M and a positive integer k, let $M^{[k]}$ be the set of all subsets of M of cardinality k.

Theorem 8. *Given r, k and n, there is a number $R_k(n,r)$ such that for all $N \geq R_k(n,r)$ the following holds:*

For each function $f\colon \{1,\dots,N\}^{[k]} \to \{1,\dots,r\}$ there exists a subset $M \subseteq \{1,\dots,N\}$ of cardinality at least n such that $f(M^{[k]})$ is a singleton.

The following estimate for the Ramsey number $R_k(l,r)$ can be found in [**3**, p. 106].

Lemma. *There is a number $c(r,k)$ depending on r and k, such that*

$$R_k(l,r) \leq P_k(c(r,k)\cdot l).$$

We can now turn to the proof of Theorem 5.

Proof of Theorem 5. The proof follows the line of James's proof in [**4**, Th. 1.1]. The main new ingredient is the use of Ramsey's Theorem to estimate the number N.

Let n, $\varepsilon > 0$, and σ be given. Define m by

$$2m\sigma < \Big(\frac{1}{1-\varepsilon}\Big)^{m-1} \tag{5}$$

and let

$$N := R_{2m}(R_{2m}(2nm+1,m),m), \tag{6}$$

where R denotes the Ramsey number introduced in the previous paragraph.

The required estimate for N then follows from Lemma 9 as follows

$$N \leq P_{2m}(c_1 P_{2m}(c_2 2nm)) \leq P_{4m}(c_3 n),$$

where c_1, c_2, and c_3 are constants depending on m, which in turn depends on σ and ε.

Replacing, e.g. σ by 2σ, we may assume that in fact $\boldsymbol{S}_N(X) < \sigma$ in order to avoid using an additional δ in the notation. If $\boldsymbol{S}_N(X) < \sigma$ then there are $A_N : l_1^N \to X$ and $B_N\colon X \to l_\infty^N$ such that $S_N = B_N A_N$ and $\|A_N\| = 1$, $\|B_N\| \leq \sigma$. Let $x_h := A_N e_h$ and $y_k := B_N^* e_k$. Note that

$$\|x_h\| \leq 1, \quad \|y_k\| \leq \sigma, \quad \text{and} \quad \langle x_h, y_k\rangle = \begin{cases} 1 & \text{if } h \leq k, \\ 0 & \text{if } h > k. \end{cases}$$

For each subset $M \subseteq \{1,\dots,N\}$, we let $\mathcal{F}_m(M)$ denote the collection of all sequences $\mathbb{F} = (F_1,\dots,F_m)$ of consecutive intervals of numbers, whose endpoints are in M, i.e.

$$F_j = \{l_j, l_j+1,\dots,r_j\}, \quad l_j, r_j \in M, \quad l_j < r_j < l_{j+1},$$

for $j = 1,\dots,m$. Note that $\mathcal{F}_m(M)$ can be identified with $M^{[2m]}$.

The outline of the proof of Theorem 5 is as follows. To each $\mathbb{F} = (F_1, \dots, F_m)$, we assign an element $x(\mathbb{F})$ which in fact is a linear combination of the elements $x_1, \dots, x_N$. Next, we extract a 'large enough' subset M of $\{1, \dots, N\}$, such that all $x(\mathbb{F})$ with $\mathbb{F} \in \mathcal{F}_m(M)$ have about equal norm. Finally, we look at special sequences $\mathbb{F}^{(1)}, \dots, \mathbb{F}^{(n)}$ and $\mathbb{E}^{(1)}, \dots, \mathbb{E}^{(n)}$ in $\mathcal{F}_m(M)$ such that

$$\Big\| \sum_{h=1}^{k} x(\mathbb{F}^{(h)}) - \sum_{h=k+1}^{n} x(\mathbb{F}^{(h)}) \Big\| \geq n \| x(\mathbb{E}^{(k)}) \|.$$

Since $\|x(\mathbb{E}^{(k)})\| \asymp \|x(\mathbb{F}^{(h)})\|$, normalizing the elements $x(\mathbb{F}^{(h)})$ yields the required elements $z_1, \dots, z_n$ to prove that $\boldsymbol{J}_n(X) < \varepsilon$.

Let us start by choosing the elements $x(\mathbb{F})$. For a sequence $\mathbb{F} \in \mathcal{F}_m(M)$, we define

$$S(\mathbb{F}) := \left\{ x = \sum_{h=1}^{N} \xi_h x_h : \sup_h |\xi_h| \leq 2,\ \langle x, y_l \rangle = (-1)^j \ \begin{array}{c} \text{for all } l \in F_j \\ \text{and } j = 1, \dots, m \end{array} \right\}.$$

By compactness, there is $x(\mathbb{F}) \in S(\mathbb{F})$ such that

$$\|x(\mathbb{F})\| = \inf_{x \in S(\mathbb{F})} \|x\|.$$

Lemma 10. *We have $1/\sigma \leq \|x(\mathbb{F})\| \leq 2m$ for all $\mathbb{F} \in \mathcal{F}_m(\{1, \dots, N\})$.*

Proof. Write $F_j = \{l_j, \dots, r_j\}$ and let

$$x := -x_{l_1} + 2 \sum_{i=2}^{m} (-1)^i x_{l_i}.$$

Then for $l \in F_j$, we have

$$\langle x, y_l \rangle = -1 + 2 \sum_{i=2}^{j} (-1)^i \cdot 1 + 2 \sum_{i=j+1}^{m} (-1)^i \cdot 0 = (-1)^j,$$

hence $x \in S(\mathbb{F})$ and $\|x(\mathbb{F})\| \leq \|x\| \leq 2m - 1$.

On the other hand,

$$1 = |\langle x(\mathbb{F}), y_{l_1} \rangle| \leq \sigma \|x(\mathbb{F})\|.$$

Hence $1/\sigma \leq \|x(\mathbb{F})\|$. □

By (5), we can write the interval $[1/\sigma, 2m]$ as a disjoint union as follows

$$\Big[\frac{1}{\sigma}, 2m\Big] \subseteq \bigcup_{i=1}^{m-1} A_i, \quad \text{where} \quad A_i := \frac{1}{\sigma}\bigg[\Big(\frac{1}{1-\varepsilon}\Big)^{i-1}, \Big(\frac{1}{1-\varepsilon}\Big)^{i}\bigg).$$

For $\mathbb{F} = (F_1, \ldots, F_m) \in \mathcal{F}_m(\{1, \ldots, N\})$ and $1 \le j \le m$, let

$$P_j(\mathbb{F}) := (F_1, \ldots, F_j) \in \mathcal{F}_j(\{1, \ldots, N\}).$$

Obviously

$$\|x(P_{j-1}(\mathbb{F}))\| \le \|x(P_j(\mathbb{F}))\| \le 2m \quad \text{for } j = 2, \ldots, m.$$

It follows that for each $\mathbb{F} \in \mathcal{F}_m(\{1, \ldots, N\})$ there is at least one index j for which the two values $\|x(P_{j-1}(\mathbb{F}))\|$ and $\|x(P_j(\mathbb{F}))\|$ belong to the same interval A_i. Letting $f(\mathbb{F})$ be the least such value j, defines a function

$$f : \{1, \ldots, N\}^{[2m]} \to \{1, \ldots, m\}.$$

Applying Ramsey's Theorem to that function, yields the existence of a number j_0 and a subset L of $\{1, \ldots, N\}$ of cardinality $|L| \ge R_{2m}(2nm + 1, m)$ such that for all $\mathbb{F} \in \mathcal{F}_m(L)$ the two values $\|x(P_{j_0-1}(\mathbb{F}))\|$ and $\|x(P_{j_0}(\mathbb{F}))\|$ belong to the same of the intervals A_i.

Next, for each $\mathbb{F} \in \mathcal{F}_m(L)$ there is a unique number i for which the value $\|x(P_{j_0}(\mathbb{F}))\|$ belongs to the interval A_i. Letting $g(\mathbb{F})$ be that number i, defines a function

$$g\colon L^{[2m]} \to \{1, \ldots, m\}.$$

Applying Ramsey's Theorem to that function, yields the existence of a number i_0 and a subset M of L of cardinality $|M| \ge 2nm + 1$ such that for all $\mathbb{F} \in \mathcal{F}_m(M)$ we have

$$\|x(P_{j_0}(\mathbb{F}))\| \in A_{i_0}, \tag{7}$$

and hence, by the choice of j_0 and L, also

$$\|x(P_{j_0-1}(\mathbb{F}))\| \in A_{i_0}. \tag{8}$$

We now define sequences

$$\mathbb{F}^{(h)} := (F_1^{(h)}, \ldots, F_m^{(h)}) \quad \text{and} \quad \mathbb{E}^{(k)} := (E_1^{(k)}, \ldots, E_{m-1}^{(k)})$$

of nicely overlapping intervals.

Write $M = \{p_1, \ldots, p_{2nm+1}\}$, where $p_1 < p_2 < \cdots < p_{2nm+1}$ and define

$$\mathbb{F}^{(h)} := (F_1^{(h)}, \ldots, F_m^{(h)}) \in \mathcal{F}_m(M) \qquad h = 1, \ldots, n$$

as follows

$$F_j^{(h)} := \begin{cases} \{p_h, \ldots, p_{n+2h-1}\} & \text{if } j = 1, \\ \{p_{n(2j-3)+2h}, \ldots, p_{n(2j-1)+2h-1}\} & \text{if } j = 2, \ldots, m-1, \\ \{p_{n(2m-3)+2h}, \ldots, p_{n(2m-1)+h}\} & \text{if } j = m. \end{cases}$$

It turns out that

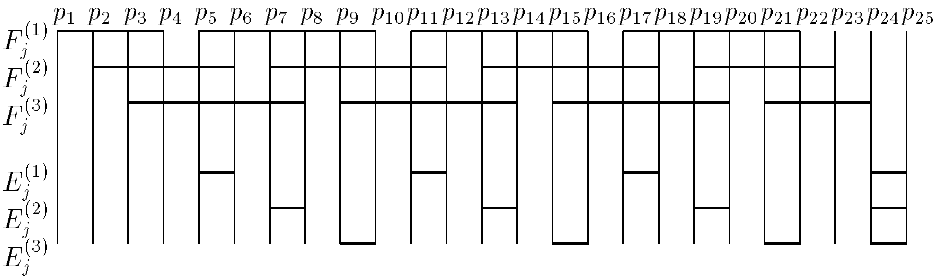
$$E_j^{(k)} := \bigcap_{h=1}^{k} F_{j+1}^{(h)} \cap \bigcap_{h=k+1}^{n} F_j^{(h)} \qquad k = 1, \ldots, n \tag{9}$$

is given by

$$E_j^{(k)} := \{p_{n(2j-1)+2k}, \ldots, p_{n(2j-1)+2k+1}\} \quad \text{if } j = 1, \ldots, m-1.$$

Hence $(E_1^{(k)}, \ldots, E_{m-1}^{(k)}) \in \mathcal{F}_{m-1}(M)$. In order to obtain an element of $\mathcal{F}_m(M)$ we add the auxiliary set $E_m^{(k)} := \{p_{2nm}, \ldots, p_{2nm+1}\}$, this can be done for $n \geq 2$, which is the only interesting case anyway since $\boldsymbol{J}_1(X) = 0$ for any Banach space X. We have $\mathbb{E}^{(k)} := (E_1^{(k)}, \ldots, E_m^{(k)}) \in \mathcal{F}_m(M)$.

The following picture shows the sets $F_j^{(h)}$ and $E_j^{(k)}$ in the case $n = 3$ and $m = 4$:



It follows from (9) that for $1 \leq k \leq n$

$$\frac{1}{n}\Big(-\sum_{h=1}^{k} x(P_{j_0}(\mathbb{F}^{(h)})) + \sum_{h=k+1}^{n} x(P_{j_0}(\mathbb{F}^{(h)}))\Big) \in S(P_{j_0-1}(\mathbb{E}^{(k)}))$$

hence

$$\Big\|\sum_{h=1}^{k} x(P_{j_0}(\mathbb{F}^{(h)})) - \sum_{h=k+1}^{n} x(P_{j_0}(\mathbb{F}^{(h)}))\Big\| \geq n\|x(P_{j_0-1}(\mathbb{E}^{(k)}))\|.$$

Let $z_h := \sigma\,(1-\varepsilon)^{i_0}\, x(P_{j_0}(\mathbb{F}^{(h)}))$. Then

$$\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| \geq n\,\sigma\,(1-\varepsilon)^{i_0}\,\|x(P_{j_0-1}(\mathbb{E}^{(k)}))\|.$$

By (7) we have $\|x(P_{j_0}(\mathbb{F}^{(h)}))\| \in A_{i_0}$, which implies $\|z_h\| \leq 1$. On the other hand, by (8) we have $\|x(P_{j_0-1}(\mathbb{E}^{(k)}))\| \in A_{i_0}$, which implies

$$\Big\|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\Big\| \geq n\,\sigma\,(1-\varepsilon)^{i_0}\,\frac{1}{\sigma}\left(\frac{1}{1-\varepsilon}\right)^{i_0-1} = n\,(1-\varepsilon).$$

Consequently $\boldsymbol{J}_n(X) \leq \varepsilon$. □

3. Problems and Examples

Example 1. $\boldsymbol{J}_n(\mathbb{R}) \geq 1 - 1/n$.

Proof. Let $|\xi_h| \leq 1$ for $h = 1, \ldots, n$. For $k = 1, \ldots, n$ define

$$\eta_k := \sum_{h=1}^{k} \xi_h - \sum_{h=k+1}^{n} \xi_h$$

and let $\eta_0 := -\eta_n$. Obviously $|\eta_k - \eta_{k+1}| \leq 2$ for $k = 0, \ldots, n-1$. Since $\eta_0 = -\eta_n$ there exists at least one k_0 such that $\operatorname{sgn} \eta_{k_0} \neq \operatorname{sgn} \eta_{k_0+1}$. Assume that $|\eta_{k_0}| > 1$ and $|\eta_{k_0+1}| > 1$, then $|\eta_{k_0} - \eta_{k_0+1}| > 2$, a contradiction. Hence there is k such that $|\eta_k| \leq 1$. This proves that

$$\inf_{1 \leq k \leq n} \Big| \sum_{h=1}^{k} \xi_h - \sum_{h=k+1}^{n} \xi_h \Big| \leq 1 = n \frac{1}{n},$$

and hence $\boldsymbol{J}_n(\mathbb{R}) \geq 1 - \frac{1}{n}$. □

Example 2. If q and ε are related by

$$\varepsilon \geq (1 - \varepsilon)^{q-1}$$

then $\boldsymbol{J}_n(l_q) \leq 4\varepsilon$ for all $n \in \mathbb{N}$.

Proof. Given $\varepsilon > 0$ find n_0 such that

$$\frac{1}{n_0} < \varepsilon \leq \frac{1}{n_0 - 1},$$

then

$$\Big(\frac{1}{n_0}\Big)^{1/q} \geq \Big(1 - \frac{1}{n_0}\Big)^{1/q} \varepsilon^{1/q} \geq 1 - \varepsilon.$$

If $n \leq n_0$, choosing

$$x_h := (\overbrace{-1, \ldots, -1}^{h}, \overbrace{+1, \ldots, +1}^{n-h}, 0, \ldots),$$

we obtain

$$\Big\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{n} x_h \Big\|_q \geq \Big\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{n} x_h \Big\|_\infty = n.$$

And since

$$\|x_h\|_q = n^{1/q} \leq n_0^{1/q} \leq 1/(1 - \varepsilon)$$

it follows that $\boldsymbol{J}_n(l_q) \leq \varepsilon$.

If $n > n_0$, there is $m \geq 2$ such that $(m-1)n_0 < n \leq mn_0$. Hence, by Properties (iii) and (iv) in the fact in Section 2 it follows that

$$\boldsymbol{J}_n(X) \leq \frac{mn_0}{n} \boldsymbol{J}_{mn_0}(X) \leq \frac{mn_0}{n} (\boldsymbol{J}_{n_0} + \frac{1}{n_0}) \leq \frac{mn_0}{n} 2\varepsilon \leq 4\varepsilon. \qquad \square$$

The main open problem of this article is the optimality of the estimate for N in Theorem 5.

Problem. *Are there $\sigma \geq 1$ and $\varepsilon > 0$ and a sequence of Banach spaces (X_n) such that*

$$\boldsymbol{S}_{f(n)}(X_n) \leq \sigma \quad \text{and} \quad \boldsymbol{J}_n(X_n) \geq \varepsilon,$$

where $f(n)$ is any function such that $f(n) > n$?

In particular $f(n) > P_m(n)$, where m is given by (5) *would show that the estimate in Theorem* 5 *for N is sharp in an asymptotic sense.*

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