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A REMARK ON ALMOST MOORE DIGRAPHS OF DEGREE THREE

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Abstract. It is well known that Moore digraphs do not exist except for trivial cases (degree 1 or diameter 1), but there are digraphs of diameter two and arbitrary degree which miss the Moore bound by one. No examples of such digraphs of diameter at least three are known, although several necessary conditions for their existence have been obtained. A particularly interesting necessary condition for the existence of a digraph of degree three and diameter $k \geq 3$ of order one less than the Moore bound is that the number of its arcs be divisible by $k+1$.

In this paper we derive a new necessary condition (in terms of cycles of the so-called **repeat permutation**) for the existence of such digraphs of degree three. As a consequence we obtain that a digraph of degree three and diameter $k \geq 3$ which misses the Moore bound by one cannot be a Cayley digraph of an Abelian group.

1. Introduction and Preliminaries

The well known **degree/diameter problem** for digraphs is to determine the largest order $n_{d,k}$ of a digraph of (out)degree at most d and diameter at most k. A straightforward upper bound on $n_{d,k}$ is the **Moore bound** $M_{d,k}$:

$$n_{d,k} \leq M_{d,k} = 1 + d + d^2 + \cdots + d^k.$$

It is well known that $n_{d,k} = M_{d,k}$ only in the trivial cases when $d = 1$ (directed cycles of length $k+1$) or $k = 1$ (complete digraphs of order $d+1$), see [**12**] or [**7**]. For $k = 2$, line digraphs of complete digraphs are examples showing that $n_{d,2} = M_{d,2} - 1$ if $d \geq 2$. On the other hand, if $d = 2$ then $n_{2,k} < M_{2,k} - 1$ for $k \geq 3$ (see [**11**]). Moreover, from the necessary conditions obtained in [**10**] it follows that, for example, $n_{2,k} < M_{2,k} - 2$ for $3 \leq k \leq 10^7$, $k \neq 274485, 5035921$. The question of whether or not equality can hold in $n_{d,k} \leq M_{d,k} - 1$ for $d \geq 3$ and $k \geq 3$ is completely open.

For convenience, a digraph of (out)degree at most d, diameter at most k (where $d \geq 3$ and $k \geq 2$) and order $M_{d,k} - 1$ will be called a (d,k)**-digraph**. It is an easy

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exercise to show that a (d,k)-digraph must, in fact, be diregular of degree d and its diameter must be equal to k.

Several necessary conditions for the existence of (d,k)-digraphs have been proved in [**2**], [**3**], [**4**], [**5**], [**6**]. In particular, for $d=3$ it was proved in [**3**] that $(3,k)$-digraphs do not exist if k is odd or if $k+1$ does not divide $\frac{9}{2}(3^k-1)$. All these conditions refer in one way or another to the so-called repeats which were first introduced in [**3**] and which we recall next.

Let G be a (d,k)-digraph. A simple counting argument shows that for each vertex u of G there exists exactly one vertex $r(u)$ in G with the property that there are two $u \to r(u)$ walks in G of length not exceeding k. The vertex $r(u)$ is called the **repeat** of u. It can be shown [**3**] that the mapping $v \mapsto r(v)$ is an **automorphism** of G. In what follows we shall therefore refer to r as the **repeat automorphism** of the (d,k)-digraph G.

Very recently, for $d=3$ it has been proved in [**6**] that all cycles of the repeat automorphism r (when written as a permutation of the vertex set of a $(3,k)$-digraph) must have the same length. However, cycles of length one are impossible, due to an earlier result of [**3**] which says that for $k \geq 3$ and $d \geq 2$ there is no (d,k)-digraph for which r is an identity automorphism.

The purpose of this note is to examine the other extreme; we show that the cycles of a repeat automorphism of a $(3,k)$-digraph cannot be too long (Section 4, Theorem 1). As a consequence of our method we shall prove that a $(3,k)$-digraph cannot be a Cayley graph of an Abelian group. We use an algebraic approach to the problem; the basics are introduced in Sections 2 and 3.

2. Algebraic Background

Let G be a digraph and let Γ be a subgroup of $\mathrm{Aut}\,(G)$, the group of all automorphisms of G, viewed as a group of permutations of the vertex set $V(G)$. In addition, let us assume that Γ is **semi-regular** on $V(G)$, that is, for any ordered pair of vertices $u,v \in V(G)$ (possibly $u=v$) there exists at most one automorphism $g \in \Gamma$ such that $g(u)=v$. Then we may define the **quotient digraph** G/Γ as follows. The vertex set $V(G/\Gamma)$ is the set of all orbits $O(u)=\{g(u);\ g \in \Gamma\}$ of the group Γ on $V(G)$. If $O(u), O(v)$ is any ordered pair of vertices of the quotient digraph G/Γ (that is, any pair of orbits of Γ on V(G); we do not exclude the case $O(u)=O(v)$) and if in the original digraph G there are t arcs emanating from u and terminating in $O(v)$, then there will be t parallel arcs in G/Γ emanating from $O(u)$ and terminating at $O(v)$. Note that in the case when $O(u)=O(v)$ the t arcs will become t loops attached at the vertex $O(u)$. The fact that quotient digraphs are well defined (i.e., incidence in the quotient graph does not depend on the choice of a particular vertex in the orbit) is an easy consequence of semi-regularity of Γ on $V(G)$. It is more important to notice that the projection $\rho\colon V(G) \to V(G/\Gamma)$ given by $\rho(u)=O(u)$ is a digraph epimorphism.

Note that if, in the situation above, the group Γ is **regular** on $V(G)$ – that is, if for any ordered pair $(u, v) \in V(G) \times V(G)$ there exists **exactly one** automorphism $g \in \Gamma$ such that $g(u) = v$ – then G is isomorphic to a Cayley digraph for the group Γ and the quotient digraph G/Γ consists of a single vertex only (with d loops attached to it if G is d-regular).

We shall soon be facing the following converse problem: Given a digraph H, what are the possible digraphs G and semi-regular subgroups $\Gamma < \mathrm{Aut}\,(G)$ for which the quotient digraph G/Γ is isomorphic to H? A complete answer can be given in terms of the so-called voltage assignments and lifts. Voltage assignments on (undirected) graphs were introduced in the early 70's [**8**] as a dual form of current graphs; the latter played a key role in proving the famous Map Color Theorem. Most of the theory (summarised in [**9**]) can be immediately transferred to digraphs, and in what follows we outline only the basic facts.

Let H be a digraph, possibly containing directed loops and/or parallel arcs. Let Γ be an arbitrary group. Any mapping $\alpha\colon\ D(H) \to \Gamma$ is called a **voltage assignment** on H. The **lift** of H by α, denoted by H^α, is the digraph defined as follows: $V(H^\alpha) = V(H) \times \Gamma$, $D(H^\alpha) = D(H) \times \Gamma$, and there is an arc (x, f) in H^α from (u, g) to (v, h) if and only if $f = g$, x is an arc from u to v, and $h = g\alpha(x)$. The mapping $\pi\colon H^\alpha \to H$ which erases the second coordinates, that is, $\pi(u, g) = u$ and $\pi(x, g) = x$ for each $u \in V(H)$, $x \in D(H)$ and $g \in \Gamma$, is called a **natural projection**. Clearly, π is a digraph epimorphism; the sets $\pi^{-1}(u)$ and $\pi^{-1}(x)$ are called **fibres** above the vertex u or above the arc x, respectively.

For any two vertices in the same fibre $\pi^{-1}(u)$ there exists an automorphism of the lift which sends the first vertex to the second. Indeed, without loss of generality, let $(u, id), (u, g) \in \pi^{-1}(u)$ be a pair of such vertices. Then it can be easily checked that the mapping $B_g\colon\ H^\alpha \to H^\alpha$, given by $B_g(v, h) = (v, gh)$ for each $(v, h) \in V(H^\alpha)$, is an automorphism of the lift H^α such that $B_g(u, id) = (u, g)$. Observe that the collection $\tilde{\Gamma} = \{B_g;\ g \in \Gamma\}$ forms a semi-regular subgroup (isomorphic to Γ) of the group $Aut(H^\alpha)$; the fibres coincide with the orbits of $\tilde{\Gamma}$.

A close connection between quotients and lifts may already be apparent from the definitions. Indeed, the basic result on semi-regular group actions on undirected graphs, which is Theorem 2.2.2 of [**9**], immediately translates to the following directed version:

Proposition 1. *Let G be a digraph and let $\Gamma < \mathrm{Aut}\,(G)$ be a semi-regular subgroup on $V(G)$. Then there exists a voltage assignment α on the quotient digraph G/Γ in the group Γ such that the lift $(G/\Gamma)^\alpha$ is isomorphic to G.*

Thus, for a given quotient digraph H, all possible digraphs G (and semi-regular groups Γ on $V(G)$) such that $G/\Gamma \simeq H$ can be re-constructed by considering voltage assignments on the digraph H and the corresponding lifts.

3. The Diameter of a Lift

We shall also be interested in recovering some properties of a lift from properties of the quotient. For this purpose we outline the connection between closed walks in the quotient and in the lift. Let α be a voltage assignment on a digraph H in a group Γ. Let $W = x_1x_2\dots x_m$ be a **walk** in H, i.e., an arc sequence in which the terminal vertex of x_{i-1} coincides with the initial vertex of x_i for each i, $2 \le i \le m$ (we allow an arc to be used repeatedly). The number m is the **length** of the walk W. The walk W is **closed** if the initial vertex of x_1 and the terminal vertex of x_m coincide. The **net voltage of** W is simply the product $\alpha(W) = \alpha(x_1)\alpha(x_2)\dots\alpha(x_m)$. For convenience, at each vertex we also admit a **trivial** closed walk of length 0 and of unit net voltage.

It is easy to see that for each walk $W = x_1x_2\dots x_m$ in H from a vertex u to vertex v and for each $g \in \Gamma$ there exists a **unique** walk $\tilde{W}$ in the lift H^α emanating from the vertex (u,g) and such that $\pi(\tilde{W}) = W$. This walk has the form $\tilde{W} = (x_1,g)(x_2,g\alpha(x_1))\dots(x_m,g\alpha(x_1)\alpha(x_2)\dots\alpha(x_{m-1}))$; it emanates in the lift from the vertex (u,g) and terminates at the vertex $(v,g\alpha(W))$. The walk $\tilde{W}$ is often called a **lift** of W.

Note that for any two distinct vertices $(u,g),(v,h)$ in $V(H^\alpha)$ there exists a path $\tilde{W}$ of length at most k from (u,g) to (v,h) if and only if the projection $W = \pi(\tilde{W})$ is a walk in the digraph H of length at most k from u to v with $\alpha(W) = g^{-1}h$. This immediately implies the following result on the diameter of the lift (cf. [**1**]):

Lemma 1. *Let α be a voltage assignment on a digraph H in a group Γ. Then $diam(H^\alpha) \le k$ if and only if for each ordered pair of vertices u,v of H (possibly $u = v$) and for each $g \in \Gamma$ there exists a walk of length $\le k$ from u to v whose net voltage is g.*

For any vertex $u \in H$ and any non-negative integer t let $\alpha[u;t]$ denote the set of all distinct voltages on closed walks in H of length t emanating from u. We now have an obvious corollary of Lemma 1:

Lemma 2. *Let α be a voltage assignment on a digraph H in a group Γ. If the diameter of the lift H^α is equal to k, then for each vertex $u \in H$,*

$$\sum_{t=0}^{k} |\alpha[u;t]| \ge |\Gamma|$$

Proof. According to Lemma 1 (the case $u = v$), if diam $(H^\alpha) = k$ then for each $u \in V(H)$ and for each $g \in \Gamma$ there exists a closed walk at u of length $\le k$ whose net voltage is equal to g. In other words, the union of all sets $\alpha[u;t]$, $0 \le t \le k$, is equal to Γ; this proves our inequality. $\square$

4. Results

Recall that for $d \geq 3$ and $k \geq 2$, by a (d, k)-digraph we understand any diregular digraph of degree d, diameter k and order $M_{d,k} - 1$. When referring to **cycles** of the repeat automorphism r we mean the cycles in the cycle decomposition of r, written as a permutation of $V(G)$.

Theorem 1. *Let G be a $(3, k)$-digraph, $k \geq 3$, let $n = |V(G)|$, and let r be the repeat automorphism of G. Then all cycles of r have equal length, smaller than $3n \log_3 k/(k + 1 - \log_3 k)$.*

Proof. Let Γ be the cyclic subgroup of $Aut(G)$ generated by r. By Theorem 3 of **[6]**, Γ acts semi-regularly on $V(G)$ with all orbits of equal size; let m be the size of each orbit. Consider the quotient digraph $H = G/\Gamma$ and let $V(H) = q$; clearly $n = |V(G)| = mq$. In order to prove our theorem it is sufficient to show that $q > (k + 1 - \log_3 k)/(3 \log_3 k)$.

According to Proposition 1, there exists a voltage assignment α on the quotient digraph H in the (cyclic) group Γ such that the lift H^α is isomorphic to the original digraph G. Although we have no information about the structure of the quotient digraph H (except that it has q vertices, each of degree three), we nevertheless may establish an upper bound on the number of distinct voltages on its closed walks as follows.

Let x_i, $1 \leq i \leq 3q$ be the collection of all arcs of H and let $\alpha(x_i) = a_i \in \Gamma$ be the corresponding voltages. Fix a vertex $u \in V(H)$ and estimate the number of elements in the set $\alpha[u; t]$ for a fixed $t \leq k$. Let W be a closed walk in H of length t, emanating from (and terminating at) u. Assume that the walk traverses j_i times the arc x_i, where $\sum_{i=1}^{3q} j_i = t$. The net voltage of W is then $\alpha(W) = \sum_{i=1}^{3q} j_i a_i$. From this we immediately see that the number of voltages appearing in the set $\alpha[u; t]$ is never greater than the number of ordered $3q$-tuples $(j_1, \ldots, j_{3q})$ of nonnegative integers whose sum is equal to t. The number of such ordered decompositions is well known to be equal to $\binom{t+3q-1}{3q-1}$. For the number of possible voltages on all closed walks at u of length $\leq k$ we therefore obtain:

$$\sum_{t=0}^{k} |\alpha[u; t]| \leq \sum_{t=0}^{k} \binom{t + 3q - 1}{3q - 1} = \binom{k + 3q}{3q} \tag{1}$$

Since diam $(H^\alpha) = k$, by Lemma 2 and the inequality (1) we have $|\Gamma| \leq \binom{k+3q}{3q}$. Recalling that the lift H^α is isomorphic to our $(3, k)$-digraph G with $n = 3(3^k - 1)/2$ vertices and that $|\Gamma| = m = n/q$, we obtain

$$\frac{3(3^k - 1)}{2q} \leq \binom{k + 3q}{3q} \tag{2}$$

In order to eliminate q, we observe that $\ell\binom{k+\ell}{\ell} < k^{\ell+1}$ for each $k \geq 3$ and $\ell \geq 1$. (Indeed, this is trivially true for $\ell = 1, 2$, and an easy induction works for $\ell \geq 2$.) Combining this inequality with (2) we finally obtain

$$3^{k+1} \leq 2q\binom{k+3q}{3q} + 3 \leq 3q\binom{k+3q}{3q} < k^{3q+1} ,$$

and hence $q > (k+1-\log_3 k)/(3\log_3 k)$. The proof is complete. $\square$

We have the following obvious corollary announced earlier.

Corollary 1. *Let G be a $(3,k)$-digraph, $k \geq 2$, and let r be the repeat automorphism of G. Then r cannot consist of a single cycle.*

Proof. If $k \geq 3$, the result follows directly from Theorem 1 because $q > 1$. For $k = 2$ it is sufficient to observe that the inequality (2) is **not** valid for $q = 1$. $\square$

The last result can be extended slightly by reformulating it in terms of Cayley digraphs. Let Γ be a (finite) group and let X be a generating set for Γ. The **Cayley digraph** $C(\Gamma, X)$ has vertex set Γ, and for any ordered pair of vertices $g, h \in \Gamma$ there is an arc emanating from g and terminating at h whenever $gx = h$ for some $x \in X$. We observe that $C(\Gamma, X)$ is a vertex-transitive digraph of degree $|X|$; the group Γ acts regularly on the vertex set of the Cayley digraph by left translations.

Corollary 2. *Let G be a $(3,k)$-digraph, $k \geq 2$. Then G cannot be a Cayley digraph of an Abelian group.*

Proof. Assume that a $(3,k)$-digraph G is isomorphic to a Cayley digraph $C(\Gamma, X)$ where Γ is an Abelian group. Since Γ acts regularly on $V(G)$, the quotient digraph G/Γ consists of precisely one vertex incident to three loops. Examining the proof of Theorem 1 one quickly sees that it is valid for all Abelian (not only cyclic) groups. The Corollary follows. $\square$

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